Electron Excitation of Nuclei*

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The cross section for an electric 2^l-pole transition produced by the electromagnetic field of fast electrons is derived in the Born approximation. Electric dipole-quadrupole interference is also obtained. Deuteron electrodisintegration is further developed, the theory now extending to 150-Mev energy transfers. Characteristic features of electron excitation experiments are mentioned, in particular the possibilities of obtaining information about nuclei and of observing the scattered electrons.

I. INTRODUCTION

NUCLEI bombarded with electrons can be excited by virtue of the interaction of the electron's electromagnetic field with the nucleons. Dipole cross sections have been calculated by Bethe and Peierls,[1] and Peters and Richman,[2] while Wick,[3] and Sneddon and Touscheck[4] also give electric quadrupole. Mullin and Guth[5] obtained the Coulomb correction to the Born approximation for nonrelativistic positive particles. Smith[6] was able to treat all electric poles at once by considering large electron momentum changes. Amaldi *et al.*[7] likewise were able to treat all pole transitions by assuming wave functions for light nuclei.

However, since transitions higher than electric quadrupole could be observed, especially at high energies, these are calculated here. Furthermore, even at moderate energies, dipole-quadrupole interference can be noticed. It is also felt that a discussion regarding the use of electrons in obtaining information about nuclei is worthwhile.

While electrons are treated relativistically, nucleons are taken to be nonrelativistic, terms of order v^2/c^2 being neglected. Nuclear recoil is of order v^2/c^2, as is the 2^{l+1}th pole cross section if the 2^lth is the first nonvanishing one. For the same reason the correction for the finite size of a nucleon can be neglected.[8] Thus the theory is valid for electron energies which are quite high and can suffer losses of the order of a couple hundred Mev, at which point one must begin to take into account finite size effects.

Electric transitions can be derived without reference to any nuclear force theory. Sachs and Austern[9] show that nuclear interaction effects need not be considered in calculations of electric transitions. Unfortunately, however, this is not the case for magnetic transitions.

The Born approximation used is generally valid for relativistic electrons. The nonrelativistic correction factor given by Mullin and Guth[5] can be used approximately here, remembering that their n is now negative. However, for high Z the lack of an accurate Coulomb correction can be serious.

It might be mentioned that μ-mesons can also electromagnetically excite nuclei. Thus the present theory can be applied with merely a change in mass.

II. EXCITATION CROSS SECTIONS

The matrix element for a transition from a nuclear state a and electron state i to a final state bf is $(bf|H'|ai)$, where

$$(f|H'|i)=\sum_n[-(\mathbf{A}_n\cdot\dot{\mathbf{r}}_n+\dot{\mathbf{r}}_n\cdot\mathbf{A}_n)q_n/2c \\ +q_n\phi_n-\mu_n\boldsymbol{\sigma}_n\cdot(\nabla\times\mathbf{A}_n)] \quad (1)$$

is the interaction of the Møller[10] potentials, $\mathbf{A}_n$ and ϕ_n, at the nth nucleon with velocity $\dot{\mathbf{r}}_n$, charge q_n, and magnetic moment μ_n.

$$\mathbf{A}_n=4\pi e\mathbf{a}(K^2-\omega^2/c^2)^{-1}\exp(i\mathbf{K}\cdot\mathbf{r}_n), \\ \phi_n=-4\pi e a_0(K^2-\omega^2/c^2)^{-1}\exp(i\mathbf{K}\cdot\mathbf{r}_n). \quad (2)$$

Here $\hbar\mathbf{K}$ and $\hbar\omega$ are momentum and energy losses of the electron, while $\mathbf{a}$ and a_0 are the relativistic spin matrix elements of the Dirac $\boldsymbol{\alpha}$ and 1, respectively.

By expanding the exponential in a Taylor series, the matrix element can be separated into terms of decreasing orders of magnitude. The electric and magnetic parts of the velocity interaction are separated by using the following relations:

$$i\omega(b|(\mathbf{K}\cdot\mathbf{r}_n)^{l-1}\mathbf{a}\cdot\mathbf{r}_n|a) \\ =\tfrac{1}{2}(b|\{\mathcal{P}_{l-1}\mathbf{K}\cdot\dot{\mathbf{r}}_n(\mathbf{K}\cdot\mathbf{r}_n)^{l-2}\}\mathbf{a}\cdot\mathbf{r}_n \\ +(\mathbf{K}\cdot\mathbf{r}_n)^{l-1}\mathbf{a}\cdot\dot{\mathbf{r}}_n+\mathbf{a}\cdot\dot{\mathbf{r}}_n(\mathbf{K}\cdot\mathbf{r}_n)^{l-1} \\ +\mathbf{a}\cdot\mathbf{r}_n\mathcal{P}_{l-1}\mathbf{K}\cdot\dot{\mathbf{r}}_n(\mathbf{K}\cdot\mathbf{r}_n)^{l-2}|a), \quad (3)$$

$$(\mathbf{K}\times\mathbf{a})\cdot(\mathbf{r}_n\times\dot{\mathbf{r}}_n)=(\mathbf{K}\cdot\mathbf{r}_n)\mathbf{a}\cdot\dot{\mathbf{r}}_n-(\mathbf{a}\cdot\mathbf{r}_n)\mathbf{K}\cdot\dot{\mathbf{r}}_n, \quad (4)$$

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[1] H. A. Bethe and R. Peierls, Proc. Roy. Soc. (London) **A148**, 146 (1935).

[2] B. Peters and C. Richman, Phys. Rev. **59**, 804 (1941).

[3] G. C. Wick, Ricerca sci. **11**, 49 (1940).

[4] I. N. Sneddon and B. F. Touscheck, Proc. Roy. Soc. (London) **A193**, 344 (1948). Our results agree with Wick.

[5] C. J. Mullin and E. Guth, Phys. Rev. **82**, 141 (1951).

[6] J. Smith, thesis, Cornell University (1951).

[7] Amaldi, Fidecaro, and Mariani, Nuovo cimento **7**, 757 (1950).

[8] This is of course permitted only for small momentum transfers. However, the contribution of large momentum transfers to the total cross section is generally quite small.

[9] R. G. Sachs and N. Austern, Phys. Rev. **81**, 705 (1951).

[10] C. Møller, Z. Physik **70**, 786 (1931); Ann. Phys. **14**, 531 (1932).

$$\mathcal{P}_{l-1}[\mathbf{K}\cdot\mathbf{r}_n]^{l-2}[(\mathbf{K}\cdot\mathbf{r}_n)\mathbf{a}\cdot\dot{\mathbf{r}}_n]+\mathcal{P}_{l-1}[\mathbf{K}\cdot\mathbf{r}_n]^{l-2}[(\mathbf{a}\cdot\dot{\mathbf{r}}_n)\mathbf{K}\cdot\mathbf{r}_n]=(l-1)[(\mathbf{K}\cdot\mathbf{r}_n)^{l-1}\mathbf{a}\cdot\dot{\mathbf{r}}_n+\mathbf{a}\cdot\dot{\mathbf{r}}_n(\mathbf{K}\cdot\mathbf{r}_n)^{l-1}],\quad(5)$$

where $\mathcal{P}_{l-1}$ indicates the sum of $l-1$ permutations of $l-1$ factors. Hence

$$(b|\mathbf{a}\cdot\mathbf{r}_n\exp(i\mathbf{K}\cdot\mathbf{r}_n)+[\exp(i\mathbf{K}\cdot\mathbf{r}_n)]\mathbf{a}\cdot\dot{\mathbf{r}}_n|a)=2\sum_{l=1}^{\infty}(b|(i^l/l!)\omega(\mathbf{K}\cdot\mathbf{r}_n)^{l-1}\mathbf{a}\cdot\mathbf{r}_n|a)+2\sum_{l=2}^{\infty}(b|(i^{l-1}/l!)\mathcal{P}_{l-1}[\mathbf{K}\cdot\mathbf{r}_n]^{l-2}[(\mathbf{K}\times\mathbf{a})\cdot(\mathbf{r}_n\times\dot{\mathbf{r}}_n)]|a).\quad(6)$$

The differential cross section is given by[11]

$$\frac{d\sigma}{d\Omega}=\frac{2\pi}{\hbar}\frac{E_i}{c^2\hbar k_i}\frac{1}{2}\sum_i^{+}\sum_f^{+}\frac{1}{2j+1}\sum_a\sum_b|(bf|H'|ai)|^2\times\frac{k_fE_f}{8\pi^3c^2\hbar^2},\quad(7)$$

where $(\hbar\mathbf{k}_i, E_i/c)$ and $(\hbar\mathbf{k}_f, E_f/c)$ are initial and final electron 4-momenta, $2j+1$ is the initial nuclear degeneracy, and the spin sums are over positive energies only. Thus from (1), (2), (6), and (7) the electric 2^l-pole and magnetic 2^{l-1}-pole differential cross sections are respectively

$$\frac{d\sigma_l}{d\Omega}=\frac{e^4E_iE_fk_f}{2(mc^2Q)^2k_i(2j+1)}\sum_i^{+}\sum_f^{+}\sum_a\sum_b\left|\frac{i^l}{l!}(b|\mathbf{Q}^{(l)}|a)\cdot\left[\mathbf{K}^{l-1}\left(a_0\mathbf{K}+\frac{\omega}{c}\mathbf{a}\right)\right]\right|^2,\quad(8)$$

$$\frac{d\sigma_{l-1}^{(m)}}{d\Omega}=\frac{e^2\mu_0^2E_iE_fk_f}{2(mc^2Q)^2k_i(2j+1)}\sum_i^{+}\sum_f^{+}\sum_a\sum_b\times\left|\frac{2(l-1)i^{l-1}}{l!}(b|\mathbf{M}^{(l-1)}|a)\cdot[\mathbf{K}^{l-2}(\mathbf{K}\times\mathbf{a})]\right|^2,\quad(9)$$

where[12]

$$Q=\hbar^2K^2/2m-\hbar^2\omega^2/2mc^2,\quad\mathbf{Q}^{(l)}=\textstyle\sum_n(q_n/e)(\mathbf{r}_n)^l,$$

$$\mathbf{M}^{(l-1)}=\frac{1}{\mu_0(l-1)}\sum_n\mathcal{P}_{l-1}\mathbf{r}_n{}^{l-2}\left(\frac{q_n}{2c}\mathbf{r}_n\times\dot{\mathbf{r}}_n+\frac{l}{2}\mu_n\boldsymbol{\sigma}_n\right),$$

$$\mu_0=e\hbar/2Mc.$$

However, only the electric cross section is unambiguous in view of the work of Sachs and Austern,[9] and in general one cannot be confident in Eq. (9).

Performing the spin sums by trace theorems, one obtains

$$\frac{d\sigma_l}{d\Omega}=\frac{e^4k_f\sum_a\sum_b}{2(l!mc^2Q)^2k_i(2j+1)}\{|(b|\mathbf{Q}^{(l)}|a)\cdot\mathbf{K}^l|^2(2E_iE_f-mc^2Q)+m\omega^2Q[(b|\mathbf{Q}^{(l)}|a)\cdot\mathbf{K}^{l-1}]^*\cdot[(b|\mathbf{Q}^{(l)}|a)\cdot\mathbf{K}^{l-1}]+2\hbar\omega E_i|(b|\mathbf{Q}^{(l)}|a)\cdot(\mathbf{K}^{l-1}\mathbf{k}_f)|^2-2\hbar\omega E_f|(b|\mathbf{Q}^{(l)}|a)\cdot(\mathbf{K}^{l-1}\mathbf{k}_i)|^2\}.\quad(10)$$

By means of the theorems of the Appendix, Eq. (10) is averaged over the random orientations of these transition moments:

$$\frac{d\sigma_l}{d\Omega}=\frac{8\pi l!^2e^4k_f(2K)^{2l-2}P_l^2}{(2l+1)!^2k_i(mc^2Q)^2}\left\{(2E_iE_f-mc^2Q)K^2+\frac{2l+1}{l}m\omega^2Q+\frac{8\pi\hbar\omega}{l^2(2l-1)}\sum_{m=1-l}^{l-1}(l^2-m^2)\times[k_f^2E_i|\mathcal{Y}_{l-1,m}(\alpha_f,\beta_f)|^2-k_i^2E_f|\mathcal{Y}_{l-1,m}(\alpha_i,\beta_i)|^2]\right\},\quad(11)$$

where α_i, β_i and α_f, β_f are the polar angles describing $\mathbf{k}_i$ and $\mathbf{k}_f$ with $\mathbf{K}$ as the polar axis, and where (23) defines P_l.

In view of the similarity between the above and the photoeffect it was possible to obtain the photoelectric cross section $\sigma_l^{(\gamma)}$ by using the above methods. The result is[13]

$$\sigma_l^{(\gamma)}=2^{2l+3}\pi^3(l+1)!(l-1)!(2l+1)!^{-2}\times\alpha\hbar\omega(\omega/c)^{2l-2}P_l^2,\quad(12)$$

α being the fine structure constant. Division of (11) by (12) gives $dN_l/d\Omega$, the number of virtual quanta per unit energy per unit solid angle for the electric 2^l-pole transition, a purely electrodynamical quantity. Integration over $d\Omega$ gives

$$\sigma_l=\int(dN_l/d\Omega)d\Omega\sigma_l^{(\gamma)}=N_l\sigma_l^{(\gamma)},\quad(13)$$

$$N_1=\frac{\alpha}{\pi\hbar\omega}\left[-2\frac{k_f}{k_i}+\frac{E_i^2+E_f^2}{(c\hbar k_i)^2}\ln\xi\right],\quad(14)$$

$$N_2=\frac{\alpha}{\pi\hbar\omega}\left[\frac{8k_f}{3k_i}\left(\frac{m^2c^4+E_iE_f}{(\hbar\omega)^2}\right)+\frac{E_i^2+E_f^2-2m^2c^4}{(c\hbar k_i)^2}\ln\xi\right],\quad(15)$$

[11] Unless otherwise stated, formulas refer to excitation to a discrete level where b is normalized to unity. However, these are readily adapted to disintegrations by normalizing b to unit energy and replacing σ by $d\sigma/dE_f$ or by normalizing b to unit energy and unit solid angle Ω_n and replacing σ by $d^2\sigma/dE_fd\Omega_n$.

[12] The notation used here is such that vectors in corresponding positions from the left edges of the two generalized dyadics separated by a dot have their scalar product formed. For example, $(\mathbf{qrs})\cdot(\mathbf{uvw})=\mathbf{q}\cdot\mathbf{u}(\mathbf{r}\cdot\mathbf{v})\mathbf{s}\cdot\mathbf{w}$; $(\mathbf{u}\cdot\mathbf{r}^2)\cdot\mathbf{s}=\mathbf{u}\cdot\mathbf{r}(\mathbf{r}\cdot\mathbf{s})$; $(\mathbf{r}^n\cdot\mathbf{u}^{n-1})\cdot(\mathbf{s}^n\cdot\mathbf{v}^{n-1})=(\mathbf{r}\cdot\mathbf{u})^{n-1}\mathbf{r}\cdot\mathbf{s}(\mathbf{s}\cdot\mathbf{v})^{n-1}$. Also $\mathcal{P}_3(\mathbf{r}^2\mathbf{s})\cdot(\mathbf{u}^2\mathbf{v})=(\mathbf{r}\cdot\mathbf{u})^2\mathbf{s}\cdot\mathbf{v}+\mathbf{s}\cdot\mathbf{v}(\mathbf{r}\cdot\mathbf{u})^2+\mathbf{r}\cdot\mathbf{u}(\mathbf{s}\cdot\mathbf{v})\mathbf{r}\cdot\mathbf{u}$.

[13] Note that for excitations to discrete levels $\sigma^{(\gamma)}$ has dimensions area times energy here.

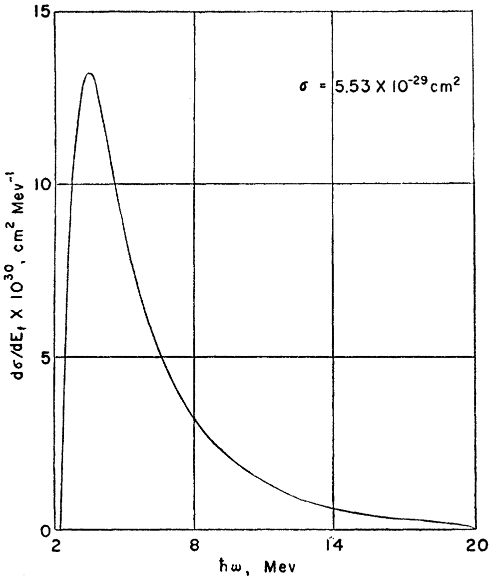

FIG. 1. Energy distribution of scattered electrons (or the outgoing nucleons) from the electric dipole disintegration of the deuteron by electrons with $E_i - mc^2 = 20$ Mev.

$$N_3=\frac{\alpha}{\pi\hbar\omega}\left[\frac{k_f}{k_i}\left(2+\frac{4E_i^2E_f^2+4E_iE_fm^2c^4-8m^4c^8}{(\hbar\omega)^4}+\frac{9E_iE_f-m^2c^4}{(\hbar\omega)^2}\right)+\frac{E_i^2+E_f^2-4m^2c^4}{(c\hbar k_i)^2}\ln\xi\right], \quad (16)$$

$$N_1^{(m)}=\frac{\alpha}{\pi\hbar\omega}\left[\frac{E_i^2+E_f^2-2m^2c^4}{(c\hbar k_i)^2}\ln\xi\right], \quad (17)$$

where

$$\xi=(E_iE_f+c^2\hbar^2k_ik_f-m^2c^4)/mc^2\hbar\omega.$$

The magnetic expression (17) is given, as it is quite likely that it can be used in a number of cases, although perhaps with limited validity.

For disintegrating nuclei a preferred nuclear moment orientation exists and the averaging of the Appendix may be omitted. By normalizing b to unit energy and unit solid angle Ω_n, one obtains (10) with $d\sigma_l/d\Omega$ replaced by $d^3\sigma_l/d\Omega dE_f d\Omega_n$. Interference between two adjacent multipole transitions is of order $(\hbar\omega/Mc^2)^{\frac{1}{2}}$ times the lower order transition. In particular the electric dipole-quadrupole interference is

$$\frac{d^3\sigma_{1\sim2}}{d\Omega dE_f d\Omega_n}=\frac{e^4k_f}{2(mc^2Q)^2k_i(2j+1)}\sum_a\sum_b$$
$$\times\mathrm{Re}\{[\mathbf{K}^3(2E_iE_f-mc^2Q)+\hbar^2\omega^2\mathbf{K}(\mathbf{k}_i\mathbf{k}_f+\mathbf{k}_f\mathbf{k}_i)$$
$$-\mathbf{K}(E_f\mathbf{k}_i+E_i\mathbf{k}_f)\hbar\omega\mathbf{K}-\hbar\omega\mathbf{K}^2(E_f\mathbf{k}_i+E_i\mathbf{k}_f)]$$
$$\cdot[(b|i\mathbf{Q}^{(2)}|a)^*(b|\mathbf{Q}^{(1)}|a)]$$
$$+m\omega^2Q[(b|i\mathbf{K}\cdot\mathbf{Q}^{(2)}|a)^*]\cdot(b|\mathbf{Q}^{(1)}|a)\}. \quad (18)$$

III. DEUTERON DISINTEGRATION

To obtain explicit cross-section values it is necessary either to evaluate P_l, which entails a knowledge of nuclear wave functions, or to infer its value from photoexcitation experiments. In the case of the deuteron, P_1 has been calculated.[14] The range parameter in the theory was taken to be such that the theoretical electric dipole cross section agrees with photo experiments[15] at 3, 6, and 18 Mev.

Equations (13) and (14) give the energy spectrum of the scattered electrons or ejected nucleons for electric dipole disintegrations. Figure 1 shows this and is also quite typical of what to expect at other E_i's. The peak occurs just below $\sigma_1^{(\gamma)}$'s maximum. It might be mentioned that Fig. 1 resembles quite closely the photonucleon yield had from the bremsstrahlung spectrum of 20-Mev electrons.

Equations (10), (17), and (18) should represent the deuteron's electrodisintegration up to energy transfers of the order of 150 Mev.[16] Discrepancies between existing theory[14] and 80- to 150-Mev photodisintegration experiments[17] would also be observable in electrodisintegration.

IV. DISCUSSION

N_1, like the bremsstrahlung spectrum, can be taken as $C(E_i)/\hbar\omega$ for approximate calculations, where $C(E_i)$ is taken to be independent of $\hbar\omega$. Thus the method of Levinger and Bethe[18] can be utilized to obtain the cross section for excitation of all electric dipole levels (which of course requires a large E_i)

$$\sigma_1\approx C(E_i)\int\frac{\sigma_1^{(\gamma)}}{\hbar\omega}d(\hbar\omega)\approx\frac{C(E_i)}{(\hbar\omega)_H}\int\sigma_1^{(\gamma)}d(\hbar\omega), \quad (19)$$

where both the harmonic mean absorption energy, $(\hbar\omega)_H$, and $\int\sigma_1^{(\gamma)}d(\hbar\omega)$ are discussed by these authors. For the deuteron with a Yukawa half-exchange potential one finds $\sigma_1\approx 3.86C(E_i)$ mb, which is a fair approximation even at 20 Mev; using $C(E_i, (\hbar\omega)_H) = 0.0117$ gives $\sigma_1\approx 0.0452$ mb as compared with 0.0553 mb in Fig. 1.

Two chief methods of observing electron excitation of nuclei are detecting the decay product (1) of the excited state itself and (2) of a state to which the ex-

[14] L. I. Schiff, Phys. Rev. **78**, 733 (1950); J. F. Marshall and E. Guth, Phys. Rev. **78**, 738 (1950).
[15] Russell, Sachs, Wattenberg, and Fields, Phys. Rev. **73**, 545 (1948); Wilson, Collie, and Halban, Nature **163**, 245 (1949); Snell, Barker, and Sternberg, Phys. Rev. **75**, 1290 (1949); Barnes, Stanford, and Wilkinson, Nature **165**, 69 (1950).
[16] The integrated forms of (10) and (18), i.e., $d^2\sigma_l/dE_fd\Omega_n$ and $d^2\sigma_{1\sim2}/dE_fd\Omega_n$, also exist. See Thie, Mullin, and Guth (to be published).
[17] T. S. Benedict and W. M. Woodward, Phys. Rev. **85**, 924 (1952); W. S. Gilbert and J. W. Rose, Phys. Rev. **85**, 766 (1952).
[18] J. S. Levinger and H. A. Bethe, Phys. Rev. **78**, 115 (1950).

cited state decays. A third possible method might be the observation of the inelastically scattered electrons.

In order to observe the latter it is necessary first to filter out unwanted electron energies. Then for nuclei excited to a discrete level there results an inelastic peak superimposed on a background of electrons which have radiated a quantum. To observe this peak for typical low Z nuclei at moderate energies, the filter must have a band width of the order of 0.001 percent of the energy being observed. However, for disintegrating nuclei the electrons can readily be detected by coincidence counting them with the decay product. Additional information can be had in some cases by this procedure.

Regarding methods (1) and (2), two upper limits on the allowable target thickness are had from the decay product's range and from photoexcitations due to γ-rays produced in the target.[19] The ratio of photon to electron excitations in a target having nt nuclei per unit area is

$$\frac{nt}{2\cos\bar{\theta}}\frac{d\sigma^{(b)}/dE_f}{N},$$

where $\bar{\theta}$ is an average multiple scattering angle and $d\sigma^{(b)}/dE_f$ is the bremsstrahlung cross section per unit energy. In view of the similarity of the latter with N_1 for electric dipole transitions the above ratio is of order $0.0002(Z^2/A)t'/\cos\bar{\theta}$ with t' being in mg/cm.

Multiplying these expressions by 2 and evaluating them instead for the γ-ray production target gives one the relative photon and electron *thin* target yields for the nucleus under investigation, i.e., yields with and without the γ-ray target intercepting the beam (assuming all γ-rays are forward). This ratio of photoexcitations to electroexcitations will determine $(d\sigma^{(b)}/dE_f)/N$ which on comparison with theoretical ratios can give information about the type of transition. However, for high Z in view of the Coulomb correction needed in both numerator and denominator, perhaps some known transition must be used as a normalizer.

APPENDIX

To average $|(b|\mathbf{u}\cdot\mathbf{r}(\mathbf{v}\cdot\mathbf{r})^{l-1}|a)|^2$, where $\mathbf{u}$ and $\mathbf{v}$ are arbitrary vectors over the randomness in nuclear moment orientations and over the axial symmetry of $\mathbf{v}$ with respect to $\mathbf{u}$, one makes the decomposition[20]

$$ur\cos\theta(\mathbf{v}\cdot\mathbf{r})^{l-1}=4\pi uv^{l-1}r^l\sum_{n=0}^{l-1}a_{l-1,n}\sum_{m=-n}^{n}\mathcal{Y}_{n,m}^*(\alpha,\beta)\times\left[\left(\frac{(n+m+1)(n-m+1)}{(2n+1)(2n+3)}\right)^{\frac{1}{2}}\mathcal{Y}_{n+1,m}(\theta,\phi)+\left(\frac{(n+m)(n-m)}{(2n+1)(2n-1)}\right)^{\frac{1}{2}}\mathcal{Y}_{n-1,m}(\theta,\phi)\right],\quad(20)$$

where $a_{l-1,n}$ is nonvanishing only for even $l-1-n$,

$$a_{l-1,n}=\frac{2^{l-1}(l-1)![(l-1+n)/2]!}{[(l-1-n)/2]!(l+n)!},$$

and where $\mathbf{u}$ is the polar axis of the θ, ϕ describing $\mathbf{r}$, and α, β describing $\mathbf{v}$.

Only the $n=l-1$ term gives a contribution to the electric 2^l-pole transition as defined here. Consequently,

$$\begin{aligned}|(b|\mathbf{u}\cdot\mathbf{r}(\mathbf{v}\cdot\mathbf{r})^{l-1}|a)|^2&\frac{(2l-1)!^2}{16\pi^2u^2(2v)^{2l-2}(l-1)!^4}\\&=\left|\sum_{m=1-l}^{l-1}\left(\frac{(l+m)(l-m)}{(2l-1)(2l+1)}\right)^{\frac{1}{2}}\mathcal{Y}_{l-1,m}^*(\alpha,\beta)\times(b|r^l\mathcal{Y}_{l,m}|a)\right|^2\\&=\frac{1}{2l+1}\left[\sum_{m=-l}^{l}|(b|r^l\mathcal{Y}_{l,m}|a)|^2\right]\times\left|\sum_{m=1-l}^{l-1}\left(\frac{(l+m)(l-m)}{(2l-1)(2l+1)}\right)^{\frac{1}{2}}\mathcal{Y}_{l-1,m}^*(\alpha,\beta)\right|^2\\&=\frac{1}{2l+1}\left[\sum_{m=-l}^{l}|(b|r^l\mathcal{Y}_{l,m}|a)|^2\right]\times\sum_{m=1-l}^{l-1}\frac{(l+m)(l-m)}{(2l+1)(2l-1)}|\mathcal{Y}_{l-1,m}(\alpha,\beta)|^2.\quad(21)\end{aligned}$$

Generalizing,

$$\sum_a\sum_b|(b|\mathbf{Q}^{(l)}|a)\cdot(\mathbf{v}^{l-1}\mathbf{u})|^2/(2j+1)=\frac{64\pi^2u^2(2v)^{2l-2}l!^4}{l^2(2l+1)!^2(2l-1)}P_l^2\sum_{m=1-l}^{l-1}(l^2-m^2)\times|\mathcal{Y}_{l-1,m}(\alpha,\beta)|^2,\quad(22)$$

where

$$P_l^2=\sum_{m=-l}^{l}|(b|\sum_n(q_n/e)r_n{}^l\mathcal{Y}_{l,m}|a)|^2/(2j+1)\quad(23)$$

is the definition of the electric 2^l-pole transition moment. Similarly one can show that

$$\sum_a\sum_b[(b|\mathbf{Q}^{(l)}|a)^*\cdot\mathbf{v}^{l-1}]\cdot[(b|\mathbf{Q}^{(l)}|a)\cdot\mathbf{v}^{l-1}]/(2j+1)=\frac{16\pi(2v)^{2l-2}}{l(2l)!^2(2l+1)}P_l^2.\quad(24)$$

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[19] Skaggs, Laughlin, Hanson, and Orlin, Phys. Rev. **73**, 420 (1948); J. S. Blair, Phys. Rev. **75**, 907 (1949). To the authors' knowledge perhaps the only experiment with a sufficiently thin target for pure electrodisintegration was the disintegration of Be^7 by M. Wiedenbeck, Phys. Rev. **69**, 236 (1945).

[20] The method is similar to that of C. J. Mullin and E. Guth, reference 5, and D. L. Falkoff, thesis, University of Michigan (1948).