with

$$P_\mu = m v_\mu \tag{3}$$

for the electromagnetic field, and

$$dP_\mu/ds = -g\partial U/\partial x^\mu, \tag{4}$$

with

$$P_\mu = (m - gU)v_\mu \tag{5}$$

for the scalar field. Here m and v_μ are the mass and the four-velocity of the particle, e and g its coupling constants with the two fields, $G_{\mu\nu}$ is the electromagnetic field tensor and U the scalar field. $G_{\mu\nu}$ and U are to be taken as the total (external plus particle) field minus the particle's symmetric field.[1,5] If it is assumed that the field of the particle is wholly retarded, this equals the external field plus the radiation field; if it is assumed to be half-retarded, half-advanced, it is just the external field, and there is no radiation damping.[6] The following results are valid in either case.

It was found that the above equations of motion allow an additional conservation law, corresponding to the conservation law (4) of I,

$$\frac{dQ}{ds} + \frac{d}{ds}\int \frac{\partial N_\mu}{\partial x_\mu} dx^0 dx^1 dx^2 dx^3 = 0, \tag{6}$$

where

$$Q = P_\mu x^\mu - ms, \tag{7}$$

and P_μ is given by (3) and (5) respectively; N_μ is the vector defined by (I.6) and (I.10) respectively. A similar law exists for point charges which possess an intrinsic angular momentum, but no dipole moment. No law corresponding to the conservation law (5) of I exists.

Thus, the system of particle and field does not obey all the conservation laws obeyed by the field alone, for the equations considered. Conversely, if we require the system to possess all the conservation properties of the field, we will not obtain the above equations of motion, which are well verified for the electromagnetic case. Therefore, it appears that instead of basing the deduction of the equations of motion on the ambiguous information provided by the conservation laws for the field alone, it may be preferable to base it on other requirements. One possibility is provided by requiring the equations of the special theory of relativity to fit within the framework of general relativity.[7]

It is planned to publish a detailed account of this work shortly.

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The Coulomb Scattering of Deuterons*

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DEVIATIONS from Rutherford scattering may be expected in the elastic scattering of deuterons by nuclei because (1) the deuteron is a loose structure with non-coincident centers of mass and charge and it may disintegrate on scattering, and (2) deuterons which penetrate through the barrier will encounter the nucleus and may initiate nuclear processes. We consider in this note the effects due to (1).

For scattering by a point Coulomb field, a simple Born approximation calculation gives for the ratio of the scattering to Rutherford scattering:

$$\frac{d\sigma}{d\sigma_R} = \left|\int d\mathbf{s}\chi_0^2(\mathbf{s})\exp(\tfrac{1}{2}i\Delta\mathbf{k}\cdot\mathbf{s})\right|^2 = S(\Delta\mathbf{k}). \tag{1}$$

Here $\mathbf{s} = \mathbf{r}_p - \mathbf{r}_n$, $\chi_0(\mathbf{s})$ is the deuteron space wave function and $\hbar\Delta\mathbf{k}$ the momentum transfer. $S(\Delta\mathbf{k})$ is then the deuteron form factor. It has the value unity for zero degree scattering but depresses the large angle scattering. For example, using the usual Hulthén wave function,

$$\chi_0(\mathbf{s}) = N[e^{-\alpha s} - e^{-\beta s}]/s, \quad \text{where} \quad \beta/\alpha = 7, \tag{2}$$

we have $S(180°) = 0.30$, for 4-Mev deuterons.

The use of the Born approximation is satisfactory only for $n = Ze^2/\hbar v \ll 1$. We make, therefore, a more accurate calculation and demonstrate thereby that the deuteron structure plays no part in the Coulomb scattering when $n \gg 1$.

With $\mathbf{k}_1$ = incident deuteron wave-vector, $\mathbf{R} = \frac{1}{2}(\mathbf{r}_n + \mathbf{r}_p)$, $V(s)$ = $n-p$ potential, and ϵ = deuteron binding energy, we have

$$\left[\nabla_R^2 + k^2 - \frac{2kn}{R} + \nabla_s^2 + V(s) - \epsilon\right]\Psi(\mathbf{R}, \mathbf{s}) = 2kn\left[\frac{1}{|\mathbf{R} - \frac{1}{2}\mathbf{s}|} - \frac{1}{R}\right]\Psi(\mathbf{R}, \mathbf{s}). \tag{3}$$

We expand Ψ on the left-hand side by $\Psi = \Sigma\chi_n(\mathbf{s})\Omega_n(\mathbf{R})$, where the $\chi_n(\mathbf{s})$ are two-particle eigenfunctions, multiply through Eq. (3) by $\chi_0(\mathbf{s})$ and integrate over $\mathbf{s}$. Then, treating the right-hand side as a perturbation, we replace Ψ by $\chi_0(\mathbf{s})\psi_1(\mathbf{R})$, where ψ_1 is the Coulomb scattering wave function for the initial vector $\mathbf{k}_1$. We then have

$$\left[\nabla_R^2 + k^2 - \frac{2kn}{R}\right]\Omega_0(\mathbf{R}) = 2kn\int \chi_0^*(\mathbf{s})\left[\frac{1}{|\mathbf{R} - \frac{1}{2}\mathbf{s}|} - \frac{1}{R}\right]\chi_0(\mathbf{s})d\mathbf{s}\psi_1(\mathbf{R}), \tag{4}$$

and now $\Omega_0(\mathbf{R})$ is the wave function describing the elastic scattering. We note immediately that the inhomogeneous term becomes very small for high n. For, because of the spherical symmetry of the deuteron distribution, $[1/|\mathbf{R} - \frac{1}{2}\mathbf{s}| - 1/R] \equiv 0$ except for $R \leqslant \frac{1}{2}s$; i.e., it contributes only when the scattering center is inside the deuteron. For high Z such deep penetration is improbable [$\psi_1(\mathbf{R})$ is small for small R]. Thus in this case the correction to Rutherford scattering is negligible. (This is precisely analogous to the situation in zero-zero electromagnetic transitions without parity change, where one finds an oscillating electromagnetic field only inside the nucleus.)

The solution of (3) is found as in reference 1, proper attention being paid to the Coulomb phase factors. For the asymptotic Green's function we have

$$K(\mathbf{R}, \mathbf{r}) \underset{R\to\infty}{\to} -\frac{1}{4\pi} e^{i(kR - n\ln 2kR)}\psi_2^+(\mathbf{r}), \tag{5}$$

where $\psi_2^+(\mathbf{r})$ is the Coulomb scattering wave function for incident wave vector $-\mathbf{k}_2 = -k\mathbf{R}/R$. Using this we find the amplitude for elastic deuteron scattering to be

$$f(\theta) = f^c(\theta) - \frac{nk}{2\pi}\int \psi_2^+(\mathbf{r})\left[\frac{1}{|\mathbf{r} - \frac{1}{2}\mathbf{s}|} - \frac{1}{r}\right]\psi_1(\mathbf{r})|\chi_0(\mathbf{s})|^2 d\mathbf{r}d\mathbf{s}, \tag{6}$$

where $f^c(\theta)$ is the usual Coulomb scattering amplitude. Using (2) we write the integral appearing in (6) as

$$-8\pi N^2\left[\int_{4\alpha}^\infty + \int_{4\beta}^\infty - 2\int_{4\rho}^\infty\right]\frac{d\gamma}{\gamma^2}\int d\mathbf{r}\psi_2^+(\mathbf{r})\psi_1(\mathbf{r})\frac{\rho^{-\gamma r}}{r},$$

where $2\rho = \alpha + \beta$. The $\mathbf{r}$ integration can now be done by a straightforward generalization of a method given by Sommerfeld[2] for the case $\gamma = 0$. Collecting the results, we have then

$$d\sigma/d\sigma_R = \Big|1 - 32\pi N^2 k^2(2\pi n)(\rho^{2\pi n} - 1)^{-1}\sin^2\tfrac{1}{2}\theta\cdot\exp(in\ln\sin^2\tfrac{1}{2}\theta) \times\left[\int_{4\alpha}^\infty + \int_{4\beta}^\infty - 2\int_{4\rho}^\infty\right]\frac{d\gamma}{\gamma^2}(\gamma^2 - 2i\gamma k)^{2n}(\gamma^2 + 4k^2\sin^2\tfrac{1}{2}\theta)^{-2n-1} \times F\left(-in, -in, 1; -\frac{4k^2}{\gamma^2}\sin^2\tfrac{1}{2}\theta\right)\Big|^2. \tag{7}$$

For small n this reduces to the Born approximation result (1), but for large n it has the value unity for all angles. We have had occasion to evaluate this numerically for the case appropriate to 14-Mev deuterons on Al, for which $n=0.8$. We find, for example, at 140°, $d\sigma/d\sigma_R=0.67$ while the Born approximation result would be 0.11.

To summarize we would predict that, for deuteron energies below the barrier, the ratio $d\sigma/d\sigma_R$ would show for the cases where n is small a steady decrease as we move towards large angles. As n increases this decrease would become less marked, and finally for $n\gg 1$ we would have simply Rutherford scattering. Not much data on Coulomb scattering seem to be available, though there is evidence that when $n\gg 1$ the Coulomb scattering is simply Rutherford.[3]

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The Born Approximation Theory of (d,p) and (d,n) Reactions*

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BUTLER[1] has given a theory for (d,p) and (d,n) reactions which has had remarkable success in explaining experimental results. The purpose of this note is to demonstrate that in all its essential features Butler's theory is equivalent to a Born approximation calculation. It is indeed clear that it should be, for Butler's theory everywhere ignores the reaction of the elastically and inelastically scattered particles as well as the scattering of the particle that is not captured; also his results are given in terms of the obvious momentum transfers of the problem. The Born approximation theory has already been given by Bhatia *et al.*,[2] but in their paper the connection with Butler's theory has not been made quite clear.

We write the case for (d,p) reactions. To avoid unessential complications we assume at first that the initial nucleus has spin 0 and that the neutron is captured under the influence of a potential $V(r_n)$ (assumed central) into the one-particle state with space dependence $\zeta_l^m(\mathbf{r})=R_l(r)Y_l^m$ with binding energy $\epsilon=t^2/2M$. Let the incident deuteron have wave vector $\mathbf{K}$ and the final proton have wave vector $\mathbf{k}$. Then $(\hbar=1)$

$$\frac{d\sigma}{d\omega}=\frac{M^2}{2\pi^2}\frac{k}{K}\frac{1}{3}\frac{j+\frac{1}{2}}{2l+1}\sum_{m_d,m_p,m_n,m}|\langle V\rangle|^2, \tag{1}$$

where $\langle V\rangle$ is the matrix element between the initial and final states and m_d, m_p, and m_n are the spin magnetic quantum numbers for the deuteron, proton, and neutron; m is the orbital magnetic quantum number for the captured neutron, and j is the final nuclear spin. The summation over spin quantum numbers gives simply a factor 3. The probability amplitude for finding a proton momentum $\mathbf{k}$ in the initial deuteron is

$$P(\mathbf{k}-\tfrac{1}{2}\mathbf{K})=\int\psi_d(\mathbf{s})\exp[i(\mathbf{k}-\tfrac{1}{2}\mathbf{K})\cdot\mathbf{s}]d\mathbf{s}$$
$$=4\pi N\left[\frac{1}{\alpha^2+(\mathbf{k}-\frac{1}{2}\mathbf{K})^2}-\frac{1}{\beta^2+(\mathbf{k}-\frac{1}{2}\mathbf{K})^2}\right], \tag{2}$$

where the last step follows on using for ψ_d the usual Hulthén wave function with constants N, α, β.

The momentum carried by the captured neutron is $\mathbf{q}=\mathbf{K}-\mathbf{k}$. The probability amplitude for capturing this neutron into the state $\zeta_l^m(\mathbf{r})$ is

$$\int\zeta_l^m(\mathbf{r})V(r)e^{i\mathbf{q}\cdot\mathbf{r}}d\mathbf{r}=\delta_{m0}[4\pi(2l+1)]^{\frac{1}{2}}i^l\int_0^\infty R_l(r)V(r)j_l(qr)r^2dr$$
$$=-\delta_{m0}[4\pi(2l+1)]^{\frac{1}{2}}\frac{q^2+t^2}{2M}\int_0^\infty R_l(r)j_l(qr)r^2dr. \tag{3}$$

The second form follows by expanding $\exp(i\mathbf{q}\cdot\mathbf{r})$, and we get a nonvanishing result only for $m=0$ by taking the axis of quantization along $\mathbf{q}$. The third form follows by eliminating $V(r)$ by using the Schrödinger equation for $R_l(r)$. The matrix element is now given by the product of factors (2) and (3).

Bhatia *et al.*[2] approximate the radial integral by

$$\int R_l(r)V(r)j_l(qr)r^2dr=j_l(qR)\int R_l(r)V(r)r^2dr, \tag{4}$$

and now the proton angular distribution is given by $|P(\mathbf{k}-\frac{1}{2}\mathbf{K})\times j_l(qR)|^2$. The difficulty here is that there is no reason why R defined by (4) should be independent of q and therefore of angle.

To proceed differently, we use the third form of (3). If we assume, as Butler implicitly does, that we may neglect the contribution to the overlap integral from $r\leqslant r_0$ (where r_0 is greater than the nuclear radius) we can, by using the equations for $R_l(r)$ and $j_l(qr)$ along with Green's theorem, write

$$(q^2+t^2)\int R_l(r)j_l(qr)r^2dr$$
$$=R_l(r_0)r_0^2\left[\frac{\partial j_l(qr)}{\partial r}-\frac{1}{R_l(r)}\frac{\partial R_l(r)}{\partial r}j_l(qr)\right]_{r_0}, \tag{5}$$

where

$$\frac{1}{R_l(r)}\frac{\partial R_l(r)}{\partial r}=\frac{1}{h_l^{(1)}(itr)}\frac{\partial h_l^{(1)}(itr)}{\partial r} \tag{6}$$

is a number defined by the l value, binding energy and r_0. Using this, we have precisely Butler's form for the proton angular distribution. The magnitude is given here in terms of the value of the captured neutron wave function on the surface r_0.

If we do not care to omit the contribution to the overlap integral for $r\leqslant r_0$, we can define the quantity

$$\bar{V}(r_0)=-\int_0^{r_0}j_l(qr)R_l(r)V(r)r^2dr\bigg/\int_0^{r_0}j_l(qr)R_l(r)r^2dr, \tag{7}$$

and then it is trivial to show that

$$\int_0^\infty R_l(r)j_l(qr)r^2dr=\left(1-\frac{q^2+t^2}{2M\bar{V}}\right)^{-1}\int_{r_0}^\infty R_l(r)j_l(qr)r^2dr. \tag{8}$$

In this case, the proton cross section contains also the angularly dependent factor $[1-(q^2+t^2)/2M\bar{V}]^{-2}$. It should be emphasized that the cross section is invariant to the choice of r_0 provided only that $r_0\geqslant r_{\text{nuclear}}$, but does of course depend on the value of the neutron potential.

Finally, we emphasize that application of the Born approximation in the low energy region is a very crude procedure. For example, the effects of scattering of the proton and deuteron are not at all small. We hope to report later some calculations of these effects.

If we take r_0 to be the nuclear radius and make the reasonable assumption that the neutron potential inside the nucleus is constant, then $\bar{V}$ is simply the potential depth. The extra factor will not disturb the most striking feature of Butler's angular distribution, namely, the angular position of the first maximum. It has a singularity at $q=(2M\bar{V}-t^2)^{\frac{1}{2}}$, but this simply removes one of the zeros of Butler's distribution. These zeros, in fact, occur when the neutron momentum transfer q equals an average wave number which a neutron could have when bound with binding energy $\epsilon=t^2/2M$ and orbital angular momentum l in a well of radius r_0. The extra factor above removes that zero which corresponds to the actual inside wave number of the captured neutron. Thus, for example, the formal Born approximation theory for a $2p$ state would not have the first zero of Butler's theory.