Construction of a Potential from a Phase Shift

RES JOST AND WALTER KOHN*
Institute for Theoretical Physics, Copenhagen, Denmark,
and
Institute for Advanced Study, Princeton, New Jersey
(Received March 21, 1952)

Two methods of constructing a potential, in the form of a series, from a given S phase shift are described. The convergence of one of these is analyzed. Examples show that it can converge comfortably even for potentials which are so strong as to have a bound state. The ambiguity of the potential, connected with the existence of bound states, is discussed in an appendix.

1. INTRODUCTION

SINCE the early work of Faxén and Holtsmark[1] the analysis of scattering data into phase shifts has been a very useful tool for the interpretation of collision processes. In the case of the scattering of two nucleons extensive investigations have been carried out, aimed at interpreting the phase shifts by means of a potential acting between the colliding particles.[2] The question, to what extent a potential can in principle be determined from a knowledge of the phase shifts and binding energies, has therefore been of considerable interest.

The problem has been recently clarified in the papers of Bargmann[3,4] and Levinson.[5] Levinson was able to show that under very general conditions a short range potential is uniquely determined by the phase shift (for all energies) corresponding to a single angular momentum l, provided there exist no bound states with the same angular momentum. Thus the phase shift corresponding to an l so high as to exclude the possibility of a bound state does at least in principle determine the potential unambiguously.

On the other hand, Bargmann has given a number of examples which show that in the presence of bound states there can be a great variety of widely different potentials giving identical phase shifts and binding energies.

It became clear therefore that, for example, for a system with bound S states, the potential is in general not uniquely determined by the S phase shift and binding energies. However, even in the absence of bound states, the Levinson uniqueness proof gave no hint as to how the underlying potential could be constructed.

An attempt in this direction was made by Fröberg.[6] Since his results were somewhat inconclusive, we thought it worthwhile to make a fresh investigation of this problem.

The main part of this paper is devoted to a method of constructing the potential, which is described in Sec. 3. In Secs. 4 and 5 conditions are established under which our procedure leads to a solution. Section 6 contains two examples in which our method is found to converge comfortably, even for nuclear potentials. It was somewhat unexpected to find that it converges even in the presence of a bound state; however, in that case we obtain only one potential out of the whole family which presumably exists.

In Sec. 7 we describe an alternative procedure which has the advantage of omitting one of the steps in our other construction. The convergence is only illustrated by an example. This section may be read independently of the rest of the paper.

The authors are well aware of the limited usefulness of these procedures for an analysis of the two-nucleon system. The difficulties are manifold. Our method, in common with the more usual fitting procedures, requires an accurate knowledge of the S phase for all energies, which is difficult to extract from scattering data, particularly at high energies. In the case of the neutron-proton system there is the additional problem of separating the singlet and triplet scattering. It is well known that the low energy scattering data suffice only to determine two characteristic constants of the potential, but not its exact shape.[7] Finally, it is quite problematical whether the two nucleon system can at all be adequately described by a potential, particularly at high energies.

Possible generalizations to describe higher angular momenta, tensor forces, and Coulomb and relativistic scattering are briefly discussed at the end of the paper.

In Appendix 2 we give a generalization of the Levinson uniqueness theorem which shows which additional parameters are necessary to determine the potential uniquely. The number of these parameters is equal to the number of bound states.

2. PRELIMINARIES

In this section we introduce the concepts which we shall use in constructing the potential from the phase shift.

* Postdoctoral N.R.C. Fellow, 1950–51.
[1] H. Faxén and J. Holtsmark, Z. Physik **45**, 307 (1927).
[2] See, for example, N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949) second edition; R. S. Christian and E. W. Hart, Phys. Rev. **77**, 441 (1950); R. S. Christian and H. P. Noyes, Phys. Rev. **79**, 85 (1950).
[3] V. Bargmann, Phys. Rev. **75**, 301 (1949).
[4] V. Bargmann, Revs. Modern Phys. **21**, 488 (1949).
[5] N. Levinson, Kgl. Danske Videnskab Selsab, Mat.-fys. Medd. **25**, No. 9 (1949).
[6] C. E. Fröberg, Arkiv Mat. Astron. Fys. **34A**, No. 28 (1948); **36A**, No. 11 (1949); Arkiv Fys. **3**, No. 1 (1951).
[7] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949); H. A. Bethe, Phys. Rev. **76**, 38 (1949).

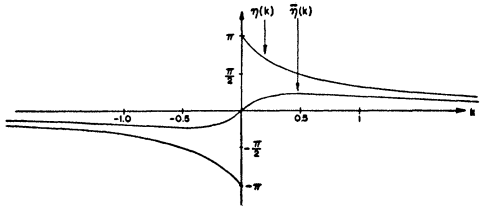

FIG. 1. The phase shift $\eta(k)$ and the modified phase shift $\bar{\eta}(k)$, corresponding to the Eckert potential for the $n-p$ ^3S state.

Restricting ourselves to S states, we start with the Schrödinger equation

$$\varphi''+k^2\varphi=V\varphi, \tag{2.1}$$

where $V(r)$ satisfies the condition

$$\int_0^\infty |V(r)|rdr<\infty. \tag{2.2}$$

Let $f(\pm k, r)$[8] be the two linearly independent solutions of (2.1) which behave at infinity like

$$\lim_{r\to\infty} e^{\pm ikr}f(\pm k, r)=1. \tag{2.3}$$

The S matrix element and the scattering phase are then related to $f(k, 0)$ by the equations

$$S(k)=e^{2i\eta(k)}=f(k, 0)/f(-k, 0), \quad k\gtrless 0. \tag{2.4}$$

Since

$$f(-k, 0)=f^*(k, 0), \tag{2.5}$$

we can write

$$\eta(k)=\mathrm{Im}[\log f(k, 0)]. \tag{2.6}$$

We now extend (2.4) and (2.6) also to negative k values. Owing to (2.5), one then has the relation $\eta(k)+\eta(-k)=2n\pi$, where we may set $n=0$. This gives

$$\eta(-k)=-\eta(k). \tag{2.7}$$

For the further development it is convenient to make the following definitions:

$$\left.\begin{aligned} z&=2ik,\\ g(z, r)=g(2ik, r)&=e^{ikr}f(k, r),\\ g(z)&=g(z, 0).\end{aligned}\right\} \tag{2.8}$$

Then (2.6) reads

$$\eta(k)=\mathrm{Im}[\log g(2ik)]. \tag{2.9}$$

The differential equation (2.1) with the initial values (2.3) can be combined in the integral equation[4]

$$g(z, r)=1+\int_r^\infty \frac{1}{z}[1-e^{-z(r'-r)}]V(r')g(z, r')dr'. \tag{2.10}$$

As shown in reference 4 this equation can be solved by iteration for $\mathrm{Re}[z]\geqq 0$, whenever (2.2) is satisfied. The resulting series converges uniformly in z and r. It follows that $g(z, r)$ and especially $g(z)$ is regular in $\mathrm{Re}[z]>0$ and continuous in $\mathrm{Re}[z]\geq 0$. The zeros, ξ_i of $g(z)$ in $\mathrm{Re}[z]>0$, lie on the real axis (corresponding to negative imaginary values of k) and are simple; they determine the energies of the bound states:[8]

$$E_i=-\tfrac{1}{4}\xi_i^2. \tag{2.11}$$

With the help of (2.2) one verifies easily that

$$\lim_{z\to\infty} g(z)=1 \tag{2.12}$$

uniformly for $\mathrm{Re}[z]\geqq 0$. Hence, by (2.9), we can set

$$\eta(\infty)=0. \tag{2.13}$$

If, furthermore,

$$\int_0^\infty r^2|V(r)|dr<\infty,$$

one can show[5] that

$$\left.\begin{aligned} \eta(+0)&=m\pi \quad \text{if} \quad g(0)\neq 0,\\ \eta(0)&=(m+\tfrac{1}{2})\pi \quad \text{if} \quad g(0)=0,\end{aligned}\right\} \tag{2.14}$$

where m is the number of bound states.

In the absence of bound states $\log g(z)$ is regular for $\mathrm{Re}[z]>0$; if further $g(0)\neq 0$, it is continuous for $\mathrm{Re}[z]\geq 0$ and its imaginary part on the imaginary z axis is given by the phase shift according to (2.9). We can therefore express $\log g(z)$ in $\mathrm{Re}[z]\geq 0$ in terms of the phase shift $\eta(k)$ by Poisson's integral

$$\log g(z)=-\frac{2i}{\pi}\int_{-\infty}^{\infty}\frac{\eta(k')}{2ik'-z}dk', \tag{2.15}$$

where we have made use of (2.12).

When the potential does have bound states, both $\eta(k)$ and the energies of the bound states are required for the construction of $g(z)$. This function now has zeros at the ξ_i and hence $\log g(z)$ has logarithmic branch points there. However, the function $\bar{g}(z)$, defined by

$$\bar{g}(z)=g(z)\prod_i^m\frac{(z+\xi_i)}{(z-\xi_i)}, \tag{2.16}$$

does not vanish for $\mathrm{Re}[z]>0$ and furthermore has the same asymptotic properties as $g(z)$. It can therefore be expressed in terms of its phase $\bar{\eta}(k)$ on the imaginary z axis:

$$\log \bar{g}(z)=-\frac{2i}{\pi}\int_{-\infty}^{\infty}\frac{\bar{\eta}(k)}{2ik-z}dk, \tag{2.17}$$

where by the definition of $\bar{g}(z)$,

$$\bar{\eta}(k)=\eta(k)-2\sum_{i=1}^m\tan^{-1}\frac{\xi_i}{2k} \tag{2.18}$$

(see Fig. 1).

[8] R. Jost, Helv. Phys. Acta **20**, 256 (1947).

Once $\bar{g}(z)$ is calculated, $g(z)$ can be obtained from it by Eq. (2.16).

It is now convenient to replace $g(z)$ by an auxiliary potential $V_1(r)$. For this purpose we iterate Eq. (2.10) which gives

$$g(z)-1=\sum_{l=1}^{\infty}\int_0^\infty dr_1\int_{r_1}^\infty dr_2\cdots\int_{r_{l-1}}^\infty dr_l\frac{1}{z^l}\times(1-e^{-zr_1})(1-e^{-z(r_2-r_1)})\cdots(1-e^{-z(r_l-r_{l-1})})\cdot V(r_1)V(r_2)\cdots V(r_l). \quad (2.19)$$

For small $V(r)$ this equation reduces to

$$g(z)-1\cong\frac{1}{z}\int_0^\infty dr(1-e^{-zr})V(r), \quad (2.20)$$

which suggests the introduction of the auxiliary potential $V_1(r)$ defined by

$$g(z)-1=\frac{1}{z}\int_0^\infty dr(1-e^{-zr})V_1(r), \quad (2.21)$$

or, from (2.19), by

$$\int_0^\infty(1-e^{-rz})[V_1(r)-V(r)]dr=\sum_{l=2}^{\infty}\int_0^\infty dr_1\cdots\int_{r_{l-1}}^\infty dr_l\frac{1}{z^{l-1}}(1-e^{-zr_1})\times(1-e^{-z(r_2-r_1)})\cdots(1-e^{-z(r_l-r_{l-1})})\cdot V(r_1)V(r_2)\cdots V(r_l). \quad (2.22)$$

Under the assumption

$$|rV(r)|<M, \quad (2.23)$$

the right-hand side of (2.22) tends to zero as z approaches infinity. This suggests that the solution $V_1(r)$ of (2.22) satisfies the condition

$$\int_0^\infty[V_1(r)-V(r)]dr=0, \quad (2.24)$$

which will be verified after the solution is obtained (see Appendix 1). Equation (2.22) now takes the form

$$\int_0^\infty e^{-zr}[V(r)-V_1(r)]=\sum_{l=2}^{\infty}\int_0^\infty dr_1\cdots\int_{r_{l-1}}^\infty dr_l\frac{1}{z^{l-1}}\times(1-e^{-zr_1})\cdots(1-e^{-z(r_l-r_{l-1})})\cdot V(r_1)V(r_2)\cdots V(r_l). \quad (2.25)$$

Next we formally carry out the Laplace-inversion[9] and obtain

$$V_1(r)=V(r)-\sum_{l=2}^{\infty}\int_0^\infty dr_1\int_{r_1}^\infty dr_2\cdots\int_{l-1}^\infty dr_l\times K(r,r_1r_2\cdots r_l)V(r_1)V(r_2)\cdots V(r_l), \quad (2.26)$$

where

$$\begin{aligned}K(r,r_1\cdots r_l)&=L^{-1}\left\{(1-e^{-zr_1})\frac{1}{z}(1-e^{-z(r_2-r_1)})\cdots\frac{1}{z}(1-e^{z(r_l-r_{l-1})});r\right\}\\&=\int_0^{r_1}dt_1\int_0^{r_2-r_1}dt_2\cdots\int_0^{r_l-r_{l-1}}dt\times L^{-1}\{z\cdot e^{-z(t_1+t_2+\cdots+t_l)};r\}\\&=\int_0^{r_1}dt_1\int_0^{r_2-r_1}dt_2\cdots\int_0^{r_l-r_{l-1}}dt_l\times\delta^l(r-t_1-t_2-\cdots-t_l)\end{aligned} \quad (2.27)$$

or

$$K(r,r_1r_2\cdots r_l)=\frac{\partial}{\partial r}\int_0^{r_1}dt_1\int_0^{r_2-r_1}dt_2\cdots\int_0^{r_l-r_{l-1}}dt_l\times\delta(r-t_1-t_2-\cdots-t_l); \quad (2.28)$$

$\delta(x)$ denotes Dirac's δ-function.

The proof that the series (2.26) for V_1 converges and satisfies Eqs. (2.25), (2.24), and therefore (2.22) is given in Appendix 1.

The results of this paragraph can be summarized as follows:

If a potential satisfies the conditions

$$\int_0^\infty r|V(r)|dr<\infty, \quad (2.29)$$

$$\int_0^\infty r^2|V(r)|dr<\infty, \quad (2.30)$$

$$r|V(r)|<M<\infty, \quad (2.31)$$

then

(a) $\eta(k)$ is an odd function which is continuous and differentiable except possibly at $k=0$ [see Eq. (2.14)] and which tends to zero as $k\to\pm\infty$.

[9] G. Doetsch, *Theorie und Anwendung der Laplace Transformation* (Dover Publications, New York, 1943).

(b) $g(z)$ is regular for $\mathrm{Re}[z]>0$ and is expressible in terms of $\eta(k)$ and the energy values of the bound states by Eqs. (2.16)–(2.18); for $z\to\infty$, $g(z)\to 1$.

(c) There exists a potential $V_1(r)$, satisfying (2.21), which obeys the inequalities

$$\int_0^\infty r|V_1(r)|dr<\infty, \tag{2.32}$$

$$r|V_1(r)|<M_1. \tag{2.33}$$

3. CONSTRUCTION OF THE POTENTIAL FROM THE PHASE

We now turn to the construction of a potential which will reproduce a given phase shift $\eta(k)$ and given energy levels.

In the presence of bound states it is by no means certain that, for given phase and binding energies, this inversion problem has a unique solution. In fact, Bargmann[4] has constructed continuous families of potentials with identical phase shifts and bound states. A complete solution of the inversion problem would consist of a construction of all potentials corresponding to a given phase and binding energies. The method of the present paper, however, yields at best a single potential out of this manifold.

On the other hand, when $\eta(0)=0$, so that no bound states exist, these complications do not arise if one admits only potentials satisfying the conditions (2.29) and (2.30), and the inversion problem has at most a single solution. This result has been proved by Levinson.[5]

Our procedure will be to construct successively the functions $g(z)$, $V_1(r)$, and $V(r)$ from Eqs. (2.16)–(2.18), (2.21), and (2.26).

We first construct from the given phase and energy levels the function $g(z)$ by means of (2.16)–(2.18). We then postulate that there exists a solution $V_1(r)$ of Eq. (2.21). In particular, if the expression

$$rV_1(r)=\lim_{R\to\infty}\frac{1}{2\pi i}\int_{x_0-iR}^{x_0+iR}e^{rz}\frac{d}{dz}z(g(z)-1)dz,\ x_0>0 \tag{3.1}$$

vanishes for $r<0$, is of bounded variation, and is absolutely integrable, it represents the solution.[9] We assume further that

$$r|V_1(r)|<M_1<\infty, \tag{3.2}$$

$$\int_0^\infty r|V_1(r)|dr=I_1<\infty. \tag{3.3}$$

It will be recalled from (c) in Sec. 2, that these conditions are in fact satisfied whenever the phase and energy levels are due to a potential $V(r)$ satisfying (2.29) and (2.31).

Finally, we formally solve the integral equation (2.26) for $V(r)$ by successive approximation. To do this, we introduce an auxiliary parameter λ, by which we multiply $V_1(r)$ and expand $V(r)$ in a power series in λ:

$$V(r)=\lambda V_1(r)+\sum_{m=2}^\infty \lambda^m V_m(r). \tag{3.4}$$

Introducing (3.4) into the integral equation gives

$$\lambda V_1(r)=V(r)-\sum_{l=2}^\infty\int_0^\infty dr_1\int_{r_1}^\infty dr_2\cdots\int_{r_{l-1}}^\infty dr_l \times K(r,r_1\cdots r_l)V(r_1)V(r_2)\cdots V(r_l) \tag{3.5}$$

and equating to zero the coefficient of λ^m, $m\geqq 2$ leads to the recursion relation

$$V_m(r)=\sum_{l=2}^m\sum_{\nu_1+\nu_2+\cdots+\nu_l=m}\int_0^\infty dr_1\int_{r_1}^\infty dr_2\cdots\int_{r_{l-1}}^\infty dr_l \times K(r,r_1r_2\cdots r_l)V_{\nu_1}(r_1)V_{\nu_2}(r_2)\cdots V_{\nu_l}(r_l). \tag{3.6}$$

When this is solved, we set $\lambda=1$ and obtain formally

$$V(r)=\sum_{m=1}^\infty V_m(r). \tag{3.7}$$

We will show in the next section, that the series (3.7) converges (almost everywhere), provided that

$$\int_0^\infty r|V_1(r)|dr\leqq 2\ln 2-1=0.38630. \tag{3.8}$$

We cannot prove that the $V(r)$ defined by (3.7) will satisfy the integral equation (2.26). However, we will show in Sec. 5 that under the condition

$$\sum_{m=1}^\infty\int_0^\infty |V_m(r)|rdr=K<\infty. \tag{3.9}$$

$V(r)$ exists (almost everywhere) and satisfies (2.22) and therefore (2.19). From this it follows that $V(r)$ reproduces the original phase $\eta(k)$. Since the condition (3.8) implies (3.9), we are sure that (3.7) represents a solution of the inversion problem.

The estimate (3.8) is probably rather crude and potentials for which it is satisfied cannot have bound states. The examples discussed in Sec. 6 converge considerably beyond this estimate, in fact well into the region of bound states.

4. THE CONVERGENCE PROOF

In this section we will prove that (3.8) implies the convergence of the series in (3.7) and we will derive properties of the $V(r)$ thus defined. To do this we need first some properties of the kernels $K(r, r_1\cdots r_l)$, defined in (2.28), to which we devote a subsection.

The Kernels $K(r, r_1 \cdots r_l)$[10]

These are conveniently discussed with the help of the non-negative quantities

$$M(r, u_1 \cdots u_l) = \int_0^{u_1} dt_l \int_0^{u_2} dt_2 \cdots \int_0^{u_l} dt_l \times \delta(r - t_1 - t_2 - \cdots - t_l). \quad u_k \geqq 0. \tag{4.1}$$

We obviously have

$$\int_0^\infty e^{-zr} M(r, u_1 \cdots u_l) dr = \prod_{k=1}^{l} \frac{1}{z}(1 - e^{-zu_k}). \tag{4.2}$$

By multiplying out the product on the right-hand side in (4.2) and Laplace-inverting term-by-term, we get

$$M(r, u_1 \cdots u_l) = \frac{1}{(l-1)!}\{[r]^l - \sum_k [r - u_k]^l + \sum_{k<s} [r - u_k - u_s]^l \cdots + (-1)^l [r - u_1 - u_2 - \cdots - u_l]^l\}, \tag{4.3}$$

where

$$[x] = \begin{cases} x & \text{for } x \geqq 0 \\ 0 & \text{for } x < 0. \end{cases} \tag{4.4}$$

It follows from (4.1) that

$$\left.\begin{aligned} &M(r, u_1 \cdots u_l) = 0 \\ \text{for} \quad &r < 0 \quad \text{and} \quad r > \sum_{k=1}^{l} u_k. \end{aligned}\right\} \tag{4.5}$$

Furthermore, it is easy to see from (4.1) that $M(r, u_1 \cdots u_l)$ as a function of r is symmetrical around the point

$$r_0 = \tfrac{1}{2} \sum_{k=1}^{u} u_k$$

and monotonically increasing for $r < r_0$ and correspondingly decreasing for $r > r_0$.

Finally, by integrating (4.1) first over t_l we get the estimate $M(r, u_1 \cdots u_l) \leqq u_1 u_2 \cdots u_{l-1}$, and since u_l may be taken as the largest u_k, it follows that

$$M(r, u_1 \cdots u_l) \leqq (u_1 u_2 \cdots u_l)^{(l-1)/l}. \tag{4.6}$$

By the definition (2.28) of the kernels we have

$$\left.\begin{aligned} &K(r, r_1 \cdots r_l) = \frac{\partial}{\partial r} M(r, u_1 \cdots u_l), \\ &u_1 = r_1 \quad u_2 = r_2 - r_1 \cdots u_l = r_l - r_{l-1}; \end{aligned}\right\} \tag{4.7}$$

but from (4.1)

$$\frac{\partial M(r, u_1 \cdots u_l)}{\partial r} = M(r, u_1 \cdots u_{l-1}) - M(r - u_l, u_1 \cdots u_{l-1}), \tag{4.8}$$

and since $r_i \geq u_i$, (4.6) gives the estimate

$$|K(r, r_1 r_2 \cdots r_l)| \leqq (r_1 r_2 \cdots r_{l-1})^{(l-2)/(l-1)} \leqq (r_1 r_2 \cdots r_l)^{(l-2)/l}. \tag{4.9}$$

We also require an estimate for

$$\int_0^\infty r |K(r, r_1 \cdots r_l)| dr \leqq \int_0^\infty r M(r, u_1 \cdots u_{l-1}) dr + \int_0^\infty r M(r - u_l, u_1 \cdots u_{l-1}) dr; \tag{4.10}$$

but

$$\int_0^\infty r M(r - \alpha, u_1 \cdots u_{l-1}) dr = -\frac{d}{dz} \int_0^\infty e^{-zr} M(r - \alpha, u_1 \cdots u_{l-1}) dr \Big|_{z=0} = -\frac{d}{dz} e^{-\alpha z} \prod_{k=1}^{l-1} \frac{1}{z}(1 - e^{-zu_k}) \Big|_{z=0} \tag{4.11}$$

and therefore

$$\int_0^\infty r |K(r, r_1 \cdots r_l)| dr \leqq -\frac{d}{dz}(1 + e^{-zu_l}) \prod_{k=1}^{l-1} \frac{1}{z}(1 - e^{-zu_k}) \Big|_{z=0} \leqq u_1 u_2 u_3 \cdots u_{l-1} (\sum_{k=1}^{l} u_k) \tag{4.12}$$

or with (4.7)

$$\int_0^\infty r |K(r, r_1 \cdots r_l)| dr \leqq r_1 r_2 \cdots r_l. \tag{4.13}$$

The explicit form of $K(r, r_1 r_2)$ is shown in Fig. 2.

The Convergence of the Series

First we notice that the conditions (3.2) and (3.3) are sufficient to ensure the existence and integrability of $rV_m(r)$ [compare (4.9) and Fig. 2]. From (3.6) we get the estimate

$$I_m = \int_0^\infty r |V_m(r)| dr \leqq \sum_{l=2}^{m} \sum_{\Sigma \nu_k = m} \int_0^\infty dr_1 \int_{r_1}^\infty dr_2 \cdots \int_{r_{l-1}}^\infty dr_l \times \int_0^\infty r |K(r, r_1 \cdots r_l)| dr |V_{\nu_1}(r_1)| \cdots |V_{\nu_l}(r_l)|, \tag{4.14}$$

[10] These kernels are discussed by E. Rufener, *Mitteilungen der Vereinigung Schweizer Versicherungsmathematiker* (to be published).

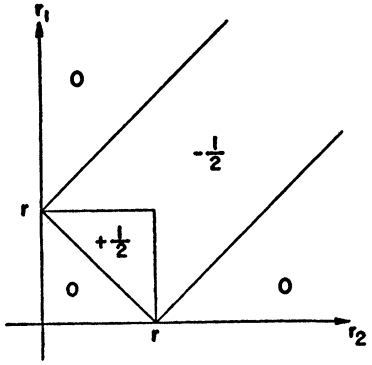

FIG. 2. The kernel $K(r, r_1r_2)$ used in the construction of the second and higher approximations [see Eq. (3.6)].

and with (4.13),

$$I_m \leqq \sum_{l=2}^{m} \sum_{\Sigma\nu_k=m} \int_0^\infty dr_1 \int_{r_1}^\infty dr_2 \cdots \int_{r_{l-1}}^\infty dr_l \prod_{k=1}^{l} r_k |V_{\nu_k}(r_k)|. \quad (4.15)$$

For symmetry reasons this can be rewritten as

$$I_m \leqq \sum_{l=2}^{m} \frac{1}{l!} \sum_{\Sigma\nu_k=m} \prod_{k=1}^{l} \left[\int_0^\infty r |V_{\nu_k}(r)| dr \right] \quad (4.16)$$

and finally

$$I_m \leqq \sum_{l=2}^{\infty} \frac{1}{l!} \sum_{\Sigma\nu_k=m} I_{\nu_1} I_{\nu_2} \cdots I_{\nu_l} (m=2, 3\cdots). \quad (4.17)$$

If we now define J_m by

$$\left.\begin{aligned} J_1 &= I_1, \\ J_m &= \sum_{l=2}^{\infty} \frac{1}{l!} \sum_{\Sigma\nu_k=m} J_{\nu_1} J_{\nu_2} \cdots J_{\nu_l} (m=2, 3\cdots), \end{aligned}\right\} \quad (4.18)$$

then we have

$$I_m \leqq J_m \quad (m=1, 2, 3\cdots). \quad (4.19)$$

To solve the recursion (4.18) we introduce the generating function

$$W(x) = \sum_{m=1}^{\infty} J_m x^m; \quad (4.20)$$

then substituting from (4.18) gives

$$W(x) - I_1 x = e^{W(x)} - 1 - W(x) \quad (4.21)$$

or

$$x = I_1^{-1}[2W(x) + 1 - e^{W(x)}]. \quad (4.22)$$

The singularity, which determines the convergence radius of (4.20), is located at the smallest root of the equation

$$dx/dW = 0 = [2 - e^{W(x)}]/I_1, \quad (4.23)$$

or

$$W(x_0) = \log 2 = 0.69315 \quad (4.24)$$

and

$$x_0 = [2 \log 2 - 1]/I_1 = 0.38630/I_1. \quad (4.25)$$

The series (4.20) will therefore converge at $x=1$, provided that

$$I_1 \leqq 0.38630; \quad (4.26)$$

note that in this case

$$W(1) \leqq 0.69315. \quad (4.27)$$

Summarizing, we have the result: If (4.26) is satisfied, then

$$\sum_{m=1}^{\infty} \int_0^\infty r |V_m(r)| dr \leqq 0.69315. \quad (4.28)$$

Applying now a theorem from the theory of real functions,[11] we can conclude that

$$\sum_{m=1}^{\infty} rV_m(r) = rV(r) \quad (4.29)$$

converges almost everywhere to an integrable function (in the sense of Lebesgue) and that

$$\int_0^\infty rV(r)dr = \sum_{m=1}^{\infty} \int_0^\infty rV_m(r)dr; \quad (4.30)$$

therefore,

$$\int_0^\infty rV(r)dr \leqq \sum_{m=1}^{\infty} I_m \leqq 0.69315 < 1, \quad (4.31)$$

and according to a well-known theorem it follows that if (4.26) holds, the potential $V(r)$ does not produce bound states.[12]

We have now constructed a $V(r)$ and shown that it leads to a finite

$$\int_0^\infty r |V(r)| dr.$$

We can, however, not be sure that it satisfies the Eq. (2.26) from which it was constructed, since our estimates for the kernels $K(r, r_1 \cdots r_l)$ do not guarantee the proper convergence of the integrals. However, to prove that a potential reproduces a given phase shift (and given bound states, when such exist), it is sufficient to show that it satisfies (2.22). This will be done in the next section.

5. PROOF THAT $V(r)$ REPRODUCES THE PHASE

The result, which we want to prove in the paragraph, is a little more general than would be necessary to prove that the $V(r)$ constructed in Sec. 4 reproduces the phase $\eta(k)$. It is the following:

If the $V_m(r)$ constructed by (3.6) have the property

[11] C. Carathéodory, *Vorlesungen ueber reelle Funktionen* (B. G. Teubner, Leipzig, 1918), p. 605, Satz 3.

[12] A potential can have a bound state only if $\int_0^\infty r|V(r)|dr \geqq 1$; see, for example, R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).

that

$$\sum_{m=1}^{\infty} I_m = K < \infty, \tag{5.1}$$

where

$$I_m = \int_0^{\infty} r |V_m(r)| dr, \tag{5.2}$$

then

$$r \cdot \sum_{m=1}^{\infty} V_m(r) = r \cdot V(r) \tag{5.3}$$

converges almost everywhere to an integrable function; $V(r)$ satisfies Eq. (2.22) and therefore reproduces the phase $\eta(k)$ and the bound states.

The existence of $V(r)$ as well as the inequality

$$\int_0^{\infty} |V(r)| r dr \leqq K \tag{5.4}$$

follows from (5.1) by an argument similar to that of the last section.

To show that $V(r)$ satisfies Eq. (2.22) we are now going to transform the left-hand side of that equation until it is equal to the right-hand side, but before we do this, we need the convergence of a series, which is related to the sequence of the I_m's.

We consider the analytical function

$$f(x) = \sum_{m=1}^{\infty} I_m x^m \tag{5.5}$$

which is, according to (5.1), regular for $|x| < 1$ and continuous for $|x| \leqq 1$. The same therefore holds for

$$F(x) = e^{f(x)} = 1 + \sum_{m=1}^{\infty} a_m x^m, \tag{5.6}$$

where

$$a_m = \sum_{l=1}^{m} \frac{1}{l!} \sum_{\Sigma \nu_k = m} I_{\nu_1} I_{\nu_2} \cdots I_{\nu_l} \geqq 0. \tag{5.7}$$

In particular, $F(1)$ exists so that

$$F(1) = 1 + \sum_{m=1}^{\infty} a_m < \infty, \tag{5.8}$$

and therefore the series $\sum_{m=1}^{\infty} a_m$ converges.[13]

Let ϵ be an arbitrarily small positive number; then we can determine an N in such a way that

$$\sum_{l=N+1}^{\infty} \frac{K^l}{l!} < \frac{\epsilon}{3}, \tag{a}$$

$$\sum_{m=N+1}^{\infty} I_m < \frac{\epsilon}{3}, \tag{b}$$

$$\sum_{m=N+1}^{\infty} a_m < \frac{\epsilon}{3}. \tag{c}$$

[13] K. Knopp, *Theorie und Anwendung der unendlichen Reihen* (Julius Springer, Berlin, 1931), third edition, p. 519.

We transform the left-hand side of (2.22) [$\mathrm{Re}[z] \geq 0$] in the following way:

$$\Lambda(z) = \frac{1}{z} \int_0^{\infty} dr (1 - e^{-zr}) [V_1(r) - V(r)]$$

$$= -\frac{1}{z} \int_0^{\infty} dr (1 - e^{-zr}) \sum_{m=2}^{N} V_m(r) + R_1(z), \tag{5.9}$$

where

$$R_1(z) = -\frac{1}{z} \int_0^{\infty} dr (1 - e^{-zr}) \sum_{m=N+1}^{\infty} V_m(r). \tag{5.10}$$

Introducing the recursions (3.6) for V_m in the first term of (5.9), we get

$$\sum_{m=2}^{N} V_m(r)$$

$$= \sum_{m=2}^{N} \sum_{l=2}^{m} \sum_{\Sigma \nu_k = m} \int_0^{\infty} dr_1 \int_{r_1}^{\infty} dr_2 \cdots \int_{r_{l-1}}^{\infty} dr_l K(r, r_1 \cdots r_l) \times V_{\nu_1}(r_1) V_{\nu_2}(r_2) \cdots V_{\nu_l}(r_l)$$

$$= \sum_{l=2} \sum_{\Sigma \nu_k \leqq N} \int_0^{\infty} dr_1 \int_0^{\infty} dr_2 \cdots \int_{r_{l-1}}^{\infty} dr_l K(r, r_1 \cdots r_l) \times V_{\nu_1}(r_1) V_{\nu_2}(r_2) \cdots V_{\nu_l}(r_l). \tag{5.11}$$

Substituting this in (5.9), interchanging the order of integration,[14] and using the definition of $K(r, r_1 \cdots r_l)$, (2.27), gives

$$\Lambda(z) = \sum_{l=2}^{N} \int_0^{\infty} dr_1 \int_{r_1}^{\infty} dr_2 \cdots \int_{r_{l-1}}^{\infty} dr_l \frac{1}{z^l} (1 - e^{-zr_1})(1 - e^{-z(r_2 - r_1)}) \cdots (1 - e^{-z(r_l - r_{l-1})}) \sum_{\Sigma \nu_k \leqq N} V_{\nu_1}(r_1) V_{\nu_2}(r_2) \cdots V_{\nu_l}(r_l) + R_1(z)$$

$$= \sum_{l=2}^{N} \int_0^{\infty} dr_1 \int_{r_1}^{\infty} dr_2 \cdots \int_{r_{l-1}}^{\infty} dr_l \frac{1}{z^l} (1 - e^{-zr_1})(1 - e^{-z(r_2 - r_1)}) \cdots (1 - e^{-z(r_l - r_{l-1})}) V(r_1) V(r_2) \cdots V(r_l) + R_2(z) + R_1(z), \tag{5.12}$$

[14] Since all our integrals and sums are absolutely convergent, the order of integrations and summations can be freely interchanged.

where

$$R_2(z)=\sum_{l=2}^{N}\int_0^\infty dr_1\int_{r_1}^\infty dr_2 \cdots \int_{r_{l-1}}^\infty dr_l \frac{1}{z^l}(1-e^{-zr_1}) \cdots (1-e^{-z(r_l-r_{l-1})}) \sum_{\Sigma\nu_k>N} V_{\nu_1}(r_1)V_{\nu_2}(r_2) \cdots V_{\nu_l}(r_l). \quad (5.13)$$

Finally,

$$\Lambda(z)=\sum_{l=2}^{\infty}\int_0^\infty dr_1\int_{r_1}^\infty dr_2 \cdots \int_{r_{l-1}}^\infty dr_l \frac{1}{z_l}(1-e^{-zr_1})(1-e^{-z(r_2-r_1)}) \cdots (1-e^{-z(r_l-r_{l-1})})V(r_1)V(r_2) \cdots V(r_l)+R_3(z)+R_2(z)+R_1(z), \quad (5.14)$$

where

$$R_3(z)=\sum_{l=N+1}^{\infty}\int_0^\infty dr_1\int_{r_1}^\infty dr_2 \cdots \int_{r_{l-1}}^\infty dr_l \frac{1}{z^l}(1-e^{-zr_1}) \cdots (1-e^{-z(r_l-r_{l-1})})V(r_1)\cdots V(r_l). \quad (5.15)$$

What remains is to estimate $R_1(z)$, $R_2(z)$, $R_3(z)$. With the help of

$$\left|\frac{1}{z}(1-e^{-z\alpha})\right| = \left|\int_0^\alpha e^{-z\alpha'}d\alpha'\right| \leqq \alpha, \quad \mathrm{Re}[z]\geqq 0, \quad (5.16)$$

one justifies easily

$$|R_1(z)| \leqq \int_0^\infty r \sum_{m=N+1}^{\infty} |V_m(r)| = \sum_{m=N+1}^{\infty} I_m < \frac{\epsilon}{3} \quad (5.17)$$

$$|R_2(z)| \leqq \sum_{l=2}^{N}\int_0^\infty dr_1\int_{r_1}^\infty dr_2 \cdots \int_{r_{l-1}}^\infty dr_l \sum_{\Sigma\nu_k>N} r_1|V_{\nu_1}(r_1)|r_2|V_{\nu_2}(r_2)| \cdots r^l|V_{\nu_l}(r_l)| \leqq \sum_{m=N+1}^{\infty}\sum_{l=2}^{m}\frac{1}{l!}\sum_{\Sigma\nu_k=m} I_{\nu_1}I_{\nu_2} \cdots I_{\nu_l} < \sum_{m=N+1}^{\infty} a_m < \frac{\epsilon}{3} \quad (5.18)$$

$$|R_3(z)| \leqq \sum_{l=N+1}^{\infty}\int_0^\infty dr_1\int_{r_1}^\infty dr_2 \cdots \int_{r_{l-1}}^\infty dr_l\, r_1|V(r_1)|r_2|V(r_2)| \cdots r_l|V(r_l)| \leqq \sum_{l=N+1}\frac{1}{l!}K^l < \frac{\epsilon}{3}, \quad (5.19)$$

where (a), (b), (c) have been used as well as (5.1), (5.7). Therefore,

$$|R_1(z)+R_2(z)+R_3(z)| < \epsilon. \quad (5.20)$$

From (5.9), (5.14), and (5.20) it follows that (2.22) is satisfied. This concludes the verification.

6. TWO EXAMPLES

$k\cot\eta(k) = -\alpha+\frac{1}{2}r_0k^2$

We shall now work out in detail a simple example which shows how the method can in principle be used for a determination of the potential. We consider a particle obeying the Schrödinger equation (2.1) whose phase shift is given by

$$k\cot\eta(k) = -\alpha+\tfrac{1}{2}r_0k^2. \quad (6.1)$$

This expression is taken from the theory of the effective range[7] and luckily is particularly easy to treat by our method.

We first discuss the case $\alpha<0$, $r_0>0$, which corresponds, for example, to singlet neutron-proton scattering. In this case one can see from (6.1) that $\eta(+0)=0$ so that the system has no bound state [compare (2.14)].

By means of the relation $\tan^{-1}x=(i/2)\log[(1-ix)/(1+ix)]$, we write $\eta(k')$ explicitly as

$$\eta(k') = \tan^{-1}[(k'/-\alpha+\tfrac{1}{2}r_0k^2)] = i/2\,\log[(z'+a)(z'-b)/(z'-a)(z'+b)], \quad (6.2)$$

where

$$z'=2ik', \quad (6.3)$$

$$a=(2/r_0)[1+(1-2\alpha r_0)^{\frac{1}{2}}], \quad b=(2/r_0)[-1+(1-2\alpha r_0)^{\frac{1}{2}}]. \quad (6.4)$$

We now calculate $g(z)$ by means of Eq. (2.15):

$$\log g(z) = -\frac{2i}{\pi}\int_{-\infty}^{\infty}\frac{\eta(k')}{2ik'-z}dk' = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dz'}{z'-z}\left[\log\frac{z'+a}{z'+b}+\log\frac{z'-b}{z'-a}\right] = -\log[(z+a)/(z+b)]; \quad (6.5)$$

since $a>b>0$, we have integrated the first and second terms by closing contours in the right and left half-planes, respectively. Hence we obtain the simple ex-

pression

$$g(z)=(z+b)/(z+a)$$
$$=1+\lambda 2a/(z+a), \tag{6.6}$$

where

$$\lambda=-(a-b)/2a. \tag{6.7}$$

Expanding the potential $V(r)$ in powers of λ,

$$V(r)=\sum_1^\infty \lambda^m V_m(r), \tag{6.8}$$

we find first, by Eq. (3.1),

$$\lambda r V_1(r)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{rz}\frac{d}{dz}z[g(z)-1]dz$$
$$=\frac{2a^2\lambda}{2\pi i}\int_{-i\infty}^{i\infty} e^{rz}\frac{1}{(z+a)^2}dz$$
$$=2a^2\lambda r e^{-ar}, \tag{6.9}$$

or

$$V_1(r)=2a^2e^{-ar}. \tag{6.10}$$

In the present case the calculation of the higher approximations is best carried out by Laplace transforming Eq. (3.6), using the first expression of Eq. (2.27) for the kernels:

$$\int_0^\infty e^{-zr}V_m(r)dr$$
$$=\sum_{l=2}^{m}\sum_{\nu_1+\nu_2+\cdots\nu_l=m}\int_0^\infty dr_1\int_{r_1}^\infty dr_2$$
$$\cdots\int_{r_{l-1}}^\infty dr_l(1-e^{-zr_1})\frac{1}{z}(1-e^{-z(r_2-r_1)})$$
$$\cdots\frac{1}{z}(1-e^{-z(r_l-r_{l-1})})V_{\nu_1}(r_1)V_{\nu_2}(r_2)$$
$$\cdots V_{\nu_l}(r_l). \tag{6.11}$$

Thus with (6.10) we find

$$\int_0^\infty e^{-zr}V_2(r)dr=2a^2\left(-\frac{1}{z+a}+\frac{2}{z+2a}\right) \tag{6.12}$$

so that

$$V_2(r)=2a^2(-e^{-ar}+2e^{-2ar}). \tag{6.13}$$

Similarly,

$$V_3(r)=2a^2(e^{-ar}-4e^{-2ar}+3e^{-3ar}), \tag{6.14}$$

etc. The total potential, therefore, is

$$V(r)=2a^2\{e^{-ar}(\lambda-\lambda^2+\lambda^3-\cdots)$$
$$+e^{-2ar}(2\lambda^2-4\lambda^3+\cdots)$$
$$+e^{-3ar}(3\lambda^3-\cdots)$$
$$+\cdots\}. \tag{6.15}$$

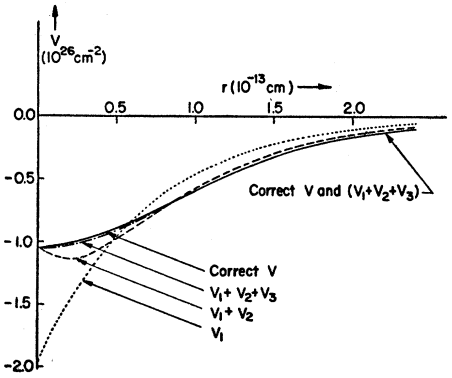

FIG. 3. Three successive approximations to the Eckert potential V, corresponding to $n-p$ singlet scattering.

The first three approximations with numerical constants corresponding to $n-p$ singlet scattering ($\alpha=-0.0422\times10^{+13}$ cm^{-1}, $r_0=2.6\times10^{-13}$ cm, $a=1.45\times10^{13}$ cm^{-1}, $\lambda=-0.469$) are plotted in Fig. 3. In the same figure the exact solution corresponding to the $g(z)$ of Eq. (6.6) is plotted, *viz.*:

$$V(r)=2\lambda(1+\lambda)a^2e^{-ar}/[1+\lambda(1-e^{-ar})]^2.^8 \tag{6.16}$$

We see from (6.16) that the radius of convergence is $|\lambda|=1$, more than twice the value appropriate for $n-p$ singlet scattering. The convergence of our procedure is therefore unexpectedly good, when compared with our criterion (3.8), which assures convergence only up to $|\lambda|=0.193$.

As a second case we consider again (6.1), this time with $\alpha>0, r_0>0$ and $2\alpha r_0<1$. This means that $\eta(+0)=\pi$ so that the system has one bound state whose energy, $E_0=-\frac{1}{4}\xi_0^2(\xi_0>0)$, is necessary for the construction of $g(z)$.

We first write down the auxiliary phase

$$\bar{\eta}(k')=\eta(k')-2\tan^{-1}(\xi_0/2k')$$
$$=-\frac{i}{2}\log\frac{(z'-b)(z'+a)(z'-\xi_0)^2}{(z'-a)(z'+b)(z'+\xi_0)^2}, \tag{6.17}$$

where a and b are given by (6.14) and $a>0$, $b<0$ (see Fig. 1).

The function $\bar{g}(z)$ is given, according to (2.17), by

$$\log\bar{g}(z)=-\frac{2i}{\pi}\int_{-\infty}^{\infty}\frac{\bar{\eta}(k')}{2ik'-z}dk'$$
$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dz'}{z'-z}\log\frac{(z'-b)(z'+a)}{(z'+\xi_0)^2}$$
$$=\log\frac{(z+\xi_0)^2}{(z-b)(z+a)}, \tag{6.18}$$

and therefore $g(z)$ is, by Eq. (2.16),

$$g(z)=\bar{g}(z)(z-\xi_0)/(z+\xi_0)$$
$$=(z^2-\xi_0^2)/[(z-b)(z+a)]. \tag{6.19}$$

In the case of the neutron-proton system one finds experimentally that $\xi_0=-b$; i.e., the bound state coincides with the absolutely smaller zero of the S matrix[15] defined by Eq. (6.1), namely,

$$S(k)=\frac{(2ik+b)(2ik-a)}{(2ik-b)(2ik+a)}. \tag{6.20}$$

Equation (6.19) now simplifies to

$$g(z)=(z+b)/(z+a), \tag{6.21}$$

which is of the same form as (6.6); however, in the present case b is negative.

The construction of V is exactly the same as before, leading again to (6.15) and (6.16). The numerical values for $n-p$ triplet scattering are: $\alpha=0.186\times10^{13}$ cm^{-1}, $r_0=1.56\times10^{-13}$ cm, $a=2.114\times10^{13}$ cm^{-1}, $\lambda=-0.606$. Therefore the convergence of our method is only slightly worse than for singlet-scattering where λ was -0.469.

It should be recalled, however, that in general, when there are bound states, there is no guarantee that the potential constructed by our method is the only solution. In the present case this ambiguity is illustrated by the family of alternative potentials which has been constructed by Bargmann.[4] Figure 4 shows the potential $V(r)$, Eq. (6.16), to which our method leads, as well as two equivalent potentials $V_2(r)$ and $V_4(r)$, selected from Bargmann's Eqs. (4.10) and (4.12) by setting $\theta=0.1$ and 0, respectively.

The Exponential Potential

In order to see whether the good convergence in the last example was accidental we have also calculated a second example.

We have taken an exponential potential,

$$V(r)=\mu e^{-r} \tag{6.22}$$

(where r is measured in convenient units), and calculated the $g(z)$ which corresponds to its phase shift and bound states. In this case it is not necessary to calculate first $\eta(k)$ and the ξ_i explicitly, since the $g(z)$ for this potential is well known:[16]

$$g(\mu,z)=1+\sum_{n=1}^{\infty}\frac{\mu^n}{n!}\frac{1}{(1+z)(2+z)\cdots(n+z)}$$
$$=1+\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{(s-1)!}\frac{1}{(s+z)}\mu^k\sum_{m=0}^{\infty}\frac{\mu^m}{m!(m+s)!}. \tag{6.23}$$

$V_1(r)$ is now calculated from (2.21)

$$g(\mu,z)-1=\frac{1}{z}\int_0^{\infty}dr(1-e^{-zr})V_1(r)$$
$$=\int_0^{\infty}dre^{-zr}\left(\int_r^{\infty}V_1(r')dr'\right), \tag{6.24}$$

which gives after Laplace inversion and differentiation

$$V_1(r)=\sum_{s=1}^{\infty}\frac{(-1)^{s-1}se^{-sr}}{(s-1)!}\mu^s\sum_{m=0}^{\infty}\frac{\mu^m}{m!(m+s)!}. \tag{6.25}$$

For the $n-p$ triplet system, we take our unit of length as $r_0=0.75\times10^{-13}$ cm and $\mu=2.13$. Then one finds

$$V_1(r)=-[0.538e^{-r}+2.061e^{-2r}+1.374e^{-3r}$$
$$+0.369e^{-4r}+0.052e^{-5r}+0.004e^{-6r}+\cdots]. \tag{6.26}$$

To calculate V_2 and V_3 one uses the easily established formulas

$$\int_0^{\infty}dr_1\int_{r_1}^{\infty}dr_2K(r,r_1,r_2)e^{-\alpha r_1}e^{-\beta r_2}$$
$$=-\frac{1}{\alpha(\alpha+\beta)}e^{-\beta r}+\frac{1}{\alpha\beta}e^{-(\alpha+\beta)r} \tag{6.27}$$

$$\int_0^{\infty}dr_1\int_{r_1}^{\infty}dr_2\int_{r_2}^{\infty}K(r,r_1r_2r_3)e^{-\alpha r_1}e^{-\beta r_2}e^{-\gamma r_3}$$
$$=\left(\frac{1}{\alpha(\beta+\gamma)(\alpha+\beta)}-\frac{1}{\alpha\beta(\alpha+\beta+\gamma)}\right)e^{-\gamma r}$$
$$+\frac{1}{\alpha\beta(\alpha+\beta+\gamma)}e^{-(\beta+\gamma)r}$$
$$-\frac{1}{\alpha(\beta+\gamma)(\alpha+\beta)}e^{-(\alpha+\beta+\gamma)r}. \tag{6.28}$$

To simplify the calculation of V_3 we have used the approximate expression

$$V_1(r)\approx-(0.6e^{-r}+3.8e^{-2.45}),$$

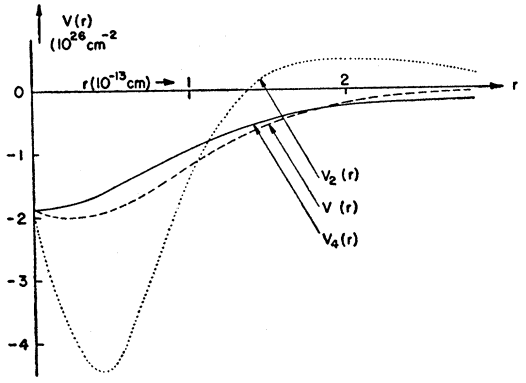

FIG. 4. Three phase equivalent potentials corresponding to the $n-p$ triplet state. All the curves go through the same point at $r=0$.

[15] For the few potentials examined so far the S matrix vanishes at points in the k plane corresponding to bound states.

[16] For example, reference 12, p. 845.

which is accurate to within a few percent. The potentials obtained in the first three approximations, which can be calculated with relatively little effort, are shown in Fig. 5, together with the exponential potential. The curves apparently converge satisfactorily towards the exponential potential.[17] It might be remarked here that for the same potential, successive Born approximation diverge up to about 20 Mev.

7. THE TANGENT METHOD

Instead of basing our construction of the potential on the integral Eq. (2.10) whose solutions behave like $f(k, r)\sim e^{-ikr}$ for large r, one can also consider integral equations corresponding to other boundary conditions. One of the most natural possibilities is

$$\varphi(0)=0, \tag{7.1}$$

$$r\to\infty:\ \varphi(r)\to\sin kr+\tan\eta\cos kr. \tag{7.2}$$

The integral equation which is equivalent to the Schrödinger equation (2.1) and these boundary conditions is

$$\varphi(r)=\sin kr+\int_0^\infty g(r, r')V(r')\varphi(r')dr',^{18} \tag{7.3}$$

where

$$g(r, r')=-\frac{1}{2k}\left(\sin k(r+r')-\sin k|r-r'|\right)$$

$$=-\frac{1}{2}\int_{|r-r'|}^{r+r'}\cos ktdt. \tag{7.4}$$

For $r\to\infty$, (7.3) gives

$$\varphi(r)=\sin kr+\left(-\frac{1}{k}\int_0^\infty \sin kr'V(r')\varphi(r')dr'\right)\cos kr \tag{7.5}$$

so that

$$-k\tan\eta=\int_0^\infty \sin krV(r)\varphi(r)dr. \tag{7.6}$$

[17] A simple function—theoretic argument shows that if our procedure converges it must converge to the exponential potential.

[18] Fröberg's method (see reference 6) is equivalent to starting with the integral equation

$$\varphi(r)=\sin(kr+\eta)+\int_r^\infty k(r, r')V(r')\varphi(r')dr',$$

where $k(r, r')=(1/k)\sin k(r'-r)$. The solution of this equation, which has the asymptotic form $\varphi(r)\to\sin(kr+\eta)$, is obtained by iteration and substituted into the equation

$$0=\sin\eta(k)+\int_0^\infty k(0, r')V(r')\varphi(r')dr',$$

which comes from the boundary condition $\varphi(0)=0$. Finally, one writes $\sin\eta(k)=\lambda F(k)$, expands both $V(r)$ and $\sin(kr+\eta)$ in powers of λ, and equates to zero the coefficients of λ^m. The resulting equations can be solved for the successive terms in the expansion of $V(r)$.

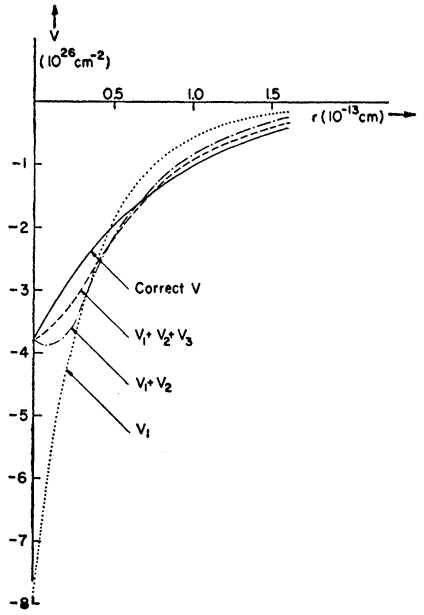

FIG. 5. Three successive approximations to the exponential potential $V(r)$ corresponding to the $n-p$ triplet state.

We now solve (7.3) by iteration and substitute the resulting series into (7.6):

$$-k\tan\eta(k)=\int_0^\infty \sin^2krV(r)dr+\sum_{n=1}^\infty\int_0^\infty dr\int_0^\infty dr_1$$

$$\cdots\int_0^\infty dr_n\sin krV(r)g(r, r_1)V(r_1)$$

$$\cdots g(r_{n-1}, r_n)V(r_n)\sin kr_n. \tag{7.7}$$

To solve this equation for $V(r)$, in terms of $\tan\eta(k)$, we write formally

$$-k\tan\eta(k)=\mu F(k), \tag{7.8}$$

$$V(r)=\sum_1^\infty \mu^m V_m(r), \tag{7.9}$$

substitute into (7.7), and equate to zero the coefficients of μ, μ^2, $\cdots$. Finally we set $\mu=1$. This gives the following set of equations:

$$-k\tan\eta(k)=\int_0^\infty \sin^2krV_1(r)dr, \tag{7.10}$$

$$0=\int_0^\infty \sin^2krV_m(r)dr+\sum_{l=2}^\infty\sum_{\Sigma\nu_k=m}\int_0^\infty dr_1$$

$$\cdots\int_0^\infty dr_l\sin kr_1V_{\nu_1}(r_1)g(r_1, r_2)V_{\nu_2}(r_2)$$

$$\cdots g(r_{l-1}, r_l)V_{\nu_l}(r_l)\sin kr_l,\quad m=2, 3, \cdots. \tag{7.11}$$

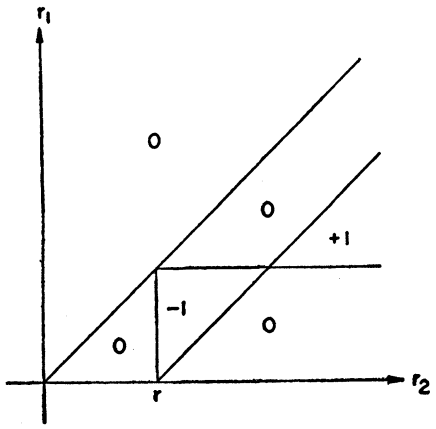

FIG. 6. The kernel $\bar{K}(r, r_1 r_2)$ used in the tangent method for the construction of the second and higher approximations [see Eq. (7.15)].

Given $\tan\eta(k)$ we can now successively solve these equations for $V_1, V_2, \cdots$. We first differentiate (7.10) with respect to k and Fourier invert, obtaining as first approximation

$$rV_1(r) = -\frac{4}{\pi}\int_0^\infty \frac{d}{dk}(k\tan\eta(k))\sin 2kr dr. \quad (7.12)$$

To solve for $V_m(r)$, $m \geq 2$, we begin by writing the first term in (7.11) as

$$\frac{1}{2}\int_0^\infty (1-\cos 2kr) V_m(r) dr \quad (7.13)$$

and assume, subject to later verification, that

$$\int_0^\infty V_m(r) dr = 0, \quad m \geq 2. \quad (7.14)$$

Now we Fourier invert (7.11) and obtain

$$V_m(r) = \sum_{l=2}^{\infty} \sum_{\Sigma \nu_k = m} \int_0^\infty dr_1 \int_0^\infty dr_2 \cdots \int_0^\infty dr_l \bar{K}(r, r_1 \cdots r_l) V_{\nu_1}(r_1) \cdots V_{\nu_l}(r_l), \quad (7.15)$$

where

$$\bar{K}(r, r_1 \cdots r_l) = -\frac{4}{\pi}\int_{-\infty}^\infty dk \cos 2kr \sin kr_1 g(r_1, r_2) \cdots g(r_{l-1}, r_l) \sin kr_l. \quad (7.16)$$

One verifies (7.14) immediately from the fact that

$$\int_0^\infty \bar{K} dr = 0.$$

In practical applications it may be most convenient to substitute (7.16) into (7.15) and do the integrations over the r_i before that over k. However, in Fig. 6 we also show $\bar{K}(r, r_1 r_2)$ as function of r, r_1, and r_2.

We do not consider it worthwhile to present here a proof for the convergence of this procedure for small enough $\eta(k)$. Such a proof can be given along the lines of Sec. IV. One can notice immediately, however, that the method does not apply to potentials with bound states since in that case $(d/dk)[k\tan\eta(k)]$ has a quadratic singularity where $\eta(k) = (\pi/2)$, and hence $V_1(r)$, Eq. (7.12), does not exist.

To illustrate how this method works, we consider again the phase shift given by (6.1), which we now write as

$$-k\tan\eta = -\mu\gamma k^2/(\beta^2+k^2), \quad (7.17)$$

where

$$\gamma = 2/r_0, \quad \beta^2 = -2\alpha/r_0, \quad (7.18)$$

and μ is eventually taken as 1. We restrict ourselves to the case where β is real and, for definiteness, positive to avoid a singularity in $\tan\eta$. From (7.12) and (7.15) we find

$$V_1(r) = -4\gamma\beta e^{-2\beta r}, \quad (7.19)$$

$$V_2(r) = -4\gamma^2 e^{-2\beta r} + 4\gamma^2\beta r e^{-2\beta r} + 4\gamma^2 e^{-4\beta r}. \quad (7.20)$$

To determine the conditions under which the series (7.9) for $V(r)$ converges when $\mu = 1$, we use the fact that the potential corresponding to (7.17) is exactly known and, since there are no bound states, even unique. With our present notation we have [compare (6.16), (6.7), (6.4), and (6.1)]

$$V(r) = 2\lambda(1+\lambda)a^2 e^{-ar}/[1+\lambda(1-e^{-ar})]^2, \quad (7.21)$$

where

$$a = \mu\gamma + (\mu^2\gamma^2 + 4\beta^2)^{\frac{1}{2}}, \quad (7.22)$$

$$\lambda = \mu^2(\gamma/2\beta)^2 - \mu(\gamma/2\beta)[\mu^2(\gamma/2\beta)^2+1]^{\frac{1}{2}}. \quad (7.23)$$

Thus we see that the expansion (7.9) for $V(r)$ converges for $\mu = 1$ only if

$$|\gamma/2\beta| < 1. \quad (7.24)$$

Comparison with (7.17) shows that for this particular potential one has convergence only if

$$|\eta(k)| < \pi/4 \quad (7.25)$$

for all k.

It is interesting to compare the convergence of this procedure with that of the method discussed earlier. At the limits of convergence of the present expansion of $V(r)$ we find

$$\begin{aligned} \mu/2\beta &= \ \ 1: \quad \lambda = 1-\sqrt{2} = -0.414, \\ \mu/2\beta &= -1: \quad \lambda = 1+\sqrt{2} = \ \ 2.414, \end{aligned} \quad (7.26)$$

so that in the case of the attractive potential $(\mu/2\beta = 1)$

our other procedure has a larger radius of convergence, while for the repulsive potential the present method is better.

8. CONCLUDING REMARKS

In conclusion we should like to survey briefly some possible generalizations of our procedure.

The simplest of these is the generalization to higher angular momenta. It is easy to give the formal expressions corresponding to our kernels $K(r, r_1r_2\cdots r_l)$. As a result of the centrifugal barrier, the potential becomes effectively weaker for higher l, and therefore the convergence of our method for a given potential is likely to become better. Also, for sufficiently high l, the uniqueness problem, connected with the existence of bound states does not occur. On the other hand, the phases for $l>0$ are not so well known experimentally, and the kernels considerably more complicated than for S states.

The inclusion of tensor forces in the framework of our procedure does not seem to be unfeasible, but is undoubtedly quite complicated.

The extension to the case of a repulsive Coulomb field seems to be promising. Jauho[19] has already generalized Levinson's results to this case, and there can be little doubt that the present method can also be adapted to it. For the proton-proton system one has the advantages of more accurate measurements and no bound state complications. These would seem to make a study of this case worth while.

Finally an investigation of the relativistic equation may be interesting for application to the scattering of high energy electrons by nuclei.

This paper was begun at the Institute for Theoretical Physics in Copenhagen and concluded at the Institute for Advanced Study in Princeton. We should like to express our great appreciation to Professors Niels Bohr and Robert Oppenheimer for making our collaboration on this problem possible. We are also very grateful to Professor V. Bargmann for many helpful discussions.

APPENDIX 1. CONVERGENCE PROOF FOR THE SERIES 2.26

We have to show that under the assumptions

$$|rV(r)|<M<\infty, \tag{A1.1}$$

$$\int_0^\infty r|V(r)|dr=I<\infty, \tag{A1.2}$$

the function $V_1(r)$ defined by (2.26) exists and has the properties

$$|rV_1(r)|<M_1<\infty, \tag{A1.3}$$

$$\int_0^\infty r|V_1(r)|dr=I_1<\infty. \tag{A1.4}$$

With the notation

$$F_l(r)=-\int_0^\infty dr_1\int_{r_1}^\infty dr_2 \cdots \int_{r_{l-1}}^\infty dr_l K(r, r_1\cdots r_l)V(r_1)\cdots V(r_l), \tag{A1.5}$$

the series in (2.26) reads

$$V_1(r)=V(r)+\sum_{l=2}^\infty F_l(r). \tag{A1.6}$$

We will first prove (A1.3). From (A1.1) and (A1.2) we find

$$A\equiv\int_0^\infty g(r)|V(r)|dr<\infty, \tag{A1.7}$$

where

$$g(r)=\begin{cases}\sqrt{r} & \text{for } 0\leq r<1\\ r & \text{for } 1\leq r\end{cases}. \tag{A1.8}$$

Analyzing first $rF_2(r)$, we get from Fig. 2

$$\begin{aligned}|rF_2(r)|&\leqq r\int_0^r dr_1\int_r^{r+r_1}dr_2|V(r_1)||V(r_2)|\\ &\quad+r\int_r^\infty dr_1\int_{r+r_1}^\infty dr_2|V(r_1)||V(r_2)|\\ &\leqq\int_0^r dr_1\int_r^{r+r_1}dr_2r_2|V(r_1)||V(r_2)|\\ &\quad+\int_r^\infty dr_1g(r_1)\int_r^\infty dr_2g(r_2)|V(r_1)||V(r_2)|,\end{aligned} \tag{A1.9}$$

and finally with (A1.1), (A1.2), and (A1.7)

$$|rF_2(r)|\leqq MI+A^2<\infty. \tag{A1.10}$$

For the estimate of $rF_l(r)$, $l\geqq 3$, we use the first estimate in (4.9) together with the property of the kernels $K(r, r_1\cdots r_l)$ to vanish for $r>r_l$ [compare (4.5)]. This gives for $rF_l(r)$

$$|rF_l(r)|\leqq\int_0^\infty dr_1\int_{r_1}^\infty dr_2\cdots\int_{r_{l-1}}^\infty dr_l g(r_1)g(r_2)\cdots g(r_l)V(r_1)V(r_2)\cdots V(r_l)\leqq A^l/l!. \tag{A1.11}$$

Introducing (A1.10), (A1.11), and (A1.1) into (A1.6) gives

$$|rV_1(r)|<M+MI+A^2+\sum_{l=3}^\infty\frac{1}{l!}A^l=M_1<\infty, \tag{A1.12}$$

which proves (A1.3).

[19] P. Jauho, Ann. Acad. Sci. Fennicae, Ser. A, **80** (1951).

The proof of (A1.4) goes exactly along the lines of the convergence proof in Sec. 4. Using (4.13) one verifies easily

$$\int_0^\infty r|V_1(r)|dr \leqq I+\sum_{l=2}^{\infty}\frac{I^l}{l!}=I_1<\infty. \quad \text{(A1.13)}$$

Furthermore, we note that as a consequence of (2.30) we also have

$$\int_0^\infty r^2|V_1(r)|dr<\infty, \quad \text{(A1.14)}$$

an equation which, however, we did not use in the text. Therefore we omit the simple proof.

Finally, it follows easily from (A1.1) and (A1.2) that all the $F_l(r)$ are integrable, and from (2.27) we get

$$\int_0^\infty F_l(r)dr=0, \quad \text{(A1.15)}$$

which together with (A1.6) gives

$$\int_0^\infty [V(r)-V_1(r)]dr=0. \quad \text{(A1.16)}$$

APPENDIX 2. A GENERALIZATION OF LEVINSON'S THEOREM[5]

Levinson has proved that for a given S phase shift and in the absence of bound states there exists at most one potential satisfying the conditions

$$\int_0^\infty r|V(r)|dr<\infty, \quad \text{(A2.1)}$$

$$\int_0^\infty r^2|V(r)|dr<\infty. \quad \text{(A2.2)}$$

We shall now give a generalization of this theorem under the same conditions (A2.1) and (A2.2). In the notation we follow the paper of Jauho.[19] For simplicity we first discuss the case of a single bound state.

We assume that for the given phase shift and binding energy there exist two potentials $V(r)$ and $\bar{V}(r)$ and corresponding wave functions $\varphi(k,r)$ and $\bar{\varphi}(k,r)$ satisfying the equations

$$\varphi''+k^2\varphi=V\varphi, \quad \text{(A2.3)}$$

$$\bar{\varphi}''+k^2\bar{\varphi}=\bar{V}\bar{\varphi}, \quad \text{(A2.4)}$$

and the boundary conditions

$$\varphi(k,0)=0, \quad \varphi'(k,0)=1, \quad \text{(A2.5)}$$

$$\bar{\varphi}(k,0)=0, \quad \bar{\varphi}'(k,0)=1. \quad \text{(A2.6)}$$

These functions are related to the functions $f(k,r)$ and $\bar{f}(k,r)$ of (2.3) by the equations

$$\varphi(k,r)=(1/2ik)\{f(k)f(-k,r)-f(-k)f(k,r)\}, \quad \text{(A2.7)}$$

$$\bar{\varphi}(k,r)=(1/2ik)\{f(k)\bar{f}(-k,r)-f(-k)\bar{f}(k,r)\}; \quad \text{(A2.8)}$$

here we have abbreviated $f(k,0)$ by $f(k)$ and $\bar{f}(k,0)$ by $\bar{f}(k)$ and used the fact that by our assumptions

$$f(k)=\bar{f}(k). \quad \text{(A2.9)}$$

$\varphi(k,r)$ and $\bar{\varphi}(k,r)$ are regular functions for all complex values of k.[8] For large $|k|$ they behave as

$$\varphi(k,r)\sim\sin kr/k, \quad \text{(A2.10)}$$

$$\bar{\varphi}(k,r)\sim\sin kr/k. \quad \text{(A2.11)}$$

Similarly we have, for large $|k|$ and $\mathrm{Im}[k]\leqq 0$,

$$f(k,r)\sim e^{-ikr}, \quad \text{(A2.12)}$$

$$\bar{f}(k,r)\sim e^{-ikr}, \quad \text{(A2.13)}$$

and therefore

$$f(k)\sim 1. \quad \text{(A2.14)}$$

Following reference 19 we discuss the expressions

$$\Phi_1(k,r)=\frac{k}{f(k)}\varphi(k,r)\int_r^\infty \bar{f}(k,\rho)F(\rho)d\rho \quad \text{(A2.15)}$$

and

$$\Phi_2(k,r)=\frac{k}{f(k)}\bar{f}(k,r)\int_0^r \varphi(k,\rho)F(\rho)d\rho, \quad \text{(A2.16)}$$

where $F(r)$ is a continuous and continuously differentiable function, which is different from zero only in a finite r interval.

Integrating $\Phi_1(k,r)$ over a contour C consisting of a large semicircle in the lower half-plane and the real axis, we get a contribution only from the zero of $f(k)$ at $k=-i\kappa$:[20]

$$\int_C \Phi_1(k,r)dk=-2\pi i\frac{(-i\kappa)}{\dot{f}(-i\kappa)}\varphi(-i\kappa,r)\times\int_r^\infty \bar{f}(-i\kappa,\rho)F(\rho)d\rho, \quad \text{(A2.17)}$$

where

$$\dot{f}(k)\equiv(df/dk). \quad \text{(A2.18)}$$

We now introduce the following abbreviations for the bound state wave function:

$$\varphi(-i\kappa,r)\equiv\varphi_0(r), \quad \text{(A2.19)}$$

$$\bar{\varphi}(-i\kappa,r)\equiv\bar{\varphi}_0(r), \quad \text{(A2.20)}$$

and the notation

$$C=[2i\kappa f'(-i\kappa,0)/\dot{f}(-i\kappa)], \quad \text{(A2.21)}$$

$$\bar{C}=[2i\kappa\bar{f}'(-i\kappa,0)/\dot{f}(-i\kappa)]. \quad \text{(A2.22)}$$

C and $\bar{C}$ are real and positive (see reference 8 p. 265). Finally, we clearly have the relations

$$\varphi_0(r)=[f(-i\kappa,r)/f'(-i\kappa,0)], \quad \text{(A2.23)}$$

$$\bar{\varphi}_0(r)=[\bar{f}(-i\kappa,r)/\bar{f}'(-i\kappa,0)]. \quad \text{(A2.24)}$$

[20] In terms of the notation of Sec. 2, $\kappa=\xi/2$.

We can now rewrite (A2.17) as

$$\int_{-K}^{+K}\Phi_1(k,r)dk-\pi i\bar{C}\varphi_0(r)\int_r^\infty\bar{\varphi}_0(\rho)F(\rho)d\rho$$
$$=-\int_{S.C.}\Phi_1(k,r)dk, \quad \text{(A2.25)}$$

where the first integral goes over the real axis and the last over the large semicircle. Levinson has shown that one can substitute in the last integral the asymptotic expressions (A2.10–14), which gives

$$-\int_{S.C.}\Phi_1(k,r)dk$$
$$=-\int_{S.C.}dk\,\sin kr\int_r^\infty e^{-ik\rho}F(\rho)d\rho+R(K), \quad \text{(A2.26)}$$

where

$$\lim_{K\to\infty}R(K)=0. \quad \text{(A2.27)}$$

An integration by parts leads to

$$\int_r^\infty e^{-ik\rho}F(\rho)d\rho=\frac{1}{ik}e^{-ikr}[F(r)+R_1(K)], \quad \text{(A2.28)}$$

where again

$$\lim_{K\to\infty}R_1(K)=0. \quad \text{(A2.29)}$$

Substituting (A2.28) into (A2.26) gives

$$-\int_{S.C.}\Phi_1(k,r)dk$$
$$=\tfrac{1}{2}F(r)\int_{S.C.}\frac{dk}{k}(1-e^{-2ik\rho})+R_2(K)$$
$$=-\tfrac{1}{2}i\pi F(r)+R_3(K), \quad \text{(A2.30)}$$

where

$$\lim_{K\to\infty}R_{2,3}(K)=0. \quad \text{(A2.31)}$$

Finally, substituting (A2.30) into (A2.25) gives

$$F(r)=2\bar{C}\varphi_0(r)\int_r^\infty\bar{\varphi}_0(\rho)F(\rho)d\rho$$
$$+\frac{2i}{\pi}\int_{-\infty}^{+\infty}\frac{kdk}{f(k)}\varphi(k,r)\int_r^\infty\bar{f}(k,\rho)F(\rho)d\rho. \quad \text{(A2.32)}$$

A similar procedure applied to (A2.16) yields

$$F(r)=2\bar{C}\bar{\varphi}_0(r)\int_0^r\varphi_0(\rho)F(\rho)d\rho$$
$$+\frac{2i}{\pi}\int_{-\infty}^{+\infty}\frac{kdk}{f(k)}\bar{f}(k,r)\int_0^r\varphi(k,\rho)F(\rho)d\rho. \quad \text{(A2.33)}$$

We now add to (A2.32) its complex conjugate which, with (A2.8), leads to

$$F(r)=2\bar{C}\varphi_0\int_r^\infty\bar{\varphi}_0(\rho)F(\rho)d\rho$$
$$+\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{k^2dk}{|f(k)|^2}\varphi(k,r)\int_r^\infty\bar{\varphi}(k,\rho)F(\rho)d\rho; \quad \text{(A2.34)}$$

similarly we obtain from (A2.33)

$$F(r)=2\bar{C}\bar{\varphi}_0(r)\int_0^r\varphi_0(\rho)F(\rho)d\rho$$
$$+\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{k^2dk}{|f(k)|^2}\bar{\varphi}(k,r)\int_0^r\varphi(k,\rho)F(\rho)d\rho. \quad \text{(A2.35)}$$

Interchanging the role of φ and $\bar{\varphi}$, C and $\bar{C}$, we find from the last equation

$$F(r)=2C\varphi_0(r)\int_0^r\bar{\varphi}_0(\rho)F(\rho)d\rho$$
$$+\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{k^2dk}{|f(k)|^2}\varphi(k,r)\int_0^r\bar{\varphi}(k,\rho)F(\rho)d\rho. \quad \text{(A2.36)}$$

If now $C=\bar{C}$, i.e., $f'(-i\kappa,0)=\bar{f}'(-i\kappa,0)$, addition of (A2.34) and (A2.36) yields

$$F(r)=C\varphi_0(r)\int_0^\infty\bar{\varphi}_0(\rho)F(\rho)d\rho$$
$$+\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{k^2dk}{|f(k)|^2}\varphi(k,r)\int_0^\infty\bar{\varphi}(k,\rho)F(\rho)d\rho. \quad \text{(A2.37)}$$

Since without the bar this is the usual completeness relation, one can now easily prove that, in fact, $\bar{\varphi}(k,r)=\varphi(k,r)$ and $\bar{\varphi}_0(r)=\varphi_0(r)$ and hence $\bar{V}(r)=V(r)$. This demonstration follows reference 5, p. 14 or reference 19, p. 39.

Thus to a given $\eta(k)$, κ, and C there corresponds, at most, one potential satisfying the conditions (A2.1) (A2.2). On the other hand, two potentials with the same $\eta(k)$ and κ but different C are, of course, necessarily different since their bound state wave functions are different. In the cases of the phase equivalent potentials of Bargmann (reference 4) the positive parameter C assumes all values between zero and infinity so that there cannot be any additional phase equivalent potentials with the same binding energies satisfying (A2.1) and (A2.2).[21]

Clearly in the general case of m bound states the potential is uniquely characterized by $\eta(k)$, κ_i and the m positive parameters C_i.

Note added in proof:—Since this paper was submitted, explicit expressions have been found for a family of potentials with the same phase shift and binding energies as those of a given potential. This family can be shown to be complete since the parameters C_i

[21] In case (b) reference 4, p. 491, one finds $f'(-i\kappa,0)=\sigma/(\beta-1)$, $1<\beta<\infty$; in case (c), $f'(-i\kappa,0)=\sigma/(1+\alpha)$, $-1<\alpha<\infty$. In both cases $\sigma>0$.

of Appendix 2 take on all values $0<C_i<\infty$ [paper submitted to The Physical Review; see also B. Holmberg, Nuovo cimento **9**, 597 (1952)].

Further, an important paper by I. M. Gel'fand and B. M. Levitan [Doklady Akad. Nauk. S.S.S.R. n Ser. **77**, 557 (1951)] has come to our attention. By adapting their methods to the present case the problem of finding the potential corresponding to given $f(k)$ and constants C_i is reduced to the solution of linear integral equations. One introduces an auxiliary potential $V^{(1)}(\gamma)$ with corresponding solutions $\varphi^{(1)}(k,\gamma)$ and bound states at $E_i^{(1)}$. It's "spectral function" $\rho^{(1)}(E)$ is defined by

$$\frac{d\rho^{(1)}(E)}{dE}=\Sigma C_i^{(1)}\delta(E-E_i^{(1)}),\quad E<0,$$

$$\frac{d\rho^{(1)}(E)}{dE}=\frac{1}{\pi}\frac{\sqrt{E}}{|\rho^{(1)}(\sqrt{E})|^2},\qquad E\geqslant 0.$$

To find the potential $V(r)$ corresponding to a given $\rho(E)$ one evaluates

$$G(r,s)=\int d[\rho(E)-\rho^{(1)}(E)]\varphi^{(1)}(\sqrt{E},r)\varphi^{(1)}(\sqrt{E},S)$$

and solves the integral equations

$$K(r,s)+G(r,s)+\int_0^r K(r,t)G(s,t)dt=0$$

for positive r. Then

$$V(r)=V_1(r)+2\frac{dK(r,r)}{dr}.$$

It is a simple consequence of this theory that for a given phase shift, the position of the bound states is completely arbitrary.

A paper on applications of the Gel'fand Levitan theory to scattering problems is in preparation.

PHYSICAL REVIEW VOLUME 87, NUMBER 6 SEPTEMBER 15, 1952

Excitation Cross Section for Helium Atoms*

SAUL ALTSHULER
Department of Physics, Iowa State College, Ames, Iowa
(Received February 28, 1952)

The excitation scattering of electrons from helium atoms is investigated in order to examine quantitatively the errors introduced by the fact that the description of the helium target is only approximately known. The cross section is calculated in Born approximation using formally equivalent matrix elements which weight the wave functions differently in space. Similar shapes for angular distribution and total cross section *vs* energy curves are obtained although absolute values differ.

SINCE exact wave functions for complex atoms are not available, there is an uncertainty introduced in scattering calculations which is distinct from those inherent in the Born approximation. The approximate wave functions, such as those of Slater, usually are well determined with respect to the energy of the state and are less well determined in regions other than those which contribute the most to the energy. For scattering problems such errors may be quite serious. A similar situation occurs in the calculation of optical transition probabilities, and investigations have been reported.[1] This difficulty has never been quantitatively examined for collision problems.

To make this study, we follow the suggestion, advanced by Bates, Fundaminsky, and Massey,[2] of employing two formally equivalent expressions for the differential cross section, both of which are within the Born approximation. These expressions are not necessarily the same in actual calculation since they weight the various regions of coordinate space differently.

The cross section for a momentum change dK for excitation of an atom from state p to state q is given in Born approximation[3] by

$$I(K)dK=\frac{8\pi dK}{k_p{}^2K^3}\left|\sum_i^n\int e^{-iKz_i}\psi_p\psi_q{}^*d\tau_1d\tau_2\cdots d\tau_n\right|^2, \quad (1)$$

where the summation is made over the atomic electrons, and where $\mathbf{K}=\mathbf{k}_q-\mathbf{k}_p$, k_p=wave number of the incident electron, k_q=wave number of the scattered electron, $k_q{}^2=k_p{}^2-2\Delta E$, and $\Delta E=E_q-E_p$, the internal energy change of the target; in all of these symbols Hartree units have been used.

The summation over the atomic electrons is made readily if product type wave functions are used. For the ground state of helium

$$\psi_p=\psi_0(N|r_1)\psi_0(N|r_2),\qquad (N=1.687)$$

while for an excited state—other than an S state—the Eckhart approximation[2] to the wave function is used, i.e.,

$$\psi_q=2^{-\frac{1}{2}}\{\psi_0(2|r_1)\psi_{nlm}(1|r_2)+\psi_0(2|r_2)\psi_{nlm}(1|r_1)\},$$

where $\psi_{nlm}(N|r)$ is the wave function of a single electron in the nlm state moving in a field of charge N.

* The research reported in this paper has been sponsored by the Geophysical Research Division of the Air Force Cambridge Research Center.

[1] D. R. Bates and A. Damgaard, Trans. Roy. Soc. (London) **A242**, 101 (1949); S. Chandrasekhar, Astrophys. J. **102**, 223 (1945).

[2] Bates, Fundaminsky, and Massey, Trans. Roy. Soc. (London) **A243**, 93 (1950).

[3] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Clarendon Press, Oxford, England, 1949), second edition, pp. 226 ff.