The Influence of Multiple Scattering on the Angular Width of Čerenkov Radiation

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(Received May 21, 1952)

One may measure the angle at which visible Čerenkov radiation appears from fast charged particles as they pass through a thin transparent plate. This angle is related to the velocity of the particle in a simple manner, and therefore offers a method of measuring the velocity. The angular width of the radiation is important in that it determines the resolution of the velocity measurement. This width contains contributions due to diffraction, energy loss of the charged particle, and multiple scattering. In this paper, the effects of multiple scattering and diffraction are considered in the following manner. The path of the particle through the transparent plate is considered to be a series of connected short straight-line segments. The radiation is calculated classically from this path, and with the aid of the results of multiple scattering theory, an ensemble average is taken over all possible paths. Two different methods are used in evaluating the radiation average. One is valid at viewing angles near the radiation maximum, and the other at viewing angles differing appreciably from the maximum. Curves are presented that give the radiation intensity as a function of the viewing angle for a variety of cases.

I. INTRODUCTION

THE Čerenkov radiation from a fast charged particle of speed v that passes through a thin transparent plate of refractive index n, appears mainly at an angle from the path of the particle given by $\theta_0 = \cos^{-1}[(1/n)\cdot(c/v)]$. If v/c is not too close to unity, measurement of θ_0 offers a convenient method for the determination of v/c, and hence, for a given rest mass, the momentum and energy of the particle. Recent experiments at Berkeley[1] with 340-Mev protons, indicate that this method is capable of high precision. It is then in order to calculate the effect of various factors on the resolution. An estimate of these factors is included in reference 1.

The natural diffraction width of the radiation is very small, being of the order of λ/L radians, where L is the thickness of the plate and λ is the wavelength of radiation viewed. One should also take into account the energy loss of the particle and the multiple scattering that occurs as the particle passes through the plate. The energy loss affects v/c in a direct and simple way. The multiple scattering changes the average direction of the particle and partially destroys the coherence of the radiation along the path. Only the effects of multiple scattering and diffraction will be considered here.

The problem then is to calculate the radiation from a charged particle moving with constant speed and undergoing multiple scattering in a thin transparent plate.

This problem may be attacked by classical radiation theory rather than by quantum radiation theory. A criterion for the validity of classical theory is that the dimension of the wave packet describing the charged particle be small compared with the reduced wavelength of the radiation emitted and remain small during the time t during which the radiation is emitted coherently. If the size of the wave packet at time t is Δx, then we have

$$\Delta x \sim \delta x + (\delta v)t,$$

where δx is the size of the packet at $t=0$. The time t may be taken as $\mathcal{L}/v$, where $\mathcal{L}$ is some length less than the thickness of the plate but substantially greater than the mean free path λ_0 of the charged particle between scattering events in the plate. A relation between δx and δv (the uncertainty in velocity of the charged particle) may be obtained as follows:

$$\delta p = m_0\delta v/(1-v^2/c^2)^{\frac{3}{2}} \sim \hbar/\delta x.$$

Therefore,

$$\Delta x \sim \delta x + (\hbar/m_0\delta x)(1-v^2/c^2)^{\frac{3}{2}}(\mathcal{L}/v).$$

The minimum size of this packet is:

$$[(\hbar/m_0 v)(1-v^2/c^2)^{\frac{3}{2}}\mathcal{L}]^{\frac{1}{2}}.$$

For an electron traversing a plate 1 mm thick and for $v \lesssim c$, we see that this is less than 10^{-6} cm, at least a factor of ten smaller than the reduced wavelength of visible radiation; it is even smaller for a proton.

II. THE RADIATION PROBLEM

(A) General

The power radiated from a classical system of currents can be calculated by evaluating the familiar retarded solution of the inhomogeneous electromagnetic wave equation. This retarded solution can be subjected to a Fourier analysis and the Fourier components of the power calculated. Such a development yields[2]

$$P_{k,\,\omega}(\mathbf{r}) = (nk^2/2\pi r^2 c) \times \left| \int J_{\perp k,\,\omega}(\mathbf{r}')\exp(-in\mathbf{k}\cdot\mathbf{r}')d\tau' \right|^2. \quad (1)$$

The propagation vector $\mathbf{k}$ is directed along the line of observation and has a magnitude $2\pi/\lambda = \omega/c$. $J_{\perp k,\,\omega}(\mathbf{r}')$ is the Fourier component of that part of the current

[1] R. L. Mather, Phys. Rev. **84**, 181 (1951).

[2] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 264.

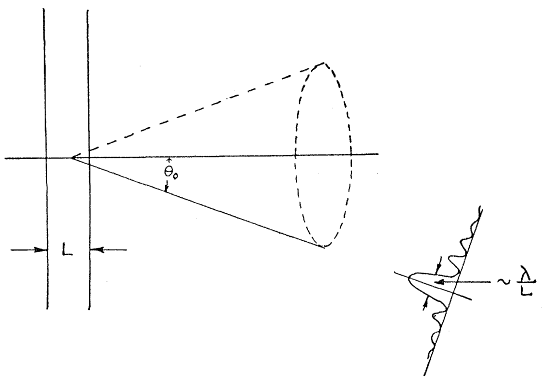

FIG. 1. The Čerenkov cone and the diffraction pattern given by Eq. (2).

that is perpendicular to $\mathbf{k}$. n is the index of refraction of the medium in which the currents are located.

The simple case of a particle passing straight through a plate with an index of refraction n has been evaluated by many authors;[3] it yields a conical shell of light which has the path of the particle as an axis and a thickness given by a diffraction pattern (see Fig. 1). For $\lambda/L \ll 1$, the pattern is given very nearly by

$$L^2[\sin(\tfrac{1}{2}nkL\kappa \sin\theta_0)/(\tfrac{1}{2}nkL\kappa \sin\theta_0)]^2, \tag{2}$$

where $\kappa \equiv \theta - \theta_0$.

In the calculation of the radiation from a particle undergoing multiple scattering, the path is assumed to consits of many short straight segments. The integral in (1) is then seen to be a sum of integrals over all the segments of the path. Therefore, one must calculate

$$\left|\sum_{\nu=1}^{N} I_\nu\right|^2, \tag{3}$$

where I_ν is the contribution due to segment ν. The segment ν is of length l_ν, goes from x_ν, y_ν, z_ν, to $x_{\nu+1}$, $y_{\nu+1}$, $z_{\nu+1}$, and has polar angles θ_ν, ϕ_ν; the particle is at the point ν at the time t_ν. $\mathbf{k}$ has polar angles θ, ϕ (see Fig. 2). It is convenient to introduce Θ_ν, the angle between the path segment and $\mathbf{k}$. From the geometry, it is seen that Θ_ν is related to θ, ϕ, etc., through $\cos\Theta_\nu = \cos\theta \cos\theta_\nu + \sin\theta \sin\theta_\nu \cos(\phi - \phi_\nu)$. From (1) we have

$$I_\nu = (e/2\pi)\sin\Theta_\nu \exp[i\omega t_\nu - ink(x_\nu \sin\theta \cos\phi + y_\nu \sin\theta \sin\phi + z_\nu \cos\theta)] \times \frac{\exp[i\omega l_\nu(1/v_\nu - n\cos\Theta_\nu/c)] - 1}{i\omega(1/v_\nu - n\cos\Theta_\nu/c)}. \tag{4}$$

The angles θ_ν are the deviations of the path segments from the direction of the incident particle and are considered to be small. Consequently, we may take

$$\cos\Theta_\nu \cong \cos\theta + \alpha_\nu \sin\theta,$$
$$\sin\Theta_\nu \cong \sin\theta - \alpha_\nu \cos\theta, \quad (\alpha_\nu \equiv \theta_\nu \cos\phi_\nu),$$

where, for convenience, we take $\phi = 0$, so that the direction of observation is in the x, z plane. Also, since the energy loss is neglected, we take $v_\nu = v$. The particle starts from the origin at time $t=0$; therefore,

$$t_\nu = v^{-1}(l_0 + l_1 + l_2 + \cdots + l_{\nu-1}) \cong v^{-1} z_\nu.$$

With these approximations, (4) becomes

$$I_\nu \cong \frac{e(\sin\theta - \alpha_\nu \cos\theta)}{2\pi} \times \exp\left[i\omega z_\nu\left(\frac{1}{v} - \frac{n\cos\theta}{c}\right) - inkx_\nu \sin\theta\right] \times \frac{\exp[i\omega l_\nu(1/v - n\cos\theta/c) - inkl_\nu\alpha_\nu \sin\theta] - 1}{i\omega(1/v - n\cos\theta/c) - ink\alpha_\nu \sin\theta}. \tag{5}$$

There are two different approaches to the evaluation of (3) by means of (5). One is an expansion valid near the radiation maximum. The other is an asymptotic expansion valid at viewing angles whose difference from the angle of the radiation maximum is much greater than the rms multiple scattering angle. The former will be treated first.

(B) The Solution Near the Radiation Maximum

It is convenient here to introduce $\kappa \equiv \theta - \theta_0$ where θ_0 is the angle of the radiation maximum (see Fig. 1). Under the small angle approximation for κ, (5) becomes

$$I_\nu \cong \frac{e \sin\theta_0}{2\pi}(1 + \kappa \cot\theta_0)\exp[ink \sin\theta_0(\kappa z_\nu - x_\nu)] \times \frac{\exp[ink \sin\theta_0(\kappa - \alpha_\nu)l_\nu] - 1}{ink \sin\theta_0(\kappa - \alpha_\nu)}. \tag{6}$$

Here it is asked that $|(\kappa - \alpha_\nu)(l_\nu/\lambda)|$ be very much less than unity. This further limits the range of validity of this solution to small κ. Equation (6) can now be written approximately

$$I_\nu \cong \frac{e \sin\theta_0}{2\pi} l_\nu \exp[ink \sin\theta_0(\kappa z_\nu - x_\nu)]. \tag{7}$$

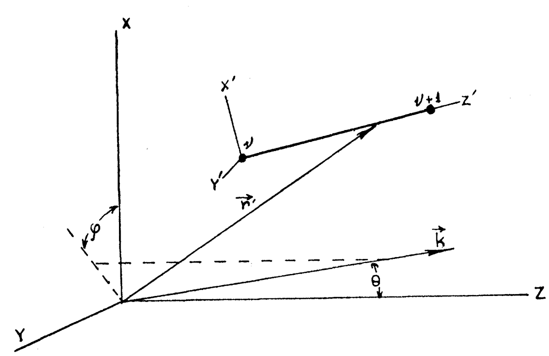

FIG. 2. The geometry involved in the development of the formula describing the radiation from a single path segment.

[3] I. Frank and I. Tamm, Compt. rend. acad. sci. U.R.S.S. **14**, 109 (1937). See also reference 2, p. 262.

Equation (3) may be written

$$\sum_{\nu=0}^{N-1}\sum_{\mu=0}^{N-1} I_\nu I_\mu^*. \quad (8)$$

Experimentally, one observes the radiation from a great number of particles. Therefore, one must calculate an ensemble average of (8) over all quantities which may be different for the paths of individual particles. This ensemble average may be computed by taking the averages of the individual terms $I_\nu I_\mu^*$ of (8) and then performing the summations indicated. With the aid of (7) we have

$$\langle I_\nu I_\mu^*\rangle_{\mathrm{Av}}\left(\frac{e\sin\theta_0}{2\pi}\right)^{-1} \cong\langle l_\nu l_\mu \exp -iB[(x_\nu-x_\mu)+\kappa(z_\nu-z_\mu)]\rangle_{\mathrm{Av}}, \quad (9)$$

where $B\equiv nk\sin\theta_0$. A change of variables will put this in a form that will allow the results of multiple scattering theory to be applied. From Fig. 3 it is clear that $x_\nu-x_\mu\cong\gamma_{\nu-\mu}+\alpha_\mu(z_\nu-z_\mu)$. With the aid of this, (9) becomes

$$\langle l_\nu l_\mu \exp -iB[\gamma_{\nu-\mu}+(\kappa+\alpha_\mu)(z_\nu-z_\mu)]\rangle_{\mathrm{Av}}. \quad (10)$$

The average is computed by multiplying by the normalized distribution functions of all the variables represented in (10) and then integrating over the allowed ranges of the variables. We therefore multiply by $p_1(l_\nu)p_2(l_\mu)p_3(\alpha_\mu, z_\mu)p_4(z_\mu)p_5(\gamma_{\nu-\mu}, z_\nu-z_\mu)p_6(z_\nu-z_\mu)$. The combination p_3p_4 is the probability that after undergoing μ collisions, the particle makes an angle α_μ with the original path. The factor p_3 is the distribution function for α_μ given that the particle has traveled a distance z_μ; p_4 is the probability that the particle has indeed traveled a distance z_μ up to the μ^{th} collision. The combination p_5p_6 is to be interpreted in the same manner. At this point it is assumed that negligible error will occur if the result of averaging over z_μ, and also $z_\nu-z_\mu$, is to replace them by their mean values directly; i.e., replace z_μ by $\mu\lambda_0$ and $z_\nu-z_\mu$ by $(\nu-\mu)\lambda_0$,[4] where λ_0 is the mean free path of a particle in the transparent plate. $p_3(\alpha_\mu, \mu\lambda_0)$ is taken from the simplest multiple scattering theory and is[5]

$$A\pi^{-\frac{1}{2}}\mu^{-\frac{1}{2}}\exp[-(A^2\alpha_\mu^2/4\mu)].$$

Examination of Fig. 3 shows that we may also use this multiple scattering theory in developing $p_5(\gamma_{\nu-\mu}, (\nu-\mu)\lambda_0)$. This yields[6]

$$\frac{\sqrt{3}A}{2\pi^{\frac{1}{2}}\lambda_0(\nu-\mu)^{\frac{3}{2}}}\exp\left\{-\left[\frac{3A^2\gamma_{\nu-\mu}^2}{4\lambda_0^2(\nu-\mu)^3}\right]\right\}.$$

[4] In support of this assumption, it can be shown that although p_4 and p_6 are rather wide functions, their widths divided by their mean values become very small as the number of segments represented by them is taken larger. Therefore, it is felt justifiable to replace p_4 and p_6 by delta-functions centered about the mean values involved.

[5] See Appendix, Eq. (A2).

[6] See Appendix, Eq. (A3).

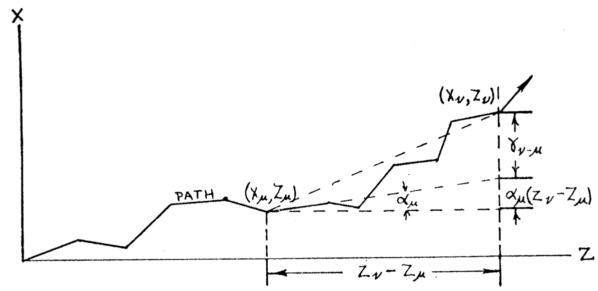

FIG. 3. Relations between variables characteristic of a give path.

$p_1(l_\nu)$ and $p_2(l_\mu)$ are $\lambda_0^{-1}\exp(-l_\nu/\lambda_0)$ and $\lambda_0^{-1}\exp(-l_\mu/\lambda_0)$, respectively.

Therefore, evaluation of (10) yields

$$\lambda_0^2\exp[-iB\lambda_0(\nu-\mu)\kappa-B^2\lambda_0^2A^{-2}(\nu-\mu)^2\{\mu+\tfrac{1}{3}(\nu-\mu)\}].$$

It is clear from this that

$$\langle I_\mu I_\nu^*\rangle_{\mathrm{Av}}=\langle I_\nu I_\mu^*\rangle_{\mathrm{Av}}^*.$$

Therefore,

$$\langle I_\nu I_\mu^*+I_\mu I_\nu^*\rangle_{\mathrm{Av}}=(e\sin\theta_0/2\pi)^2 2\lambda_0^2\cos[B\lambda_0(\nu-\mu)\kappa] \times\exp[-B^2\lambda_0^2A^{-2}(\nu-\mu)^2\{\mu+\tfrac{1}{3}(\nu-\mu)\}]. \quad (11)$$

With the help of (11), a little consideration shows that the ensemble average of (8) may be written

$$2\lambda_0^2\left(\frac{e\sin\theta_0}{2\pi}\right)^2\sum_{l=0}^{N-1}\cos(B\lambda_0 l\kappa)\exp\left[-\left(\frac{B\lambda_0}{A}\right)^2\frac{l^3}{3}\right] \times\sum_{\mu=0}^{N-l-1}\exp\left[-\left(\frac{B\lambda_0}{A}\right)^2 l^2\mu\right], \quad (12)$$

where we have written: $l\equiv\nu-\mu$.

It will now be shown that the summations over l and μ may be approximated by integrals. The factor in (12) following the cosine, falls to e^{-1} for $l=(\sqrt{3}A/B\lambda_0)^{\frac{2}{3}}$. For this value of l, the exponent of the last factor in (12) is: $-(3B\lambda_0/A)^{\frac{2}{3}}\mu$. Here, the coefficient of μ can be seen to be small compared to unity in practical cases, so that the replacement of the sum over μ by an integral is justified. By a similar argument, one can show that the sum over l can also be replaced by an integral; the criteria in this case are that $B\lambda_0\kappa$ and $(B\lambda_0/\sqrt{3}A)^{\frac{2}{3}}$ both be small compared to unity. Equation (12) becomes:

$$2\lambda_0^2\left(\frac{e\sin\theta_0}{2\pi}\right)^2\int_{l=0}^{N-1}\cos(B\lambda_0 l\kappa)\exp\left[-\left(\frac{B\lambda_0}{A}\right)^2\frac{l^3}{3}\right] \times\int_{\mu=0}^{N-l-1}\exp\left[-\left(\frac{B\lambda_0}{A}\right)^2 l^2\mu\right]d\mu dl. \quad (13)$$

The integration over μ is elementary. It is convenient to express (13) in terms of some new variables. Let us

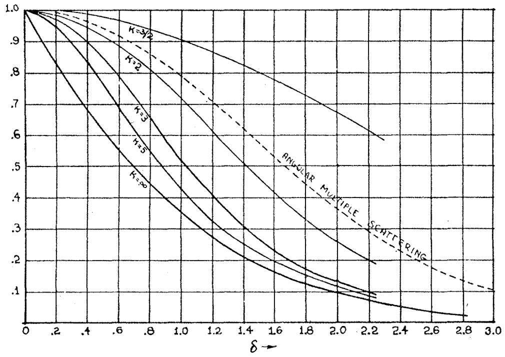

FIG. 4. Curves, normalized to unity at $\delta=0$, showing the angular shape of the radiation pattern. The abscissa is the difference between the viewing angle and the angle at which the radiation maximum appears, and it is measured in units of the rms multiple scattering which occurs in the plate. The dashed curve is a plot of the exponential of Eq. (A2) in the appendix, and shows the projected angular distribution of particles emerging from the plate. The curve for $K=\infty$ is the projected angular multiple scattering averaged over the thickness of the plate.

take:

$$u\equiv B\lambda_0 N^{\frac{1}{2}}l/A,$$
$$K\equiv(3B\lambda_0 N^{\frac{3}{2}}/A)^{\frac{1}{3}}=[(3BL/2^{\frac{1}{2}})((2N)^{\frac{1}{2}}/A)]^{\frac{1}{3}},$$
$$\delta\equiv A\kappa/N^{\frac{1}{2}}.$$

The rms value of the angular multiple scattering in the plate is $A^{-1}(2N)^{\frac{1}{2}}$. Therefore, it is seen that δ measures the viewing angle κ in terms of the rms multiple scattering. K is a constant characteristic of L/λ and the rms multiple scattering. In terms of these variables, (13) becomes

$$6\lambda_0^2\left(\frac{e\sin\theta_0}{2\pi}\right)^2\frac{N^2}{K^3}\int_0^{\frac{1}{3}K^3}\cos(\delta u)$$
$$\times\exp[-(u/K)^3]\frac{1-\exp[-u^2+3(u/K)^3]}{u^2}du. \quad (14)$$

The integral in (14) has been evaluated numerically for several values of K. The results, normalized[7] to unity are shown in Fig. 4. These curves are convenient for $K\gtrsim1$, but for smaller K, it is best to express the results in terms of the variable $BN\lambda_0\kappa$ as is done in the elementary theory. [See Eq. (2).]

The case of very large K can be interpreted as the result of there being coherence only over a few adjoining segments. It is possible to arrive at the large K case in another manner. The power radiated by a few coherent adjoining segments can be found [by the simple theory Eq. (2)] to be proportional to Δz, the length of the group of segments. The radiation maximum due to this group of segments will appear at an angle differing from θ_0 by the projection of the average angle of the group of segments from the mean path, onto the plane determined by the line of observation and the mean path. Thus, to calculate the angular distribution of radiation from the entire plate, we multiply the power radiated from a group of segments, proportional to Δz, by the angular multiple scattering probability and sum over the thickness of the plate.

$$\sum_{r=0}^{L/\Delta z}\frac{A\lambda_0^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}(r\Delta z)^{\frac{1}{2}}}\exp\left(-\frac{A^2\lambda_0\kappa^2}{4r\Delta z}\right)(\Delta z),$$

where $\kappa\equiv\theta-\theta_0$. In the limit of small Δz, we can replace $(r\Delta z)$ by z and

$\sum_r(\cdots)\Delta z$ by $\int(\cdots)dz$:

$$\int_0^{N\lambda_0}\frac{A\lambda_0^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}z^{\frac{1}{2}}}\exp\left(-\frac{A^2\lambda_0\kappa^2}{4z}\right)dz$$
$$=(\lambda_0N^{\frac{1}{2}}A/2\pi^{\frac{1}{2}})\tfrac{1}{2}\delta\int_{(\delta/2)^2}^{\infty}v^{-\frac{3}{2}}e^{-v}dv, \quad (15)$$

where δ is defined as before and $v\equiv4z/(A^2\lambda_0\kappa^2)$. This integral is related to the incomplete gamma-function in a simple manner and may be expressed as a special case of the confluent hypergeometric series.[8] This series, multiplied by $(\delta/2)$, can be shown to be equal to the series expansion of

$$\frac{2}{\pi^{\frac{1}{2}}}\int_0^{\infty}\cos(\delta u)\frac{1-\exp(-u^2)}{u^2}du,$$

i.e., proportional to (14) for $K=\infty$. This serves as a check on (14). Another check on (14) is the case of no scattering. In this limit, the simple theory result, Eq. (2), is obtained.

(C) The Asymptotic Form

The radiation maximum occurs at a viewing angle θ_0 such that $\cos\theta_0=c/(nv)$. Since in this section we seek a solution away from this maximum, we may extract $(v^{-1}-nc^{-1}\cos\theta)$ from the denominator of (5) and expand the remaining factor in a power series, retaining only the first two terms. The limitation on the final result imposed by this approximation may be seen to restrict the result to values of κ much greater than any α_i that occurs, hence much greater than the angular rms scattering that occurs in the plate. As in the case of the solution near the radiation maximum, we note that the lateral deviation of a particle from its path between two collisions is small compared to a wavelength of the light observed. This allows us to replace $\exp(-inkl_\nu\alpha_\nu\sin\theta)$ by the first two terms of a power series expansion. With these two approximations, (5) becomes

$$I_\nu\cong(e\sin\theta/2\pi ia)\exp i(az_{\nu+1}-bx_\nu)$$
$$\times[(1-e^{-ial_\nu})(1+\alpha_\nu g)-ib\alpha_\nu l_\nu], \quad (16)$$

[7] In normalizing, we have divided by the value of the integral for $\delta=0$. These values are tabulated in Sec. (B) of the Appendix.

[8] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (The Macmillan Company, New York, 1946), p. 341.

where, for simplicity, we have introduced

$$a\equiv\omega[(1/v)-(n\cos\theta/c)],\qquad b\equiv\omega n\sin\theta/c,$$

$$g\equiv\frac{n-c\cos\theta/v}{c[(1/v)-(n\cos\theta/c)]\sin\theta}.$$

One may now use (16) to form

$$\sum_{\nu=0}^{N-1} I_\nu\cong(e\sin\theta/2\pi ia)\{-(1+\alpha_0 g)\exp[i(az_0-bx_0)]-g\sum_{\nu=1}^{N-1}[\eta_\nu\exp\{i(az_\nu-bx_\nu)\}]+(1+\alpha_{N-1}g)\exp\{i(az_N-bx_N)\}\}.\quad(17)$$

In forming this expression, it is necessary to interchange $\exp(-ib\alpha_\nu l_\nu)$ with $1-ib\alpha_\nu l_\nu$ occasionally, and also to neglect other terms of order α_ν^2 as compared with α_ν. Use has also be made of $z_{\nu+1}\cong z_\nu+l_\nu$ and $x_{\nu+1}\cong x_\nu+\alpha_\nu l_\nu$. The angle η_ν is defined as the projection of the angle of scattering of the charged particle at collision ν. This is clearly $\eta_\nu\equiv\alpha_\nu-\alpha_{\nu-1}$.

We are interested in the average of the absolute square of (17). In examining the square, we find many terms that are linear in any one particular η_i. The average of such terms will be zero, since the probability of any η_i (i.e., the projected cross section for a single collision) is an even function of η_i. We may take $\alpha_0=x_0=z_0=0$, since the particle is taken to be incident on the plate at the origin and directed along the z axis. The average of the square of (17) is then

$$\langle|\sum_{\nu=0}^{N-1}I_\nu|^2\rangle_{\mathrm{Av}}=(e\sin\theta/2\pi a)^2\{\langle|(1+\alpha_{N-1})\times\exp[i(az_N-bx_N)]-1|^2\rangle_{\mathrm{Av}}+g^2\sum_{\nu=1}^{N-1}\langle\eta_\nu^2\rangle_{\mathrm{Av}}\}$$
$$=2(e\sin\theta/2\pi a)^2\{1-\langle(1+g\alpha_{N-1})\cos(aL-bx_N)\rangle_{\mathrm{Av}}+g^2\langle\alpha_{N-1}^2\rangle_{\mathrm{Av}}\}.\quad(18)$$

In obtaining (18), use has been made of $\langle\alpha_{N-1}\rangle_{\mathrm{Av}}=0$, $z_N\equiv L$, and

$$\alpha_{N-1}\equiv\sum_{\nu=1}^{N-1}\eta_\nu.$$

The number of segments N is large, and hence, one may substitute α_N for α_{N-1} in (18). The averaging indicated in (18) is performed by utilizing the distribution function (A) and integrating over α_N and x_N. The result is

$$2\left(\frac{e\sin\theta}{2\pi a}\right)^2\left\{1-\left(\cos aL+\frac{gbN^2\lambda_0}{A^2}\sin aL\right)\times\exp\left(-\frac{b^2\lambda_0^2N^3}{3A^2}\right)+\frac{2g^2N}{A^2}\right\}.\quad(19)$$

Equation (19) may be written in terms of K and δ. The limitation on this form being that κ should be small so that one may replace $\sin\kappa$ by κ, etc. This yields

$$2\left(\frac{e\sin\theta_0}{2\pi}\right)^2\frac{9L^2}{\delta^2K^6}\left\{1-\left[\cos\left(\frac{K^3\delta}{3}\right)+\frac{K^3}{3\delta}\sin\left(\frac{K^3\delta}{3}\right)\right]\times\exp\left(-\frac{K^6}{27}\right)+\frac{2}{\delta^2}\right\}.$$

The exponential factor destroys the diffraction wiggles for large K.

The asymptotic solution was developed under the assumption that the viewing angle κ is large compared with the angular rms multiple scattering. In the limit of no scattering, the solution should be valid very near the radiation maximum. By allowing the scattering to go to zero in Eq. (19), i.e., by allowing NA^{-2} to go to zero, it can be seen that the result of the simple theory is obtained.

III. DISCUSSION OF RESULTS

The curves of Fig. 4 are the results of the calculation of the radiation near the usual maximum (see Sec. II-B). The asymptotic solution (Sec. II-C) is valid only for $\delta\gg1$ and, therefore, does not appear in Fig. 4. All the curves of Fig. 4 are symmetrical about the vertical axis. The abscissa δ is proportional to $\kappa\equiv\theta-\theta_0$, where θ is the viewing angle with respect to the direction of the path of the charged particle, and θ_0 is the viewing angle of the usual maximum. δ is related to κ through $\delta\equiv\kappa\sqrt{2}/(\langle\theta^2\rangle)^{\frac{1}{2}}$, where $(\langle\theta^2\rangle)^{\frac{1}{2}}$ is the rms multiple scattering angle that corresponds to the thickness of the plate. The dashed curve of Fig. 4 shows the projected angular distribution of particles emerging from the plate. This is the Gaussian from Eq. (A2) of the appendix.

The parameter K may be readily determined for a particular case in the following manner. First calculate the rms multiple scattering angle $(\langle\theta^2\rangle)^{\frac{1}{2}}$. K is then obtained from

$$K\equiv[(6\pi/\sqrt{2})n\sin\theta_0(L/\lambda)(\langle\theta^2\rangle)^{\frac{1}{2}}]^{\frac{1}{3}},$$

where L is the thickness of the plate, λ is the wavelength of the light observed, n is the refractive index of the plate, and θ_0 is the viewing angle at which the usual maximum occurs. As mentioned in the last paragraph of Sec. II-B, the angular distribution of radiation in the case of large K may be obtained by averaging the projected angular multiple scattering distribution function over the length of the path L. The results for large K will therefore fall within the dashed curve of Fig. 4. From the above definition of K, we may look on the curves for progressively smaller K as the radiation distribution if longer wavelength radiation is observed. The broadening is due to the effects of partial incoherence and, for sufficiently small K, due to the diffraction over the entire path length.

The author is indebted to Professor L. I. Schiff for suggesting this problem and for his very helpful aid and interest throughout the entire calculation.

APPENDIX

(A) Multiple Scattering Theory

Multiple scattering theories give the probability that a particle that has suffered a great number of collisions in a slab of material has a projected angle of emergence between θ and $\theta+d\theta$, and has been deviated from its original path by a distance between y and $y+dy$. The simplest of these theories make use of the fact that fast particles being scattered by the electric field of the nucleus are scattered predominantely forward. This allows approximations to be made that yield the distribution function originally due to Fermi[9]

$$F(z,y,\theta)=\frac{\sqrt{3}A^2\lambda_0}{2\pi z^2}\times\exp\left[-\frac{A^2\lambda_0}{z}\left\{\left(\theta-\frac{3y}{2z}\right)^2+\frac{3}{4}\left(\frac{y}{z}\right)^2\right\}\right], \quad \text{(A1)}$$

where z is defined as the thickness of the slab and is expressed in the same units as λ_0, the mean free path of a particle in the material. $A^2\lambda_0$ is a constant characteristic of the single scattering process and takes on slightly different values when one makes different assumptions regarding the screening of the Coulomb potential and uses different methods in calculating the single scattering cross section.[10] Equation (A1) may be integrated over y to form the distribution in θ alone. We then have

$$G(z,\theta)=\frac{A\lambda_0^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}z^{\frac{1}{2}}}\exp\left[-\frac{A^2\lambda_0\theta^2}{4z}\right]. \quad \text{(A2)}$$

Alternatively, (A1) may be integrated over θ to yield the distribution in y.

$$H(z,y)=\frac{\sqrt{3}A\lambda_0^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}z^{\frac{3}{2}}}\exp\left[-\frac{3A^2\lambda_0y^2}{4z^3}\right]. \quad \text{(A3)}$$

(B) Normalizing Factors

The factors noted in reference 7 are the values of

$$\int_0^{\frac{1}{3}K^3}\left[\exp\left\{-\left(\frac{u}{K}\right)^3\right\}\right]\cdot\frac{1-\exp[-u^2+3(u/K)^3]}{u^2}du.$$

The values of these integrals are

K	∞	5	3	2	3/2
Value of the integral:	$\pi^{\frac{1}{2}}$	1.49	1.28	0.933	0.519.

[9] B. Rossi and K. Griesen, Revs. Modern Phys. **13**, 267 (1941).

[10] See reference 5 in W. T. Scott, Phys. Rev. **76**, 213 (1949).