Pseudoscalar Mesons with Applications to Meson-Nucleon Scattering and Photoproduction*

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The scattering of π^+ and π^- mesons by protons and the production of π^+ mesons by gamma-rays incident on protons are studied with the quantized theory of pseudoscalar mesons with pseudoscalar coupling. The Dyson transformation is applied to the Hamiltonian to yield a new representation in which the velocity-independent aspects of the nucleon dynamics are more readily identified. A large "core" term appears in the meson-nucleon interaction in this representation and is incorporated into the zero-order Hamiltonian. The remaining interaction terms are treated perturbation-wise or according to the Heitler damping procedure. It is found that even after separating out the large core term, this method of approach fails to give agreement between theory and experiment. It does, however, improve on the results of the straight perturbation approximation and brings the theory in closer accord with the observations. It is suggested that a direct test of the role of the "core" term is possible in the process of double meson production by gamma-rays incident on protons.

I. INTRODUCTION

THERE are but a very few physical processes that are both measurable and sufficiently simple from the point of view of meson theory that one can avoid phenomenological methods, and, using only the Yukawa field equations, attempt to correlate the theory with the experimental data. It is important to study these processes and to attempt to uncover what, if any, aspects of the present field theory formalism can be used in establishing a quantitative check between theory and experiment. Perhaps characteristic discrepancies will be useful in pointing the way to appropriate modification of the theory.

In this paper we study the scattering of π^+ and π^- mesons by protons and the production of π^+ mesons by gamma-rays incident on protons. The quantized pseudoscalar meson and Dirac nucleon fields are assumed to interact by means of the nonderivative pseudoscalar coupling. Results obtained on the basis of a derivative pseudovector meson-nucleon interaction are compared. Perturbation calculations of the scattering and photoproduction cross sections assuming weak coupling between the meson and nucleons have already been given,[1] and they do not fit the experimental data. The aim of the present work is to narrow the gap between theory and experiment by going beyond the perturbation approximation rather than by adopting a phenomenological approach.

We apply the Dyson[2] transformation to the Hamiltonian for pseudoscalar mesons with pseudoscalar coupling. This yields a new representation in which the velocity-independent aspects of the nucleon dynamics are more readily identified.[3] The meson-nucleon interaction as described in the transformed Hamiltonian can be separated into one large term suggestive of a repulsive short-range core interaction between the mesons and nucleons plus other terms, one of which is the usual derivative coupling form. The large core term is incorporated into the zero-order Hamiltonian, and the remaining interaction terms are treated perturbation-wise or according to the Heitler damping procedure. It is found that even after separating out the large core term, a perturbation approach to the remaining portion of the meson-nucleon interaction fails to give agreement between theory and experiment.[4] With this method of approach it does prove possible, however, to improve on the straight perturbation approximation results in the pseudoscalar theory and to bring the theoretical results closer to the observations. We are encouraged to believe that the quantized field theory of charge-symmetric pseudoscalar mesons with pseudoscalar coupling to Dirac nucleons may prove fully adequate for quantitative analysis of elementary π-meson and single nucleon interactions when suitable methods for calculating in the intermediate coupling region are developed.

II. HAMILTONIAN

We treat the mesons as elements of a charge-symmetric pseudoscalar field. This choice is suggested by recent data and analyses of meson interactions with nucleons and deuterons.[5] We assume that the neutrons and protons (nucleons) represent the two charge states of a Dirac spinor field. The meson and nucleon fields are taken to interact by means of the nonderivative pseudoscalar coupling. Ward, Matthews, and Salam[6] have shown that this form of coupling has the advantage of yielding a theory that is renormalizable in the sense of Tomonaga, Schwinger, Feynman, and

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[1] References to the literature will be given in the following.
[2] F. J. Dyson, Phys. Rev. 73, 929 (1948).
[3] L. L. Foldy, Phys. Rev. 84, 168 (1951) and private communication. Foldy has recently developed a somewhat different transformation to a representation with similar features to the one we discuss here. See also G. Wentzel [Phys. Rev. 86, 802 (1952)] for a brief discussion of this transformation and its application.
[4] See conclusion in letter by G. Wentzel mentioned in reference 3.
[5] R. E. Marshak, Revs. Modern Phys. 23, 137 (1951).
[6] P. T. Matthews and A. Salam, Revs. Modern Phys. 23, 311 (1951).

Dyson. On the other hand, it is known that the derivative (pseudovector) coupling form gives a nonrenormalizable theory.

Including interactions with the electromagnetic field, the Hamiltonian is thus written as ($\hbar=c=1$):

$$H=H_0+H_f+H_e;$$

$$H_0=\tfrac{1}{2}\sum_{\alpha=1}^{3}\int(\pi_\alpha{}^2+|\nabla\phi_\alpha|^2+\mu^2\phi_\alpha{}^2)d\mathbf{r}+\int\bar{\psi}(\gamma\cdot\nabla+M)\psi d\mathbf{r}+\frac{1}{2}\int(\mathfrak{E}^2+\mathfrak{H}^2)d\mathbf{r},$$

$$H_f=if\sum_{\alpha=1}^{3}\int\bar{\psi}\gamma_5\tau_\alpha\phi_\alpha\psi d\mathbf{r}, \tag{1}$$

$$H_e=-e\int[\mathbf{A}\cdot(\phi_2\nabla\phi_1-\phi_1\nabla\phi_2)-\Phi(\pi_2\phi_1-\pi_1\phi_2)]d\mathbf{r}+e\int\bar{\psi}\tau_P(-i\gamma\cdot\mathbf{A}+\gamma_4\Phi)\psi d\mathbf{r}+e^2/2\int A^2(\phi_1{}^2+\phi_2{}^2)d\mathbf{r}.$$

M and μ are the nucleon and meson masses, respectively; f and e represent the meson and electromagnetic (for positive charge) coupling constants; ψ and $\bar{\psi}=\psi^*\gamma_4$ are the quantized nucleon field amplitudes (ψ^* is the Hermitian conjugate of ψ); π_α and ϕ_α, with $\alpha=1, 2, 3$, are the conjugate momenta and amplitudes of the charge symmetric meson field; $\mathbf{A}$ and Φ are the vector and scalar potentials of the electromagnetic field, related to the field strengths by $\mathfrak{H}=\text{curl}\mathbf{A}$, $\mathfrak{E}=-\text{grad}\Phi-\partial\mathbf{A}/\partial t$; $\gamma=i\alpha\beta$, $\gamma_4=\beta$, $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$ are the usual Dirac spinors; τ_α, $\tau_p=(1+\tau_3)/2$ are, similarly, the two by two isotopic spin (charge) matrices. The nonvanishing commutation relations in the Schrödinger representation for the field amplitudes are

$$\begin{aligned}&[\pi_\alpha(\mathbf{r}),\phi_\beta(\mathbf{r}')]=-i\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'),\\&[\mathfrak{E}_i(\mathbf{r}),\mathfrak{H}_{jk}(\mathbf{r}')]=i(\delta_{ij}\partial/\partial x_k-\delta_{ik}\partial/\partial x_j)\delta(\mathbf{r}-\mathbf{r}'),\\&[\psi_\mu{}^*(\mathbf{r}),\psi_\nu(\mathbf{r}')]_+=\delta_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}'),\end{aligned}\tag{2}$$

where μ and ν specify different spin and charge solutions of the Dirac equations.

The Schrödinger equation reads

$$i\partial\Omega/\partial t=H\Omega, \tag{3}$$

where Ω is the functional specifying the state of the fields. We proceed now by making a canonical transformation to a representation in which the velocity-independent aspects of the nucleon motion are more readily identified. By means of a time independent canonical transformation,

$$\Omega=\exp(-iS)\Psi, \tag{4}$$

Eq. (3) becomes

$$\begin{aligned}i\partial\Psi/\partial t&=\exp(iS)H\exp(-iS)\Psi\\&=(H+i[S,H]+i^2/2![S,[S,H]]+i^3/3![S,[S,[S,H]]]+\cdots)\Psi.\end{aligned}\tag{5}$$

We choose S to be[7]

$$S=f/2M\int\bar{\psi}\gamma_4\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\psi d\mathbf{r}. \tag{6}$$

This is the canonical transformation suggested by Dyson[2] in demonstrating the equivalence of pseudoscalar and pseudovector coupling forms through the first order in f and used more recently by Lepore[8] in discussing higher order processes. The repeated commutators of Eq. (5) may be summed exactly with the above choice of S to give, with $\sigma_i=-\gamma_5\alpha_i(i=1, 2, 3)$,

$$\begin{aligned}H'&=\exp(iS)H\exp(-iS)=H_0+H_e\\
&+\frac{f}{2M}\int\psi^*[(\sigma_i\nabla_i\boldsymbol{\phi}-\gamma_5\boldsymbol{\pi})\cdot\boldsymbol{\tau}-e\sigma_iA_i(\boldsymbol{\tau}\times\boldsymbol{\phi})_3]\psi d\mathbf{r}\\
&+M\int\psi^*\gamma_4\left[\left(1+i\frac{f}{M}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\times\exp[-i(f/M)\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}]-1\right]\psi d\mathbf{r}\\
&+\left(\frac{f}{2M}\right)^2\int\psi^*\gamma_5\left[\tfrac{1}{2}(\sigma_i\nabla_i\boldsymbol{\phi}-\gamma_5\boldsymbol{\pi})\cdot\boldsymbol{\phi}\times\boldsymbol{\tau}\frac{\sin^2\chi}{\chi^2}+\frac{1}{2}\frac{\sin^2\chi}{\chi^2}(\sigma_i\nabla_i\boldsymbol{\phi}-\gamma_5\boldsymbol{\pi})\cdot\boldsymbol{\phi}\times\boldsymbol{\tau}\right.\\
&\left.-e\sigma_iA_i(\boldsymbol{\phi}\times(\boldsymbol{\tau}\times\boldsymbol{\phi}))_3\frac{\sin^2\chi}{\chi^2}\right]\psi d\mathbf{r}+\frac{2}{3}\left(\frac{f}{2M}\right)^3\\
&\times\int\psi^*\left[\tfrac{1}{2}(\sigma_i\nabla_i\boldsymbol{\phi}-\gamma_5\boldsymbol{\pi})\times\boldsymbol{\phi}\cdot\boldsymbol{\phi}\times\boldsymbol{\tau}\frac{2\chi-\sin2\chi}{4\chi^3/3}+\frac{1}{2}\frac{2\chi-\sin2\chi}{4\chi^3/3}\boldsymbol{\phi}\times\boldsymbol{\tau}\cdot(\sigma_i\nabla_i\boldsymbol{\phi}-\gamma_5\boldsymbol{\pi})\times\boldsymbol{\phi}\right.\\
&\left.-e\sigma_iA_i\phi^2(\boldsymbol{\phi}\times\boldsymbol{\tau})_3\frac{2\chi-\sin2\chi}{4\chi^3/3}\right]\psi d\mathbf{r}\\
&+\frac{1}{2}\left(\frac{f}{2M}\right)^2\int|\psi^*\gamma_5\boldsymbol{\tau}\psi|^2d\mathbf{r}-\frac{2}{3}\left(\frac{f}{2M}\right)^4\\
&\times\int|\psi^*\gamma_5\boldsymbol{\tau}\times\boldsymbol{\phi}\psi|^2\frac{\chi^2-\sin^2\chi\cos^2\chi}{4\chi^4/3}d\mathbf{r}\\
&+\frac{1}{2}\left(\frac{f}{2M}\right)^4\int|\psi^*\boldsymbol{\tau}\times\boldsymbol{\phi}\psi|^2\frac{\sin^4\chi}{\chi^4}d\mathbf{r}.\\
&+\frac{1}{2}\left(\frac{f}{2M}\right)^3\int(\psi^*\boldsymbol{\tau}\psi\cdot\psi^*\gamma_5\boldsymbol{\tau}\times\boldsymbol{\phi}\psi\\
&-\psi^*\gamma_5\boldsymbol{\tau}\psi\cdot\psi^*\boldsymbol{\tau}\times\boldsymbol{\phi}\psi)\frac{\sin^3\chi\cos\chi}{\chi^3}d\mathbf{r},\end{aligned}\tag{7}$$

[7] In the following we apply vector notation in isotopic spin space and use component notation for ordinary space vectors.

[8] J. V. Lepore, Phys. Rev. 87, 209 (1952). The interaction representation is used in this work. We stay in the Schrödinger representation here, since we do not make a straight perturbation expansion.

where $\chi=(f/2M)\boldsymbol{\tau}\cdot\boldsymbol{\phi}$, and expansions of all factors which involve χ contain only χ^{2n}, with n an integer and $\chi^2=(f/2M)^2\phi^2$. The transformed Hamiltonian H' can be analyzed to show where perturbation calculations may be expected to be valid and for what processes the pseudoscalar and pseudovector coupling theories will yield different results. In Eq. (7) H_0 and H_e are the same expressions as in Eq. (1) for the uncoupled fields and their electromagnetic interactions. The third term, which we denote hereafter as H_{pv}, is the usual derivative coupling form of the meson-nucleon Hamiltonian and shows the equivalence of the two couplings in the pseudoscalar theory through terms of first order in the coupling constant. The coupling constant g normally used for the derivative interaction is related to the constant f used here by

$$g/\mu=f/2M. \tag{8}$$

An expansion of the fourth term in H', hereafter referred to as H_c, in powers of f, gives to leading order

$$(f^2/2M)\int\bar{\psi}\psi\phi^2d\mathbf{r}. \tag{9}$$

This expression is analogous to the quadratic term, $(e^2/2m)A^2$, in the vector potential that appears in a two-component reduction of the Dirac equation.[9] The importance of this term has been recognized and emphasized in the analysis of the nucleon-nucleon interaction problem.[8,10,11] Even in a perturbation expansion in powers of f it has been shown to give a large spin independent contribution to nuclear forces. The relative importance of this term may be understood by inspection of Eq. (7). The H_c term is proportional to the large nucleon mass M, whereas M appears in other terms there describing the meson-nucleon interaction only in the combination (f/M). Also the characteristic length associated with the meson field is $1/\mu$, and we may expect the other terms in H' to be reduced relative to H_c by various powers of the ratio $\mu/M\cong 0.15$.

In the elastic scattering of very low energy mesons by nucleons, matrix elements of Eq. (9) are known to give the dominant contribution in a perturbation calculation. This explains much of the difference in scattering cross sections between the pseudoscalar and pseudovector coupling theories as calculated by Ashkin, Simon, and Marshak.[12] The contribution of the quadratic term, Eq. (9), is analogous to the Thomson scattering of light by charged particles in the low energy limit. Since nucleon recoil may be neglected at low energies, Eq. (9) may be written as

$$(f^2/2M)\int K(\mathbf{r}-\mathbf{r}_N)\phi^2d\mathbf{r}, \tag{10}$$

where $\int K(\mathbf{r}-\mathbf{r}_N)d\mathbf{r}=1$ represents a fixed nucleon "source" at $\mathbf{r}_N$. The form and sign of Eq. (10) suggest a short-range interaction of mesons with a repulsive nucleon core. Indeed, we note that Eq. (10), together with the mass term, $\frac{1}{2}\mu^2\int\phi^2d\mathbf{r}$ of the meson self-field can be interpreted as giving the meson an effective mass $\mu'=(\mu^2+f^2K(\mathbf{r}-\mathbf{r}_N)/M)^{\frac{1}{2}}$ inside the nucleon source. For a constant source of radius equal to a nucleon Compton wavelength $1/M$, this gives

$$\mu'=\mu(1+3(f^2/4\pi)(M/\mu)^2)^{\frac{1}{2}}. \tag{11}$$

An analogous result has been obtained by Schiff[13] in his nonlinear meson theory. The nonlinearity in the meson equations was observed there to give rise to a strong short-range repulsion between nuclear matter and small amplitude meson waves in the vicinity. In order to estimate the effect of the mass change due to the core, we may tentatively assign a value to $f^2/4\pi$ of the order of 7,[14] as suggested by perturbation calculations of the anomalous magnetic moment of the neutron[15] and of the nucleon-nucleon interaction.[16] The effective mass of the meson interacting with the nucleon source thus appears to be approximately 30μ. It increases for smaller source radii. For a large source radius, of the order of $5/2M$, as suggested by Lévy[11] and Jastrow,[17] the effective mass is still approximately 8μ. These surprisingly large results for μ' suggest two things: (a) It is exceedingly dangerous to attempt to draw any conclusions from meson theoretic calculations based on a perturbation expansion in powers of f, and (b) it is necessary to consider the terms of higher order in f in H_c.

The fifth and sixth terms in Eq. (7) (we denote the fifth term by H_π hereafter) are given in a form that exhibits both their relation to H_{pv} and their invariance under a gauge transformation of the electromagnetic potentials. This latter property is seen to follow directly from the usual gauge invariant prescription

$$\nabla\phi_1\rightarrow\nabla\phi_1-e\mathbf{A}\phi_2,$$
$$\nabla\phi_2\rightarrow\nabla\phi_2+e\mathbf{A}\phi_1,$$
$$\nabla\phi_3\rightarrow\nabla\phi_3,$$

with which the potentials are introduced into a Hamiltonian theory. The trigonometric factors that multiply

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[9] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 317.

[10] R. P. Feynman, hectographed notes of lectures at California Institute of Technology on "High Energy Phenomena and Meson Theories," January (1951), unpublished.

[11] M. M. Lévy, Phys. Rev. **86**, 806 (1952).

[12] Ashkin, Simon, and Marshak, Prog. Theor. Phys. **5**, 634 (1950). See also M. Peshkin, Phys. Rev. **81**, 425 (1951).

[13] L. I. Schiff, Phys. Rev. **84**, 1 (1951); see also L. L. Foldy, reference 3.

[14] By Eq. (8) this corresponds to a $g^2/4\pi\sim 0.04$.

[15] J. M. Luttinger, Helv. Phys. Acta **21**, 483 (1948); M. Slotnick and W. Heitler, Phys. Rev. **75**, 1645 (1949); K. M. Case, Phys. Rev. **76**, 1 (1949); S. D. Drell, Phys. Rev. **76**, 427 (1949); L. L. Foldy, Phys. Rev. **83**, 688 (1951); B. D. Fried, Phys. Rev. **86**, 434 (1952).

[16] K. M. Watson and J. V. Lepore, Phys. Rev. **76**, 1157 (1949).

[17] R. Jastrow, Phys. Rev. **81**, 636 (1951); **87**, 209 (1952).

these terms in Eq. (7) reduce to unity in the weak coupling limit, $\chi \ll 1$, but are much smaller than one for larger values of χ (or of the coupling constant f), decreasing at least as $1/\chi^2$ for χ larger than one. The fifth term H_π contributes to the matrix elements of order f^2 in a perturbation calculation of meson-nucleon scattering. The seventh term in Eq. (7) has the form of a nucleon-nucleon direct interaction and will not operate in the processes with which this paper is concerned.[18] The remaining terms in the Hamiltonian, expressed in the form of Eq. (7), are quadratic in the bilinear form $(\bar{\psi}O\psi)$, where O is any operator, and involve only f^3 and higher powers of the coupling constant. If the meson-nucleon interaction is weak and the perturbation procedure of expanding in powers of f is valid, then these higher order terms may be expected to add only a minor contribution, relative to H_{pv} and H_c, to matrix elements for elementary processes such as meson-nucleon scattering and photoproduction of mesons. In this weak coupling limit all the trigonometric factors, such as $(\chi^2 - \cos^2\chi \sin^2\chi)3/4\chi^4$, approach unity. If, on the other hand, we go to the strong coupling limit of large f, these trigonometric factors may be expected to be considerably reduced, the one exhibited above decreasing as $1/\chi^2$. However, since these terms are seen to be proportional to a high power of f, it is difficult to estimate their relative importance in the Hamiltonian when the weak coupling approximation breaks down. We have no methods at present that permit a complete study of them in the form in which they are displayed in Eq. (7).

From the Hamiltonian in the form of Eq. (7), one can see which basic processes will have identical or different matrix elements in a lowest order perturbation treatment of the pseudoscalar and derivative coupling forms. As mentioned above, the quadratic term [Eq. (9)] resulting from a reduction of the series H_c in Eq. (7) contributes to the matrix elements of order f^2 for the elastic scattering of mesons by nucleons only for a pseudoscalar coupling theory, as does H_π. They are not present in the usual derivative coupling theory. For photoproduction of mesons by gamma-rays incident on protons, and two coupling forms are identical to order ef in the matrix elements.[19] In the calculation of the anomalous magnetic moments of nucleons the matrix elements of order ef^2 are identical for both coupling forms.[20] However, the matrix elements of this order for the electron-neutron interaction differ for the two types of coupling and converge only in the pseudoscalar coupling theory in virtue of the H_π term in Eq. (7).[21]

We have seen that the leading order contribution [Eq. (9)] in the expansion of H_c is quite large. We are thus motivated to study the higher order terms. An attempt to estimate their contribution in the nonrelativistic limit is developed as follows. We separate H_c of Eq. (7) into terms proportional to even and odd powers of f. The even terms contain $\gamma_5^2=1$ and $\tau_i^2=1$ and thus may be written, as in Eq. (10), as an interaction of mesons with a classical nucleon source density in the limit of a nonrelativistic treatment of the nucleons. The odd terms (f^3 is the lowest order of the coupling constant that appears), on the other hand, are proportional to γ_5, which has matrix elements between positive energy nucleon states of order $|v_N| \ll 1$. Neglecting the odd terms in our approximation, we write

$$H_c \approx M \int K(\mathbf{r}-\mathbf{r}_N)\{\cos 2\chi + 2\chi \sin 2\chi - 1\} d\mathbf{r}. \quad (12)$$

The matrix element of Eq. (12) for meson scattering and pair creation or annihilation can be evaluated by the methods of Glauber.[22] We obtain, in the notation of Glauber,

$$\begin{aligned}\langle H_c \rangle_2 &= M \int K(\mathbf{r}-\mathbf{r}_N) \langle \cos 2\chi + 2\chi \sin 2\chi - 1 \rangle_2 d\mathbf{r}, \\ &= M \int K(\mathbf{r}-\mathbf{r}_N)(f^2/2M^2)\langle \phi^2 \rangle_2 \\ &\qquad\qquad \langle \cos 2\chi - 2\chi \sin 2\chi \rangle_0 d\mathbf{r}, \\ &= M \int K(\mathbf{r}-\mathbf{r}_N)(f^2/2M^2)\langle \phi^2 \rangle_2 \\ &\quad \times [1-(f^2/M^2)\langle \phi^2 \rangle_0] \exp[-(f^2/2M^2)\langle \phi^2 \rangle_0] d\mathbf{r}. \quad (13)\end{aligned}$$

Since nucleon recoil has been neglected, we use a maximum meson momentum cutoff of $k_m \approx M$ in evaluating $\langle \phi^2 \rangle_0$, obtaining $3M^2/8\pi^2$. This gives

$$\begin{aligned}\langle H_c \rangle_2 &= (f^2/2M) \int K(\mathbf{r}-\mathbf{r}_N)\langle \phi^2 \rangle_2 d\mathbf{r} \\ &\quad \times [1-(3/2\pi)f^2/4\pi] \exp[-(3/4\pi)f^2/4\pi]. \quad (14)\end{aligned}$$

If we compare Eqs. (10) and (14), we see that the higher order terms are responsible for a reduction of roughly

$$[1-(3/2\pi)f^2/4\pi] \exp[-(3/4\pi)f^2/4\pi]. \quad (15)$$

This reduction factor is plotted in Fig. 1. It vanishes at $f^2/4\pi = 2.1$. For $f^2/4\pi = 7$ it is equal to -0.44; the minus sign indicates that the interaction is one of attraction, in contrast to the repulsive core result indicated by Eq. (10). For a maximum momentum cutoff of $k_m = 2M/5$, corresponding to a larger core radius,[11,17] the reduction factor is equal to $+0.38$.

[18] Such terms are discussed by R. K. Osborn, Phys. Rev. **86**, 370 (1952).

[19] B. Araki, Prog. Theor. Phys. **5**, 507 (1950); K. M. Case, Phys. Rev. **76**, 14 (1949).

[20] B. D. Fried, Ph.D. thesis, University of Chicago, Department of Physics, June (1952), unpublished.

[21] The terms H_π and H_{pv} give individually diverging contributions to the matrix elements of order ef^2 describing the electron-neutron interaction. It is only their sum that converges in this order of a perturbation calculation to the pseudoscalar result. See also B. D. Fried, Phys. Rev. **88**, 1152 (1952), and reference 20.

[22] R. J. Glauber, Phys. Rev. **84**, 395 (1951).

If we treat H_c in Eq. (12) as a self-energy term and calculate its vacuum expectation value, we obtain by the same methods a reduction

$$[(3/4\pi)f^2/4\pi]^{-1}\{[1+(3/2\pi)f^2/4\pi] \times\exp[-(3/4\pi)f^2/4\pi]-1\}, \quad (16)$$

owing to the higher order terms. This factor is also plotted in Fig. 1 for comparison with Eq. (15).

In view of the fact that only the first-order scattering matrix element is evaluated in Eq. (14) and that the effect of nucleon recoil, neglected in this procedure, is simulated by a maximum momentum cutoff $k_m \approx M$, we can draw only qualitative suggestions from this calculation. Clearly, for values of $f^2/4\pi$ as large as those quoted above, the entire S matrix, must be studied and the renormalization techniques employed. However, it is immediately obvious that the power series expansion from which Eqs. (10) and (11) are derived is not to be trusted. Only for small effective values of χ can the trigonometric factors $\cos 2\chi + 2\chi \sin 2\chi - 1$ in Eq. (12) be approximated by $2\chi^2$. For larger values of χ, this factor oscillates with period 2π and increases in amplitude in proportion to χ. Also, unless the effective value of χ is much smaller than unity, a perturbation calculation is, at best, of questionable validity. We see from Eq. (15) that for scattering, it is necessary that $f^2/4\pi < 1.0$ for perturbation methods to be applicable.

We make here the basic assumption that will guide our treatment of H_c in the following sections of this paper. In studying the elastic scattering of mesons by nucleons and meson photoproduction in energy ranges which admit a nonrelativistic description of the nucleon source, we replace H_c of Eq. (7) by

$$\lambda(f^2/2M)\int \bar{\psi}\psi\phi^2 d\mathbf{r}, \quad (17)$$

with λ a constant of magnitude less than one, whose value remains to be specified. For positive (negative) values of λ, Eq. (17) corresponds to a repulsive (attractive) short-range meson-nucleon interaction. The "constant" λ in general may have different values depending on the manner in which Eq. (17) operates in various processes. Thus, the two reduction factors in Eqs. (15) and (16), plotted in Fig. 1, apply when H_c is treated as a scattering and self-energy term, respectively. However, in applying the Hamiltonian in Eq. (7) to scattering and photoproduction in the following paragraphs, we shall be able to deduce only that the predictions of the pseudoscalar meson theory suggest qualitatively some features of the observed cross sections when $\lambda < 1$ is introduced into the core term H_c.

We take as a working Hamiltonian with which to discuss current accelerator experiments on meson scat-

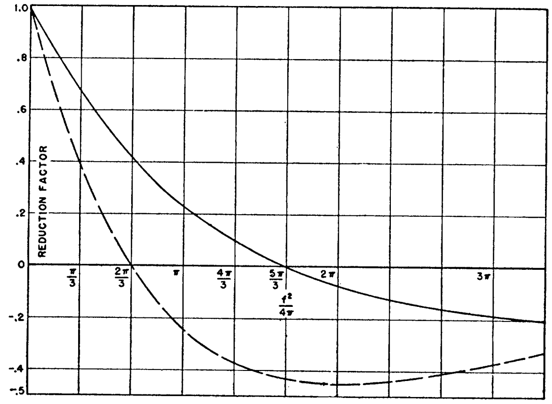

FIG. 1. Reduction in the core term due to higher order contributions plotted *vs* $f^2/4\pi$. The dashed curve represents the factor in Eq. (15) expressing the reduction for a scattering or pair process. The solid curve represents the factor in Eq. (16) for the reduction in the self-energy.

tering and photoproduction from nucleons,

$$H=H_0+H_e+f/2M\int \psi^*[\sigma_i\nabla_i\boldsymbol{\phi}\cdot\boldsymbol{\tau}-e\sigma_iA_i(\boldsymbol{\tau}\times\boldsymbol{\phi})_3]\psi d\mathbf{r}$$

$$+\lambda f^2/2M\int\bar{\psi}\psi\phi^2 d\mathbf{r}+(f/2M)^2\int\psi^*\boldsymbol{\pi}\cdot\boldsymbol{\tau}\times\boldsymbol{\phi}\psi d\mathbf{r}. \quad (18)$$

Terms proportional to the nucleon velocity are neglected since their contributions are small compared to the effects we are interested in studying.

III. ELASTIC SCATTERING

We consider in this section the elastic scattering of mesons by protons. The following three processes have been investigated at Columbia[23] and Chicago[24–27]

$$\begin{aligned} P+\pi^+&\rightarrow P+\pi^+ \quad &(a),\\ P+\pi^-&\rightarrow P+\pi^- \quad &(b), \quad (19)\\ P+\pi^-&\rightarrow N+\pi^0 \quad &(c).\end{aligned}$$

The observations on the angular and energy dependence of the cross sections for these processes are shown in references 26 and 27.

Theoretical interpretation of these curves has succeeded only when phenomenology has been used. Brueckner[28] has interpreted the results in terms of nucleon isobars, and the Chicago group in terms of scattering phase shifts.[27] Our aim here is to see to what extent a straight meson field theoretic approach succeeds in fitting the data.

Perturbation calculations fail to predict the observed behavior. A thorough relativistic treatment of this

[23] Isaacs, Sachs, and Steinberger, Phys. Rev. **85**, 803 (1952).
[24] Anderson, Fermi, Long, Martin, and Nagle, Phys. Rev. **85**, 934 (1952).
[25] Fermi, Anderson, Lundby, Nagle, and Yodh, Phys. Rev. **85**, 935 (1952).
[26] Anderson, Fermi, Long, and Nagle, Phys. Rev. **85**, 936 (1952).
[27] Anderson, Fermi, Nagle, and Yodh, Phys. Rev. **86**, 793 (1952).
[28] K. Brueckner, Phys. Rev. **86**, 106 (1952).

problem has been given by Ashkin, Simon, and Marshak.[12] They calculate the matrix elements for scattering through order f^4 in the pseudoscalar coupling theory and through order g^2 with derivative coupling (in this case the higher order terms diverge and are not renormalizable). The sharp rise in cross section as a function of meson energy is suggested by the calculation with the derivative coupling form H_{pv}. We write the differential cross section in the center-of-mass system for this case, in the low energy approximation that neglects nucleon recoil, as

$$\begin{aligned} d\sigma(\pi^+) &= (g^2/4\pi)^2(4q^4/\mu^2\omega^2)d\Omega/\mu^2 \quad &(a),\\ d\sigma(\pi^-) &= d\sigma(\pi^+) &(b),\\ d\sigma(\pi^0) &= 2\cos^2\theta d\sigma(\pi^+) &(c), \end{aligned} \qquad (20)$$

where $d\sigma(\pi^+)$, $d\sigma(\pi^-)$, and $d\sigma(\pi^0)$ are the differential cross sections for the three processes as given in Eq. (19). The coupling constant g for the derivative coupling theory is defined by Eq. (8), ω and q are the meson energy and momentum, and θ is the meson scattering angle in the collision center-of-mass system. The theoretical prediction of a spherically symmetric π^+ scattering cross section is in serious disagreement with the experimental results, as is the ratio of the elastic π^+ to elastic plus charge exchange π^- total cross sections. The experimental ratio is of the order of 2 at 137 Mev, whereas the above formulas predict a ratio of 3/5 for all energies.

The cross sections derived on the basis of a weak coupling pseudoscalar coupling theory differ from the results in Eq. (20), as mentioned earlier, in virtue of the pair term H_c of Eqs. (7) and (10) and the fifth term H_π to order f^2 in Eq. (7). The results are

$$\begin{aligned} d\sigma(\pi^+) &= (f^2/4\pi)^2[(1+\omega/2M)^2+q^4/4M^2\omega^2 \\ &\quad -(q^2/M\omega)(1+\omega/2M)\cos\theta]d\Omega/M^2 \quad &(a),\\ d\sigma(\pi^-) &= (f^2/4\pi)^2[(1-\omega/2M)^2+q^4/4M^2\omega^2 \\ &\quad +(q^2/M\omega)(1-\omega/2M)\cos\theta]d\Omega/M^2 &(b),\\ d\sigma(\pi^0) &= 2(f^2/4\pi)^2[(\omega/2M)^2+(q^4/4M^2\omega^2)\cos^2\theta \\ &\quad -(q^2/M\omega)(\omega/2M)\cos\theta]d\Omega/M^2 &(c). \end{aligned} \qquad (21)$$

These formulas are calculated from Eq. (7) with neglect of nucleon recoil[29] or from Eq. (18) in the perturbation limit $\lambda\rightarrow 1$. The second term in the right-hand member of these expressions represents the contribution of H_{pv} and is seen to correspond to the cross-section result of Eq. (20) when relation (8) is inserted. The core term H_c contributes the factor unity in the first terms in the right members of $d\sigma(\pi^+)$ and $d\sigma(\pi^-)$ but does not operate in calculations of the charge exchange scattering to this order in f. The factor $\omega/2M$ in the first terms results from H_π, the fifth term in Eq. (7). The third terms in the right members are cross terms that appear when the matrix elements are squared to give a cross section.

Again we see that these cross sections stand in disagreement with the experimental data. Whereas the experimental points indicate a sharp rise in cross section with energy and a strong $\cos^2\theta$ angular dependence for π^+ mesons with energies of the order of 100 Mev, the expressions (21a) and (21b) are essentially energy and angle independent in this energy range in virtue of the dominant contribution from the core term H_c. Also the ratio of the total cross sections $\sigma(\pi^+):\sigma(\pi^-):\sigma(\pi^0)\cong 1:1:\omega^2/2M^2$ is seen to differ considerably from observation.

It is clear from the above that the first step in an attempt to reconcile the pseudoscalar coupling theory with observation must reduce the core contribution relative to the p wave contribution of H_{pv}, as expressed in Eq. (20), so that the rapid rise of the experimental cross section with meson energy may be understood. As discussed in the previous section, we may expect such a reduction when higher order terms in f are considered, and we have lumped their effect into a constant λ given in Eq. (18). A perturbation calculation on the basis of this Hamiltonian gives scattering cross sections that differ from those given by Eq. (21) only in that the constant $\lambda<1$ replaces unity whenever the latter appears there, i.e., $(1\pm\omega/2M)\rightarrow(\lambda\pm\omega/2M)$. The charge exchange cross section $\sigma(\pi^0)$ is unchanged since H_c does not contribute to it. The energy variation indicated by these results for $\lambda\leq\mu/2M\sim 1/13$ suggests the sharp experimentally observed rise with energy, but it is still somewhat too gradual. Also, the predicted angular asymmetry is not in accord with the data. We reserve further discussion of this latter point till later. We note here, however, that it is possible to achieve a more accurate treatment of H_c and that it is reasonable to expect an improved calculation of H_c to increase the relative importance of the p wave contribution H_{pv}, thereby sharpening the predicted energy rise of the cross sections.

The argument for this develops as follows. Consider the wave equation for the meson amplitudes as obtained from the Hamiltonian [Eq. (18)] in the absence of electromagnetic fields. If we neglect for the moment all terms but those obtained from H_0+H_c, the wave equation is

$$[\Delta+k^2-\lambda(f^2/M)K(\mathbf{r})]\phi_{\mathbf{k}\alpha}=0, \qquad (22)$$

for an eigenstate of energy $\omega=(k^2+\mu^2)^{\frac{1}{2}}$. The straight perturbation procedure given above for evaluating the scattering matrix elements corresponds to solving Eq. (22) by ordinary Born approximation. That is, the zero-order equation and solutions for the meson amplitudes are

$$(\Delta+k^2)\phi_{\mathbf{k}\alpha}{}^0=0, \qquad (23)$$

$$\phi_{\mathbf{k}\alpha}{}^0=(2\omega L^3)^{-\frac{1}{2}}[a_{\mathbf{k}\alpha}\exp(i\mathbf{k}\cdot\mathbf{r})+\text{complex conj.}], \qquad (24)$$

where L^3 is the normalization volume, and $a_{\mathbf{k}\alpha}(a_{\mathbf{k}\alpha}{}^*)$

[29] Recoil corrections to Eqs. (21a) and (21b) due to the core term are of the same order of magnitude as the ω^2/M^2 terms in the square brackets and must be included if the formulas are to be discussed with this order of accuracy.

is the meson annihilation (creation) operator. The third term in Eq. (22) is then treated as a small perturbation. However, the usual criterion for application of the Born approximation to scattering of a particle of mass μ by a potential of radius a and height V_0 requires[30]

$$|\mu V_0 a^2| \ll 1 \tag{25}$$

for $ka \ll 1$, where k is the particle momentum. The core term of Eq. (22) corresponds to a square potential of radius a and height $(\lambda f^2/2\mu M)(3/4\pi a^3)$, so that the criterion (25) becomes $|(3/2)\lambda f^2/4\pi| \ll 1$ for slow mesons with $ka \approx k/M \ll 1$, where a has been chosen the order of a nucleon Compton wavelength. We shall see below, however, that it will be necessary to take $\lambda f^2/4\pi > 1$ in order to approximate data on meson scattering and photoproduction. It, therefore, becomes desirable to abandon the Born approximation. We solve Eq. (22), instead, by the method of phase shifts as applied in ordinary single particle scattering by a fixed potential. Since we have $ka \ll 1$ for the processes under discussion, we can confine our attention to the s and p wave phase shifts. This method of calculation gives directly the amplitude of the scattered wave due to H_c. We also use solutions of Eq. (22) as the zero-order meson amplitudes in terms of which to calculate the scattering matrix elements of $H_{pv}+H_\pi$.

Since the matrix elements of H_{pv} are proportional to gradϕ inside the nucleon core, whereas those of H_π depend on the amplitude, the ratio of the p wave scattering of H_{pv} to the s wave scattering of H_π is increased over the value obtained when perturbation methods were applied to the core. In other words, the effect of treating the core in the zero-order wave equation is to cut down the amplitude of the meson field at the nucleon source and to thereby increase the ratio of p to s wave scattering.

In order to compare the solutions of Eq. (22) with the ones exhibited above for Eqs. (23)–(24), we expand the latter in radial and angular functions

$$\phi_{\mathbf{k}\alpha}{}^0 = (2\omega L^3)^{-\frac{1}{2}}[a_{\mathbf{k}\alpha} \textstyle\sum_l c_l j_l(kr) P_l(\cos\theta) + \text{complex conj.}],$$

where j_l is a spherical Bessel function and P_l is a Legendre polynomial of order l. The radial solutions of Eq. (22) inside and outside the core give in place of $c_l j_l(kr)$

$$\begin{aligned} l=0 \quad & r\leqslant a\cdots c_0 \,\mathrm{sech}\beta a\; i_0(\beta r),\\ & r> a\cdots c_0[j_0(kr)+ka(1-\tanh\beta a/\beta a)n_0(kr)],\\ l=1 \quad & r\leqslant a\cdots -c_1 ka\,\mathrm{csch}\beta a\; i_1(\beta r),\\ & r> a\cdots c_1[j_1(kr)+(ka)^3\{\tfrac{1}{3}-\coth\beta a/\beta a\\ &\qquad +1/(\beta a)^2\}n_1(kr)], \end{aligned} \tag{26}$$

where

$$\beta a = \left[3\lambda\left(\frac{f^2}{4\pi}\right)\left(\frac{1}{Ma}\right) - k^2a^2\right]^{\frac{1}{2}} \approx \left[3\lambda\left(\frac{f^2}{4\pi}\right)\left(\frac{1}{Ma}\right)\right]^{\frac{1}{2}}$$

is approximately constant if $\lambda \gg (k^2/3M^2)(f^2/4\pi)^{-1}$ for $a\approx 1/M$; η_l is the spherical Neumann function, and i_l is the spherical Bessel function of imaginary argument defined so that $i_0(z)=\sinh z/z$ and $i_1(z)=\sinh z/z^2 - \cosh z/z$; $c_l=(2l+1)i^l$. Corrections to these solutions are of order $(ka)^2 \ll 1$. The s wave scattering amplitude due to the core alone is then given by [31]

$$-a(1-\tanh\beta a/\beta a) = k^{-1}\sin\delta_0. \tag{27}$$

Furthermore, we can use the solutions given in Eq. (26) rather than the plane waves [Eqs. (23)–(24)] in calculating the scattering matrix elements due to $H_{pv}+H_\pi$. An additional improvement is possible if we modify the perturbation method to include radiation damping in the manner first developed by Heitler.[32] Unfortunately, there exists no quantitative procedure (short of an exact solution) by which to gauge the accuracy of the Heitler approximation, which (approximation) takes into account only the "resistive" effect of higher order processes.[33] It will be seen, however, that as a result of invoking the Heitler damping theory, we are able to bring the calculated cross sections one step closer to the observed scattering data.

Goldberger[34] and others have already applied the Heitler damping procedure to the calculation of meson scattering by nucleons for the pseudovector theory in which only H_{pv} operates in the meson-nucleon interaction and H_0 describes the propagation of free mesons. The work we wish to present here is an extension of the methods given by Lippman and Schwinger[35] and by Goldberger to the case in which the zero-order Hamiltonian is taken as H_0 plus the core term H_c, and the interaction is given by $H_{pv}+H_\pi$. But first we present for discussion the differential cross sections calculated in the straight pseudovector coupling theory. With neglect of nucleon recoil, they are

$$\begin{aligned} d\sigma(\pi^+) &= 4(g^2/4\pi)^2\frac{q^4}{\omega^2\mu^2}\left(\frac{1+4x^2\cos^2\theta}{(1+2x^2)^2+x^2}\right)\frac{d\Omega}{\mu^2},\\ d\sigma(\pi^-) &= 4(g^2/4\pi)^2\frac{q^4}{\omega^2\mu^2}\left(\frac{1}{1+9x^2}\right)\frac{d\Omega}{\mu^2},\\ d\sigma(\pi^0) &= 8(g^2/4\pi)^2\frac{q^4}{\omega^2\mu^2}\left(\frac{\cos^2\theta}{1+2x^2}\right)\frac{d\Omega}{\mu^2}, \end{aligned} \tag{28}$$

where $x=\frac{2}{3}(g^2/4\pi)(q^3/\omega\mu^2)$ corresponds to the x of

[30] L. I. Schiff, reference 13, p. 168.

[31] For $\beta a \gg 1$ this reduces to the strong coupling limit given by Wentzel, (see reference 3) and references to previous work given there. Recently the wave equation of the form Eq. (22) has been studied more thoroughly in connection with the nonlinear meson theory, as formulated by Schiff (reference 13), by D. R. Yennie (to be published) and by E. M. Henley, unpublished. The methods we use here are sufficiently precise for our purposes since the core in our equation is the heavy nucleon.

[32] W. Heitler, Proc. Cambridge Phil. Soc. **37**, 291 (1941).

[33] J. M. Blatt, Phys. Rev. **72**, 461 (1947).

[34] M. Goldberger, Phys. Rev. **84**, 929 (1951); E. Corinaldesi and G. Field, Phil. Mag. **41**, 364 (1950); Lepore, Ruderman, and Wolff, unpublished.

[35] B. Lippman and J. Schwinger, Phys. Rev. **79**, 469 (1950).

Goldberger, except for a factor 2 that results from using the charge-symmetric meson theory instead of the pure charged theory. In the limit $x\rightarrow 0$, Eqs. (28) are seen to reduce directly to the perturbation results presented in Eqs. (20). If we adjust the coupling constant to fit an experimental value of 120 mb for the total π^+ elastic cross section at a meson kinetic energy of 135 Mev in the lab system, we find $g^2/4\pi\approx 0.4$, corresponding by Eq. (8) to a pseudoscalar coupling constant of $f^2/4\pi\approx 70$. Several interesting features emerge in Eqs. (28). The variation of the total cross sections for π^+ and for π^- mesons as a function of meson energy indicates a sharp rise in the region 60 to 150 Mev, followed by a general flattening at higher energies in accord with the data on π^- scattering (elastic plus charge exchange). A choice of $g^2/4\pi>0.5$ or $g^2/4\pi<0.3$ shifts the energy plateau for the calculated cross section to too high or too low an energy value, respectively. However, if we were to adjust the angular distribution of $d\sigma(\pi^+)$ to fit the observed distribution, of roughly $1+3\cos^2\theta$ at an energy of 135 Mev, we would require $g^2/4\pi\approx 0.9$. For $g^2/4\pi=0.4$ the predicted distribution is $1+0.6\cos^2\theta$. The ratio of $\sigma(\pi^+):\sigma(\pi^-)+\sigma(\pi^0)$ is still less than unity, in disagreement with the data which indicate this ratio to be 1.4 at 60 Mev and 3 above 100 Mev, but is increased slightly to 0.72 from the perturbation result of 0.6. In accord with observation, $d\sigma(\pi^-)$ is isotropic.

Proceeding now to a discussion of the damping theory as applied to the pseudoscalar Hamiltonian we write first, in Goldberger's notation,[34] the scattering matrix for elastic scattering due to H_c:

$$R_{\mathbf{q}\mathbf{q}_0}{}^{(c)}=-(\Phi_{\mathbf{q}},\,H_c\Psi_{\mathbf{q}_0}{}^{(c)+}),\tag{29}$$

where $\mathbf{q}_0$ and $\mathbf{q}$ denote, respectively, the momenta of the initial and scattered meson, and $\Psi_{\mathbf{q}_0}{}^{(c)+}$ is defined by the integral equation

$$\Psi_{\mathbf{q}_0}{}^{(c)+}=\Phi_{\mathbf{q}_0}+\Delta^{-1}H_c\Psi_{\mathbf{q}_0}{}^{(c)+},\tag{30}$$

with $\Delta=\omega+i\epsilon-H_0$. This solution incorporates the boundary condition of an incident free wave, $(\Phi_{\mathbf{q}_0})$, plus an outgoing scattered wave due to the core term H_c. The R matrix defined as above is related to the scattering phase shift δ_0, given in Eq. (27), by

$$R_{\mathbf{q}\mathbf{q}_0}{}^{(c)}=(2\pi/\omega qL^3)\sin\delta_0\exp(i\delta_0).\tag{31}$$

The total R matrix for elastic scattering is

$$R_{\mathbf{q}\mathbf{q}_0}=-(\Phi_{\mathbf{q}},\,\{H_c+H_{pv}+H_\pi\}\Psi_{\mathbf{q}_0}{}^{+})\tag{32}$$

with H_c, H_{pv}, and H_π as in Eq. (18) and $\Psi_{\mathbf{q}_0}{}^{+}$ defined by

$$\Psi_{\mathbf{q}_0}{}^{+}=\Phi_{\mathbf{q}_0}+\Delta^{-1}(H_c+H_{pv}+H_\pi)\Psi_{\mathbf{q}_0}{}^{+}.\tag{33}$$

It is useful to define $\Psi_{\mathbf{q}_0}{}'=\Psi_{\mathbf{q}_0}{}^{+}-\Psi_{\mathbf{q}_0}{}^{(c)+}$. By Eqs. (30) and (33), it satisfies the relation

$$\Psi_{\mathbf{q}_0}{}'=\Delta^{-1}(H_{pv}+H_\pi)\Psi_{\mathbf{q}_0}{}^{(c)+}+\Delta^{-1}(H_c+H_{pv}+H_\pi)\Psi_{\mathbf{q}_0}{}'.\tag{34}$$

There are two elements in the prescription yielding the Heitler damping theory approximation to the solution of Eq. (34).

(1) Only the lowest order matrix elements which contribute to the scattering, as found by an iteration procedure, are to be kept (first order for H_π, second order for H_{pv}).

(2) The energy denominator

$$\Delta^{-1}=(\omega+i\epsilon-H_0)^{-1}=(\omega-H_0)^{-1}\mathbf{P}-i\pi\delta(\omega-H_0)$$

is replaced by its second term $-i\pi\delta(\omega-H_0)$, indicating that contributions from terms off the energy shell are neglected. Use of the above prescription and Eqs. (29), (32), and (34) allows us to express the R matrix of $H_{pv}+H_\pi$ as

$$\begin{aligned}R_{\mathbf{q}\mathbf{q}_0}{}'&=R_{\mathbf{q}\mathbf{q}_0}-R_{\mathbf{q}\mathbf{q}_0}{}^{(c)}\\&=-(\Phi_{\mathbf{q}},\,\{H_\pi+H_{pv}\Delta^{-1}H_{pv}\}\Psi_{\mathbf{q}_0}{}^{(c)+})\\&\quad+i\pi\textstyle\sum_{\mathbf{q}'}\delta(\omega-\omega_{\mathbf{q}'})B_{\mathbf{q}\mathbf{q}'}R_{\mathbf{q}'\mathbf{q}_0}{}',\end{aligned}\tag{35}$$

where

$$B_{\mathbf{q}\mathbf{q}'}=-(\Phi_{\mathbf{q}},\,\{H_c+H_\pi+H_{pv}\Delta^{-1}H_{pv}\}\Phi_{\mathbf{q}'})\tag{36}$$

is the lowest order Born approximation scattering matrix. Equation (35) is the desired extension of the usual Heitler integral equation, expressed in terms of the known solutions $\Psi_{\mathbf{q}_0}{}^{(c)+}$. These are essentially the state functionals for meson field amplitudes given as solutions of Eq. (22). In the absence of H_c, which we treat by the previously discussed phase shift method, Eq. (35) reduces to Goldberger's result.

It is possible to solve Eq. (35) exactly with the trial form

$$R_{\mathbf{q}\mathbf{q}_0}{}'=-L^{-3}(s_0q^2+s_1\mathbf{q}\cdot\mathbf{q}_0+is_2\boldsymbol{\sigma}\cdot\mathbf{q}_0\times\mathbf{q}),\tag{37}$$

since all higher order terms in σ can be reduced to these. We consider specifically the scattering of π^+ mesons by protons. The Born approximation scattering matrix is just

$$B_{\mathbf{q}\mathbf{q}_0}(\pi^+)=-(f^2/2M\omega L^3)(\lambda+\omega/2M-\boldsymbol{\sigma}\cdot\mathbf{q}_0\boldsymbol{\sigma}\cdot\mathbf{q}/2\omega M),\tag{38}$$

corresponding to the cross section (21a) with $\lambda\rightarrow 1$. The first term on the right member of Eq. (35) differs from the Born approximation result because of the appearance of $\Psi_{\mathbf{q}_0}{}^{(c)+}$, defined by Eq. (30), in place of $\Phi_{\mathbf{q}_0}$. This means that in calculating the matrix elements of H_π and H_{pv} we insert the solutions written in Eq. (26) for the meson field amplitude that annihilates the incident meson state of momentum $\mathbf{q}_0$ in place of the usual plane wave, Eqs. (23)–(24). The effect of thus including the core H_c in the zero-order Hamiltonian is to reduce the s and p wave scattering matrix elements by the factors

$$\begin{aligned}r_s&=3(\beta a)^{-2}(1-\tanh\beta a/\beta a),\\r_p&=3\coth\beta a/\beta a(1-\tanh\beta a/\beta a),\end{aligned}\tag{39}$$

respectively, relative to the Born approximation. These factors neglect terms $(ka)^2\ll 1$ and reduce to unity for $\beta a\rightarrow 0$. Introducing Eqs. (36)–(39) into Eq. (35), we obtain after some algebra

$$s_0{}^+=(f/2M)^2(r_s/q^2)\times[1+(i/2\pi)(f/2M)^2q\omega(1+2M\lambda/\omega)]^{-1},$$
$$s_1{}^+=-(f/2M)^2(r_p/\omega^2)(1+2ix)/(1-ix+2x^2),$$
$$s_2{}^+=-(f/2M)^2(r_p/\omega^2)(1-ix+2x^2)^{-1}.$$

The complete R matrix is now known since, by Eqs. (27) and (31),

$$R_{\mathbf{qq}0}{}^{(c)}(\pi^+)=-(2\pi/\omega L^3)a(1-\tanh\beta a/\beta a),$$

to order $(ka)^2\ll 1$. The procedure is entirely similar in the cases of elastic and charge exchange scattering of π^- mesons by protons. Relating the R-matrix to the differential scattering cross sections by

$$d\sigma=(L^6\omega^2/4\pi^2)\langle|R|^2\rangle_{\mathrm{Av}}d\Omega,$$

we obtain finally

$$d\sigma(\pi^+)=a^2\left(1-\frac{\tanh\beta a}{\beta a}\right)^2\left\{1+\frac{\omega}{2M\lambda}\frac{2+\omega/2M\lambda}{1+y_+{}^2}+\frac{\beta^2a^2\coth^2\beta a}{4\lambda^2}\frac{q^4}{\omega^2M^2}\frac{1+4x^2\cos^2\theta}{(1+2x^2)^2+x^2}-\frac{\beta a\coth\beta a}{\lambda}\frac{q^2}{\omega M}\frac{\cos\theta}{(1+2x^2)^2+x^2}\times\left[1+\frac{\omega}{2M\lambda}\frac{1-x(3+4x^2)y_+}{1+y_+{}^2}\right]\right\}d\Omega,$$

$$d\sigma(\pi^-)=a^2\left(1-\frac{\tanh\beta a}{\beta a}\right)^2\left\{1-\frac{\omega}{2M\lambda}\frac{2-\omega/2M\lambda}{1+y_-{}^2}+\frac{\beta^2a^2\coth^2\beta a}{4\lambda^2}\frac{q^4}{\omega^2M^2}\frac{1}{1+9x^2}+\frac{\beta a\coth\beta a}{\lambda}\frac{q^2}{\omega M}\times\frac{\cos\theta}{1+9x^2}\left[1-\frac{\omega}{2M\lambda}\frac{1+3xy_-}{1+y_-{}^2}\right]\right\}d\Omega,$$

$$d\sigma(\pi^0)=2a^2\left(1-\frac{\tanh\beta a}{\beta a}\right)^2\left\{\frac{(\omega/2M\lambda)^2}{1+2(f^2/4\pi)^2(q\omega/2M^2)^2}+\frac{\beta^2a^2\coth^2\beta a}{4\lambda^2}\frac{q^4}{\omega^2M^2}\frac{\cos^2\theta}{1+2x^2}-\frac{\beta a\coth\beta a}{\lambda}\frac{q^2}{\omega M}\times\frac{\cos\theta}{1+2x^2}\frac{\omega}{2M\lambda}\frac{1-2x(f^2/4\pi)(q\omega/2M^2)}{1+2(f^2/4\pi)^2(q\omega/2M^2)^2}\right\}d\Omega,\quad(40)$$

where $y_\pm$ represent

$$y_\pm=\frac{f^2}{4\pi}\lambda\frac{q}{M}\left(1\pm\frac{\omega}{2M\lambda}\right).$$

The total cross sections, integrated over angle, are

$$\sigma(\pi^+)=4\pi a^2\left(1-\frac{\tanh\beta a}{\beta a}\right)^2\left\{1+\frac{\omega}{2M\lambda}\frac{2+\omega/2M\lambda}{1+y_+{}^2}+\frac{\beta^2a^2\coth^2\beta a}{4\lambda^2}\frac{q^4}{\omega^2M^2}\frac{1+4x^2/3}{(1+2x^2)^2+x^2}\right\},$$

$$\sigma(\pi^-)=4\pi a^2\left(1-\frac{\tanh\beta a}{\beta a}\right)^2\left\{1-\frac{\omega}{2M\lambda}\frac{2-\omega/2M\lambda}{1+y_-{}^2}+\frac{\beta^2a^2\coth^2\beta a}{4\lambda^2}\frac{q^4}{\omega^2M^2}\frac{1}{1+9x^2}\right\},$$

$$\sigma(\pi^0)=8\pi a^2\left(1-\frac{\tanh\beta a}{\beta a}\right)^2\left\{\frac{(\omega/2M\lambda)^2}{1+2(f^2/4\pi)^2(q\omega/2M^2)^2}+\frac{1}{3}\frac{\beta^2a^2\coth^2\beta a}{4\lambda^2}\frac{q^4}{\omega^2M^2}\frac{1}{1+2x^2}\right\}.\quad(41)$$

We comment here that proton recoil has been neglected in these calculations. This means that terms reduced by ω/M relative to unity are omitted. They have been explicitly calculated and shown to be indeed small and not to materially affect the above results. The terms proportional to ω/M in the above formulas appear only in combination with the constant $1/\lambda>1$.

In analyzing the above formulas we make the following observations:

(1) The data indicate that p wave scattering (due to H_{pv}) dominates above ~60 Mev and, therefore, in order to approximate the energy variation of the experimental cross sections, we must take $0.3\leqslant g^2/4\pi\leqslant 0.5$ as we did in the discussion below Eqs. (28); or by Eq. (8), $53\leqslant f^2/4\pi\leqslant 89$.

(2) If we restrict $g^2/4\pi$ to lie within the above-indicated limits, there remain two constants λ and a (or β) to adjust in order to bring these cross sections in as close agreement as possible with some of the data. If we take a to correspond to the nucleon Compton wavelength, $a=1/M$, we find that we must take $\lambda=\mu/2M=0.075$ to fit the magnitude of π^- elastic plus charge exchange scattering for mesons in the energy range of 135–215 Mev. The data and theoretical curve are presented in Fig. 2. The data of Anderson *et al.* and of Steinberger *et al.* are seen to fall somewhat below the theoretical curve for meson energies below 120 Mev. We may also take a larger source radius of $a=2/M$. Choosing $\lambda=\mu/M=0.15$ we fit the data for the higher energy π^- mesons, but the curve is again high below 120 Mev. This curve is also plotted in Fig. 2.

(3) The cross section for π^+ scattering given in Eq. (41) is smaller by a factor of roughly $\frac{3}{4}$ than $\sigma(\pi^-)+\sigma(\pi^0)$ with the above-indicated choice of parameters and thus is in disagreement with the observation. If we attempt to fit $\sigma(\pi^+)$ with the data, the π^- cross sections are then much too large.

(4) The data indicate a ratio of $\sigma(\pi^0):\sigma(\pi^-)=2:1$ for incident π^- mesons of energy above 100 Mev. The formulas in Eq. (41) yield a value for this ratio of approximately 1.2 at 132 Mev, increasing to 2 at 200 Mev with both of the above sets of parameters. The calculated ratio is perhaps on the low side of the experimental determination.

(5) The data on the angular distributions of the scattered π^+, π^-, and π^0 mesons are too crude to permit a detailed comparison with the above formulas. The experiments suggest, however, that at 135 Mev, the π^- elastic scattering is essentially isotropic, the charge exchange cross section varies largely as $\cos^2\theta$, and the π^+ cross section has the rough form of $1+3\cos^2\theta$ as predicted in a phenomenological analysis that assumes the scattering to take place predominantly in a spin $\frac{3}{2}$ state. We tabulate here the angular distributions predicted by Eqs. (40) with the above choices of parameters. For $g^2/4\pi=0.4$, $a=1/M$, and $\lambda=\mu/2M$, we obtain at a meson kinetic energy of 132 Mev:

$$d\sigma(\pi^-)\sim[1+0.35\cos\theta]d\Omega,$$
$$d\sigma(\pi^0)\sim[1+1.3\cos\theta+36\cos^2\theta]d\Omega,$$
$$d\sigma(\pi^+)\sim[1-0.20\cos\theta+0.49\cos^2\theta]d\Omega;$$

at the same energy, for $g^2/4\pi=0.4$, $a=2/M$, and $\lambda=\mu/M$ we obtain:

$$d\sigma(\pi^-)\sim[1-0.04\cos\theta]d\Omega,$$
$$d\sigma(\pi^0)\sim[1+1.3\cos\theta+36\cos^2\theta]d\Omega,$$
$$d\sigma(\pi^+)\sim[1-0.51\cos\theta+0.39\cos^2\theta]d\Omega.$$

The calculated distributions for π^- elastic and charge exchange scattering are in satisfactory agreement with present experimental data. However, the $\cos^2\theta$ contribution to π^+ scattering is too small by a factor ~ 6 relative to the data.

It would appear from the above discussions that it is possible to select a physically reasonable set of three parameters—the meson-nucleon coupling strength, the depth and the radius of a repulsive nucleon core potential for meson interactions—which permits reasonable agreement with experiment on the elastic and charge exchange scattering of π^- mesons. On the other

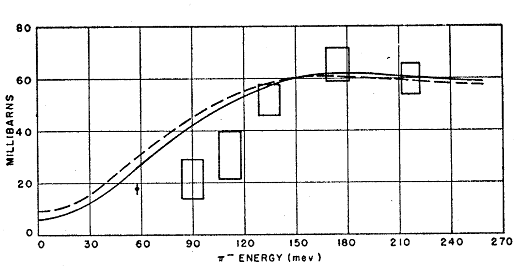

FIG. 2. Elastic plus charge exchange scattering of π^- mesons by protons as a function of π^- energy. The blocks represent the data obtained at Chicago and the low energy point was obtained at Columbia. The theoretical curves are deduced from Eq. (41) with $g^2/4\pi=0.4$. The solid curve is for $a=1/M$, $\lambda=\mu/2M$ and the dashed curve is for $a=2/M$, $\lambda=\mu/M$.

hand, the calculated π^+ cross section is considerably too small in magnitude and too flat in angular distribution. It is encouraging to note, however, that by going somewhat beyond a straight weak coupling perturbation approximation, it has been possible to narrow the gap between theory and experiment. An analysis of the perturbation theory suggests that an improved, intermediate coupling calculation of the meson-nucleon interaction may be expected to increase the ratio of π^+ to π^- scattering by protons. For π^- scattering, the proton must absorb the incident π^- meson before emitting the scattered one, *viz.*,

$$\pi^-+p\rightarrow N\rightarrow\pi^{-\prime}+p'.$$

The energy denominator for this process in a second-order perturbation calculation is $E_p+\omega-E_N\sim\omega$. For π^+ scattering, the final meson is emitted first:

$$\pi^++p\rightarrow\pi^++\pi^{+\prime}+N\rightarrow\pi^{+\prime}+p',$$

and the energy denominator is $E_p-\omega-E_N\sim-\omega$. However, a bound state of one of the mesons about the neutron in the intermediate step would reduce the energy denominator and thereby increase the cross section.[36] Such a state would correspond to a nucleon isobar. This mechanism would also serve to increase the ratio $\sigma(\pi^0):\sigma(\pi^-)$ which we observed to be on the small side in the preceding discussion, since charge exchange scattering proceeds in either of the two ways,

$$\pi^-+p\underset{N}{\rightarrow}\pi^-+\pi^{0\prime}+p'\rightarrow\pi^{0\prime}+N'.$$

IV. PHOTOPRODUCTION

We next apply the Hamiltonian [Eq. (18)] to a brief discussion of the photoproduction of π^+ mesons by gamma-rays of several hundred Mev energy incident on protons. The zero-order Hamiltonian is taken to be

$$H_0+\lambda f^2/2M\int K(\mathbf{r}-\mathbf{r}_N)\phi^2d\mathbf{r}, \qquad (42)$$

and $H_{pv}+H_e$ is treated as a small perturbation. This calculation differs from the previous work of Brueckner[37] and Araki[38] by the inclusion of the core term in the zero-order Hamiltonian in Eq. (42). The core term alters the relative amounts of s and p wave production and hence alters the angular distribution and excitation function for π^+ production.

A repulsive core with $\lambda>0$ suppresses the s wave relative to the p wave production, thereby increasing the angular variation of the differential cross section. It appears from the experimental and perturbation-

[36] An actual calculation has shown that the introduction of a phenomenological constant $\Gamma=\omega/\omega'>1$, with ω' the reduced energy denominator, in both $\pi^++p\rightarrow\pi^++\pi^{+\prime}+N\rightarrow\pi^{+\prime}+p'$ and $\pi^-+p\rightarrow\pi^-+\pi^{0\prime}+p'\rightarrow\pi^{0\prime}+N'$, makes agreement with experiment at 135 Mev possible in both magnitude and angular distribution.

[37] K. Brueckner, Phys. Rev. **79**, 641 (1950).

[38] G. Araki, Prog. Theor. Phys. **5**, 507 (1950).

weak-coupling curves given by Steinberger and Bishop[39] (Figs. 17 and 20 of this reference) that such an increase in the relative p wave contribution may be desirable. We shall see below that it is also possible to improve the agreement between the theoretical curve and the data on the excitation function for π^+ production by increasing the relative p wave contribution in the pseudoscalar theory. We remark here that suppression of the s wave in π^+ production operates so as to increase the theoretical ratio of π^0 and π^+ photoproduction cross sections, since π^0 production occurs predominantly in the p state.

Foldy[40] has given a nonrelativistic perturbation calculation of π^+ photoproduction. Plane wave solutions to the Klein-Gordon equation for the meson amplitudes [Eqs. (23)–(24)] are used in calculating the matrix elements for this process. For the differential cross section in the pseudoscalar theory Foldy obtains (both the pseudoscalar and pseudovector coupling terms yield the same result)

$$d\sigma_F=\frac{2e^2}{\mu^2}\frac{g^2}{4\pi}\frac{q}{k}\left(1-\frac{2q^2\mu^2}{\omega_{\mathbf{q-k}}{}^4}\sin^2\theta\right)d\Omega_q, \qquad (43)$$

for incident photons of momentum k, and π^+ mesons of momentum $q=(k^2-\mu^2)^{\frac{1}{2}}$ and energy $\omega_{\mathbf{q-k}}$ emerging at an angle θ with the incident photon direction. The meson-nucleon coupling constant in Eq. (43), $g^2/4\pi$, is for the derivative coupling form and corresponds to Foldy's[41] $\frac{1}{2}g^2$. If we calculate the π^+ photoproduction cross section with the zero-order Hamiltonian taken as Eq. (42), the meson amplitudes are taken as the solutions [Eq. (26)] of the modified wave equation (22), so that the s and p wave contributions are reduced, respectively, by r_s and r_p as given in Eq. (39). The resulting cross section is then

$$d\sigma=\frac{e^2}{2M^2}\frac{f^2}{4\pi}\left(\frac{3}{\beta a}[\coth\beta a-1/\beta a]\right)^2\frac{q}{k}\left[\left(\frac{\tanh\beta a}{\beta a}\right)^2-\frac{2q^2\mu^2-2(1-\tanh\beta a/\beta a)q^2\omega_{\mathbf{q-k}}{}^2}{\omega_{\mathbf{q-k}}{}^4}\sin^2\theta\right]d\Omega_q. \qquad (44)$$

In the limit $\beta a\rightarrow 0$ this reduces to Eq. (43), with the coupling constants f and g related by Eq. (8). There are two constants in Eq. (44), $g^2/4\pi$ and βa, to be adjusted to give the best fit with the data on the angular and energy variation of the cross section. In the discussion of the preceding paragraph on meson scattering we specified $g^2/4\pi=0.4$ and $\beta a=4$. This choice of parameters in Eq. (44) gives a photoproduction cross section with an energy variation appropriate for the data but too small in magnitude by roughly 35 percent, and with too sharp and too forward an angular distribution. If we keep $g^2/4\pi=0.4$, we find that $\beta a=2.3$ gives a cross section of magnitude in accord with the data. The energy distribution of photomesons at 90° in the laboratory and the angular distribution at a photon energy of 255 Mev as calculated from Eq. (44) are compared with the data in Figs. 3 and 4. The results of a straight perturbation calculation as calculated from Eq. (43) and normalized to the data by a choice of $g^2/4\pi=1/11$ are also presented for comparison. The fit between theory and experiment on the energy distribution of π^+ mesons is considerably improved over the straight perturbation result (corresponding to $\beta a=0$). The curve for the angular distribution of π^+ mesons predicts more scattering in the angular interval 60°–120° than indicated by the rather limited data. On the other hand, the straight perturbation result seems to indicate relatively too much s wave production and to predict too many mesons in the backward direction. It is tempting to suggest that one should choose a smaller value of βa in the neighborhood of 1.5 and thereby obtain a theoretical angular distribution median between the two curves drawn and thus in better accord with the data. If we do this, however, the energy distribution of mesons at 90° becomes more similar to the perturbation result and deviates from the shape of the excitation curve given by Steinberger and Bishop. When more quantitative data become available it will be possible to analyze more accurately the relative roles of s and p wave π^+ meson photoproduction.

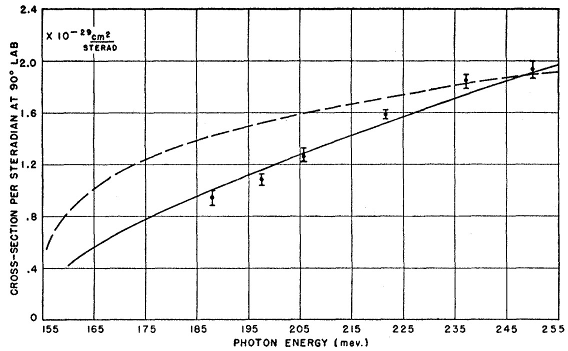

FIG. 3. The energy distribution of photomesons at 90° in the laboratory as a function of incident gamma-energy. The solid theoretical curve results from Eq. (44) with $g^2/4\pi=0.4$ and $\beta a=2.3$. The dashed curve is the perturbation result [Eq. (43)] fitted to the data with $g^2/4\pi=1/11$.

The fact that the value of βa used to fit the magnitude of the photoproduction cross section is smaller than for the π^- scattering cross section suggests that the amplitude of the nucleon-meson core, λ, is smaller in the photoproduction than in the scattering process, for a fixed radius. This result indicates that the higher order terms in Eq. (7) that are lumped into the constant λ in Eq. (18) operate somewhat differently in these two physical processes, and it is a further manifestation of

[39] J. Steinberger and A. S. Bishop, Phys. Rev. 86, 171 (1952).
[40] L. L. Foldy, Phys. Rev. 76, 372 (1949).
[41] The factor $\frac{1}{2}$ appears since we use the charge-symmetric meson theory in this work, whereas Foldy calculated with a charged meson theory.

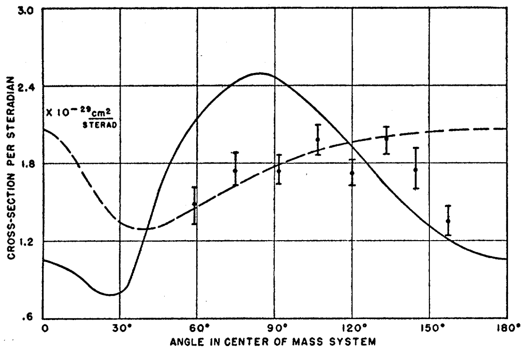

FIG. 4. The angular distribution of photomesons in the center-of-mass system for an incident gamma-energy of 255 Mev. The solid theoretical curve results from Eq. (44) with $g^2/4\pi=0.4$ and $\beta a=2.3$. The dashed curve is the perturbation result [Eq. (43)] fitted to the data with $g^2/4\pi=1/11$.

the failure of a perturbation approach to meson theoretic calculations.

V. CONCLUSION

In conclusion we summarize briefly the aims, methods, and deductions of this work. We have sought to harmonize the predictions of the pseudoscalar meson theory with the observations on meson scattering and photoproduction from protons. Cross sections previously derived for these processes with the assumption of weak meson-nucleon coupling disagree with the experimental data. On the other hand, Brueckner[28] and Watson[42] have achieved striking success by means of a phenomenological approach. In this paper we have adopted a field theoretic approach to the calculations and improved on the weak coupling perturbation approximation in two instances. First, we have used the Dyson transformation to a new representation for the pseudoscalar theory that permits a large meson-nucleon interaction term carrying the factor M—the core term H_c—to be treated with the zero-order Hamiltonian with methods that do not limit the meson-nucleon coupling strength to be small. In this representation we can also readily identify the velocity-independent features ($\boldsymbol{\sigma}$, $\boldsymbol{\tau}$) of the nucleon dynamics. Secondly, we have improved on the perturbation results in the scattering calculations by applying the Heitler damping theory.

[42] K. Brueckner and K. M. Watson, Phys. Rev. 86, 923 (1952).

The formulas derived by this method come closer to fitting the observations than do the perturbation weak-coupling results but are far from satisfactory. It appears that more powerful techniques of calculation in the intermediate coupling region must be developed to provide an adequate basis for the study of the meson-nucleon interaction, even in a representation that permits separation of the large core term.[4]

In particular the higher order terms in the exact Hamiltonian, Eq. (7), remain to be studied. It is of interest to note here that the core term H_c and the constant λ which we incorporate with it in Eq. (17) may be checked directly in the very near future, when electron accelerators are operating in the 400-Mev region, by the process of double meson production by gamma-rays incident on protons. Thus, at 400 Mev, the cross section for production of π^- mesons in the reaction

$$\gamma+p\rightarrow\pi^++\pi^-+p'$$

is calculated to be[43] $\sigma\sim(g^2/4\pi)^2 10^{-27}$ cm² in a weak coupling perturbation calculation. The dominant contribution results from the core term H_c for which $\lambda=1$ in the weak coupling approximation. This cross section result is proportional to λ^2, and thus a direct measurement of the photoproduced π^- serves to limit the possible range of values. This is an important point for the pseudoscalar theory, since we have seen that small values of $\lambda\sim\mu/M$ to $\mu/2M$ are necessary in the interpretation of the scattering and single photoproduction data.

As discussed at the end of Secs. III and IV, the failure to match the π^+ meson scattering data and the difference in the vales of parameters introduced to fit the calculations on π^- scattering and on π^+ photoproduction indicate that an intermediate coupling approximation is necessary. We are encouraged by our results, however, to believe that charge-symmetric pseudoscalar meson theory with renormalizable pseudoscalar coupling may prove adequate for a quantitative analysis of the processes discussed.

We wish to thank Dr. J. V. Lepore and Mr. R. D. Lawson for valuable discussion.

[43] R. D. Lawson (to be published) has made a perturbation study of the double photomeson production processes in the pseudoscalar theory with both direct and derivative coupling.