value of the energy from all the 47 plates 9.48±0.15 Mev.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the encouragement and guidance of Professors E. M. McMillan, A. C. Helmholz, and J. G. Hamilton during the course of this work. Sincere thanks are also due Dr. Louis Rosen, for making this experiment possible by the use of the multiplate camera; Dr. J. C. Allred and Dr. R. O. Bondelid for their assistance in making the runs; Dr. Alice H. Armstrong for many helpful discussions.

The reading of the photographic plates was done in part at the Microscopy Laboratory at Los Alamos, chiefly by Mrs. J. Gammel and Mrs. M. Downs. Mrs. R. Gibbs and Mrs. T. M. Putnam assisted in the final counting of the tracks.

Thanks are also due the members of the Crocker Laboratory staff who contributed in many ways towards the completion of this work.

PHYSICAL REVIEW VOLUME 87, NUMBER 6 SEPTEMBER 15, 1952

Interaction between Electron and One-Dimensional Electromagnetic Field

NATHAN ROSEN
Department of Physics, University of North Carolina, Chapel Hill, North Carolina
(Received May 14, 1952)

For a single electron interacting with the quantized transverse electromagnetic field it is found that, in the one-dimensional case, the Schrödinger equation can be put into a form like that of a system of coupled harmonic oscillators. From the classical frequencies of the normal modes of oscillation of such a system the quantal energy can be determined. While the perturbation method gives a logarithmic divergence in the interaction energy, one finds by the present method that the energy diverges like the square root of a logarithm.

A QUESTION which does not appear to have received a satisfactory answer is whether the divergences arising in the interaction between a point electron and the quantized electromagnetic field are inherent in the problem or are due to the methods of calculation employed. The purpose of this paper is to attempt to throw some light on this question by considering a simplified case in which the electron and the field are constrained to one-dimensional motion. This may be regarded as a continuation of earlier work on one-dimensional fields.[1]

Let us consider first the more general case of a single electron, described by the Dirac equation, interacting with a quantized electromagnetic radiation field. Thus we do not make use of the hole theory of the vacuum. Nor do we concern ourselves here with the interaction with the longitudinal part of the field; this will be regarded as already included in the mass term of the Dirac equation. If we think of the system as satisfying a periodicity condition in a fundamental cube of side L, we can expand the field in a set of monochromatic plane polarized waves of proper frequencies with quantized amplitudes.[2] Let us label these modes of oscillation of the field by an integer λ. The Hamiltonian for the system can then be written

$$H=c\boldsymbol{\alpha}\cdot\mathbf{p}+mc^2\beta-e\boldsymbol{\alpha}\cdot\textstyle\sum_\lambda \mathbf{A}_\lambda+H_0, \tag{1}$$

where $\mathbf{p}$ is the electron momentum, H_0 is the Hamiltonian of the field,

$$H_0=\tfrac{1}{2}\textstyle\sum_\lambda(P_\lambda^2+\omega_\lambda^2Q_\lambda^2-\hbar\omega_\lambda), \tag{2}$$

and

$$\mathbf{A}_\lambda=c\pi^{\frac{1}{2}}L^{-\frac{3}{2}}\boldsymbol{\varepsilon}_\lambda[Q_\lambda(e^{i\mathbf{k}_\lambda\cdot\mathbf{r}}+e^{-i\mathbf{k}_\lambda\cdot\mathbf{r}}) +(i/\omega_\lambda)P_\lambda(e^{i\mathbf{k}_\lambda\cdot\mathbf{r}}-e^{-i\mathbf{k}_\lambda\cdot\mathbf{r}})] \tag{3}$$

is the vector potential for the degree of freedom labeled by λ at the position of the electron specified by the vector $\mathbf{r}$. Here $\mathbf{k}_\lambda$ is $2\pi/L$ times a vector with integer components (positive and negative) depending on λ, $\omega_\lambda=c|\mathbf{k}_\lambda|$, and $\boldsymbol{\varepsilon}_\lambda$ is the polarization vector, orthogonal to $\mathbf{k}_\lambda$. The amplitudes Q_λ and P_λ are pairs of conjugate variables satisfying the usual commutation relations for coordinates and momenta. In H_0 the zero-point energy has been subtracted.

We now consider the Schrödinger equation for a stationary state of energy E,

$$H\psi=E\psi, \tag{4}$$

where the operator H is given by (1) with $\mathbf{p}=-i\hbar\nabla$, $P_\lambda=-i\hbar(\partial/\partial Q_\lambda)$.

To solve this equation let us write the wave function ψ as an expansion of the form

$$\psi=\textstyle\sum_{(n)} a_{n_1n_2\cdots n_\lambda\cdots}U_{n_1n_2\cdots n_\lambda\cdots} \times\exp[i(\mathbf{p}'/\hbar-n_1\mathbf{k}_1-n_2\mathbf{k}_2\cdots)\cdot\mathbf{r}]. \tag{5}$$

Here

$$U_{n_1n_2\cdots n_\lambda\cdots}=u_{n_1}(Q_1)u_{n_2}(Q_2)\cdots u_{n_\lambda}(Q_\lambda)\cdots, \tag{6}$$

[1] N. Rosen, Phys. Rev. **71**, 833 (1947), and **76**, 202 (1949); W. Sollfrey and G. Goertzel, Phys. Rev. **83**, 1038 (1951).

[2] See, e.g., W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition.

where $u_n(Q)$ is a harmonic oscillator wave function with quantum number $n(=0, 1, 2, \cdots)$ describing the state of the unperturbed mode of oscillation,

$$\tfrac{1}{2}(P^2+\omega^2Q^2-\hbar\omega)u_n=n\hbar\omega u_n; \tag{7}$$

$\mathbf{p}'$ is a constant vector, the total momentum of the system; and $a_{n_1n_2}\cdots n_\lambda\cdots$ is a constant spinor expansion coefficient. The summation $\sum_{(n)}$ is carried out over all the states of all the degrees of freedom. Substituting the expression (5) into Eq. (4) and making use of the relations

$$\begin{aligned}&[Q+(i/\omega)P]u_n=(2n\hbar/\omega)^{\frac{1}{2}}u_{n-1},\\ &[Q-(i/\omega)P]u_n=(2(n+1)\hbar/\omega)^{\frac{1}{2}}u_{n+1},\end{aligned} \tag{8}$$

one finds that with some rearrangement of terms one gets an equation which leads to the following relation among the expansion coefficients:

$$\begin{aligned}&[c\mathbf{p}'\cdot\boldsymbol{\alpha}+mc^2\beta+\hbar\textstyle\sum_\lambda n_\lambda(\omega_\lambda-c\mathbf{k}_\lambda\cdot\boldsymbol{\alpha})-E]a_{n_1n_2}\cdots n_\lambda\cdots\\ &\quad-K\textstyle\sum_\lambda\boldsymbol{\varepsilon}_\lambda\cdot\boldsymbol{\alpha}[(2(n_\lambda+1)\hbar/\omega_\lambda)^{\frac{1}{2}}a_{n_1n_2}\cdots(n_\lambda+1)\cdots\\ &\qquad+(2n_\lambda\hbar/\omega_\lambda)^{\frac{1}{2}}a_{n_1n_2}\cdots(n_\lambda-1)\cdots]=0,\end{aligned} \tag{9}$$

where

$$K=ce\pi^{\frac{1}{2}}L^{-\frac{3}{2}}. \tag{10}$$

Now define a function φ, by

$$\varphi=\textstyle\sum_{(n)}a_{n_1n_2}\cdots n_\lambda\cdots U_{n_1n_2}\cdots n_\lambda\cdots, \tag{11}$$

and an operator

$$\begin{aligned}H'=&c\mathbf{p}'\cdot\boldsymbol{\alpha}+mc^2\beta+\tfrac{1}{2}\textstyle\sum_\lambda(P_\lambda^2+\omega_\lambda^2Q_\lambda^2-\hbar\omega_\lambda)\\ &\times[1-(c/\omega_\lambda)\mathbf{k}_\lambda\cdot\boldsymbol{\alpha}]-2K\textstyle\sum_\lambda Q_\lambda\boldsymbol{\varepsilon}_\lambda\cdot\boldsymbol{\alpha},\end{aligned} \tag{12}$$

and consider the equation

$$H'\varphi=E\varphi. \tag{13}$$

A simple calculation shows that this also leads to Eq. (9) as the relation among the expansion coefficients. Thus we see that the energy levels associated with the Hamiltonian H' are the same as those of H. (One can also discuss the transformation from H to H' as a unitary transformation from a representation in which the electron coordinates are diagonal to one in which the total momentum is diagonal.) The obvious advantage of Eq. (12) is that the electron coordinates have been eliminated.

Since the solution of Eq. (13) for the general case still presents difficulties, let us now go over the one-dimensional problem. Let the vector $\mathbf{p}'$ have only a Z component which will be denoted simply by p, and let the vectors $\mathbf{k}_\lambda$ also be along the Z axis, the polarization vectors $\boldsymbol{\varepsilon}_\lambda$ being in the X and Y directions. Equation (13) can then be written

$$[cp\alpha_z+mc^2\beta+H_0(1-\alpha_z)-2K(Q_x\alpha_x+Q_y\alpha_y)]\varphi=E\varphi, \tag{14}$$

where Q_x and Q_y are the two parts of $\sum Q_\lambda$ for $\boldsymbol{\varepsilon}_\lambda$ in the X and Y directions, respectively.

Multiplying both sides of the equation by $(1+\alpha_z)$ or $(1-\alpha_z)$ and using the properties of the Dirac matrices, one obtains relations which can be written

$$\begin{aligned}&(E-cp)(1+\alpha_z)\varphi\\ &\qquad=[mc^2\beta-2K(Q_x\alpha_x+Q_y\alpha_y)](1-\alpha_z)\varphi,\\ &(E+cp-2H_0)(1-\alpha_z)\varphi\\ &\qquad=[mc^2\beta-2K(Q_x\alpha_x+Q_y\alpha_y)](1+\alpha_z)\varphi.\end{aligned} \tag{15}$$

Eliminating $(1+\alpha_z)\varphi$ between these equations and writing $\chi=(1-\alpha_z)\varphi$, one obtains

$$(E-cp)(E+cp-2H_0)\chi=[m^2c^4+4K^2(Q_x^2+Q_y^2)]\chi. \tag{16}$$

If we write

$$E'=\tfrac{1}{2}[E+cp-m^2c^4/(E-cp)],\quad \sigma=4K^2/(E-cp), \tag{17}$$

this equation takes the form

$$[H_0+\tfrac{1}{2}\sigma(Q_x^2+Q_y^2)]\chi=E'\chi, \tag{18}$$

which looks like the Schrödinger equation for a system consisting of two sets of coupled oscillators (the coupling being given by the products $Q_\lambda Q_{\lambda'}$, present in Q_x^2 and Q_y^2).

Let us investigate the classical frequencies of the normal modes of oscillation of such a system. It will be enough for us to consider only the part of the system corresponding to polarization in the X direction, since the other part will have the same frequency distribution. For this case, if we let W be the energy, then classically

$$W=\tfrac{1}{2}\textstyle\sum_n(P_n^2+\omega_n^2Q_n^2)+\tfrac{1}{2}\sigma\sum_{m,n}Q_mQ_n, \tag{19}$$

with

$$\omega_n=\gamma|n|,\quad \gamma=2\pi c/L,\quad n=\pm1,\pm2,\cdots,$$

and the frequencies of the system are the roots ω of the secular equation[3]

$$\begin{vmatrix}\sigma+\gamma^2-\omega^2 & \sigma & \sigma & \cdots\\ \sigma & \sigma+\gamma^2-\omega^2 & \sigma & \cdots\\ \sigma & \sigma & \sigma+4\gamma^2-\omega^2 & \cdots\\ \cdot & \cdot & \cdot & \cdot\end{vmatrix}=0, \tag{20}$$

where the diagonal elements occur in equal pairs. Denoting the determinant by $F(\omega^2)$, one finds that it can be written

$$\begin{aligned}F(\omega^2)=&[\prod_{n=1}^{\infty}\gamma^2n^2]^2(\gamma^2/\pi^2\omega^2)\sin^2(\pi\omega/\gamma)\\ &\times\{1+\sigma/\omega^2-(\pi\sigma/\gamma\omega)\cot(\pi\omega/\gamma)\},\end{aligned} \tag{21}$$

so that the roots of (20) are given by the roots of the equations

$$\sin(\pi\omega/\gamma)=0\quad(\omega\neq0), \tag{22a}$$

$$(\omega^2+\sigma)\sin(\pi\omega/\gamma)=(\pi\sigma\omega/\gamma)\cos(\pi\omega/\gamma). \tag{22b}$$

[3] E. T. Whittaker, *Analytical Dynamics* (Cambridge University Press, London, 1937, or Dover Publications, New York, 1944), fourth edition, p. 179.

It follows (because there are two waves traveling in opposite direction for each frequency) that half of the frequencies, those given by (22a), remain unchanged by the interaction. If we denote the roots of (22b) by ω_n' ($n=1, 2, 3, \cdots$), numbered in increasing order, then the quantal energy of the system in its lowest state is given by

$$E'=\hbar\sum_{n=1}^{\infty}(\omega_n'-\omega_n), \tag{23}$$

which has been doubled to take account of the two kinds of polarization.

The question now is whether the series in (23) converges. From (22b) one sees that for large values of ω,

$$\omega_n'-\omega_n\sim\sigma/n\gamma,$$

so that the series in (23) diverges logarithmically. Let us write

$$E'=(\hbar\sigma/\gamma)w, \tag{24}$$

where $w\sim\Sigma(1/n)$. From (17) one then obtains the relation

$$E^2-c^2p^2-m^2c^4=(4\hbar/\gamma)K^2w. \tag{25}$$

Although in the three-dimensional case K was taken as in Eq. (10), in the one-dimensional problem it is better to take

$$K=ce\pi^{\frac{1}{2}}L^{-\frac{1}{2}}S^{-\frac{1}{2}}, \tag{10a}$$

where we think of the electron and field as moving in a (long) cylinder of length L and cross-sectional area S. The right-hand member of (25) becomes $(2e^2\hbar c/S)w$ and one gets, for the positive energy state,

$$E\sim e(2\hbar c/S)^{\frac{1}{4}}w^{\frac{1}{4}}, \tag{26}$$

so that the energy diverges like the square root of a logarithm.

Thus we see that the interaction of the electron with the transverse field gives a divergence in the energy in the one-dimensional case. It is of some interest to note that, if one were using a perturbation calculation based on the assumption that the interaction between electron and field is weak, one would regard the right-hand member of (25) as small; and one would then get, taking $p=0$ for simplicity,

$$E\sim mc^2+(2\hbar K^2/\gamma mc^2)w=mc^2+(e^2\hbar/mcS)w, \tag{27}$$

which in view of the actual logarithmic divergence of w leads to a worse type of infinity than that of Eq. (26).

Although the three-dimensional case is much more complicated and involves features not present in the simple case discussed here, the result obtained in the latter suggests rather strongly that the transverse self-energy of the point electron in the three-dimensional case is inherently divergent.

In conclusion, I should like to express my indebtedness to Professor Wayne A. Bowers for helpful discussions.

PHYSICAL REVIEW VOLUME 87, NUMBER 6 SEPTEMBER 15, 1952

Capture of μ-Mesons in Heavy Elements*

J. W. KEUFFEL, F. B. HARRISON,† T. N. K. GODFREY, AND GEORGE T. REYNOLDS
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received May 5, 1952)

Measurements have been made of the mean lifetimes of μ-mesons in several heavy elements. Time lags between the arrival of a cosmic-ray meson in the target and its subsequent absorption—as signaled by the neutrons and gamma-rays following capture—are measured using large liquid scintillation counters and a chronotron timing circuit. The timing uncertainty is about 2×10^{-9} sec, and the counting rates are such that a mean life can be determined to an accuracy of 10 percent in about a week's run at sea level. A short-lived component of the decay curves was found and identified as due to neutrons from proton-induced stars. Errors from this effect were avoided. Our latest results for the mean lives, in mμsec, are

Fe 163±27, Hg 58±4, Cu 116±9, Pb 76±4, Sb 99±11, Bi 68±5.

The results are in agreement with the Wheeler Z_{eff}^4 law up through Cu but disagree by a factor 3 for the heavier elements. They are in reasonable agreement with Kennedy's calculations (see following paper). The difference between the mean lives for Hg and Pb is also qualitatively predicted by Kennedy on the basis of a shell model of the nucleus. Our results, together with Kennedy's calculation for Pb, allow us to conclude that the μ-meson-nucleon coupling constant has the same value, within about 25 percent, as recent values of the coupling constants of beta-decay and of the spontaneous disintegration of the μ-meson.

I. INTRODUCTION

IT is now well established that the capture of negative μ-mesons by atomic nuclei is characterized by (1) a weak meson-nucleon coupling and (2) the transfer of most of the meson rest energy following nuclear capture to a light neutral particle.

That the μ-meson interacts weakly with neutrons and protons was convincingly demonstrated by the classic experiment of Conversi, Pancini, and Piccioni.[1]

* Assisted by the joint program of the AEC and ONR.

† Now at Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

[1] Conversi, Pancini, and Piccioni, Phys. Rev. **71**, 209 (1947).