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Cascade Theories with Ionization Loss

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Analytical solutions have previously been given for the number distribution functions and for the general moments of the electron-photon and nucleon cascades neglecting ionization losses (approximation A). Solutions are now given for the moments of the electron-photon and proton-neutron cascades taking into account energy loss, via ionization, by electrons and protons (approximation B). The diffusion equations for the differential moment functions, which yield the required factorial moments by a simple integration over the energy variables, are transformed by Laplace-Mellin transforms to matrix recurrence relations, the general solution of which is obtained in the form of power series. From these series, solutions for the moments in a form suitable for numerical calculations are obtained by a generalization of the method used by Bhabha and Chakrabarty for the first moments of the electron-photon cascade and by Messel in the proton-neutron cascade. To a first approximation, the solutions for the moments in approximation B are expressed as a correction factor multiplying the solutions obtained in approximation A.

1. INTRODUCTION

ANALYTICAL solutions have recently[1-5] been given for the fluctuation problems arising in nucleon and electron-photon cascade theories in approximation A (neglecting ionization loss). In the above references, analytical expressions were obtained for the general number distribution functions as well as for their factorial moments. It is the purpose of the present paper to give solutions for the (n, m)th factorial moments of the electron-photon and nucleon cascades when energy loss by ionization is accounted for (approximation B). In this case it is necessary to distinguish the protons and neutrons in the nucleon cascade, and to emphasize this the cascade will be called the "proton-neutron" cascade.

The method used in this paper is the following:—The diffusion equation for the differential moment function is solved in series form. From this probability function the (n, m)th factorial moments are obtained by a simple integration over the energy variables. In a manner similar to that used by Bhabha and Chakrabarty[6,7] and Messel[8,9] for the first moments in electron-photon and proton-neutron cascade theory, respectively, the series solution for the moments is transformed to a new series, the first term of which gives an approximate formula for the moments in a form suitable for numerical calculations.

The general results, not unexpectedly, are exceedingly complicated. The amount of work required to compute the second moments is not prohibitive providing one has the aid of an electronic brain. A program is at present being set up for such a calculation.

2. THE ELECTRON-PHOTON CASCADE

(a) The Diffusion Equations

Using the same notation as previously,[5] we let $q_{n,m}^{(j)}(E_0; E_1, \cdots, E_n; E_{n+1}, \cdots, E_{n+m}; x)$ be the differential moment function expressing the probability that after a depth of x cascade units a primary (j) of energy E_0 has given rise to n electrons with energies in the ranges E_k, dE_k, $k=1, \cdots, n$ in any order, to m photons with energies in the ranges E_{n+l}, dE_{n+l}, $l=1, \cdots m$ in any order, and to any numbers of electrons and photons with arbitrary energies. For $j=1$, the primary is an electron, for $j=2$, a photon. It is assumed that the electrons suffer a constant energy loss β by ionization. For the cross sections, the well-known Bethe-Heitler expressions in the full-screening approximation will be used: $w^{(1)}(E_k, E_l)$ for bremsstrahlung, $w^{(2)}(E_k, E_l)$ for pair production, and $\alpha^{(1)}$ and $\alpha^{(2)}$ for the corresponding total cross sections.

The last-collision diffusion equation[4] satisfied by $q_{n,m}^{(j)}$ is

$$\left(\frac{\partial}{\partial x}+n\alpha^{(1)}+m\alpha^{(2)}\right)q_{n,m}^{(j)}(E_0; E_1, \cdots, E_n; E_{n+1}, \cdots, E_{n+m}; x)$$
$$=\sum_{C_1^n}\sum_{C_1^m} q_{n,m-1}^{(j)}(E_0; E_1', \cdots, E_{n-1}', E_n'+E_{n+m}'; E_{n+1}', \cdots, E_{n+m-1}'; x)w^{(1)}(E_n', E_{n+m}')$$
$$+\sum_{C_2^n} q_{n-2,m+1}^{(j)}(E_0; E_1', \cdots, E_{n-2}'; E_{n+1}, \cdots, E_{n+m}, E_{n-1}'+E_n'; x)w^{(2)}(E_{n-1}', E_n')$$

(continued on next page)

[1] H. Messel, Proc. Phys. Soc. (London) **A65**, 465 (1952).
[2] H. Messel and J. W. Gardner, Phys. Rev. **84**, 1256 (1951).
[3] H. Messel and R. B. Potts, Proc. Phys. Soc. (London) **A65**, 473 (1952).
[4] H. Messel and R. B. Potts, Proc. Phys. Soc. (London) (to be published).
[5] H. Messel and R. B. Potts, Phys. Rev. **86**, 847 (1952).
[6] H. J. Bhabha and S. K. Chakrabarty, Proc. Roy. Soc. (London) **A181**, 267 (1943).
[7] H. J. Bhabha and S. K. Chakrabarty, Phys. Rev. **74**, 1352 (1948).
[8] H. Messel, Phys. Rev. **83**, 21 (1951).
[9] H. Messel, Phys. Rev. **83**, 26 (1951).

$$+\sum_{C_1^n}\int_0^\infty q_{n-1,\,m+1}{}^{(j)}(E_0;E_1',\cdots,E_{n-1}';E_{n+1},\cdots,E_{n+m},U;x)2w^{(2)}(U-E_n',E_n')dU$$

$$+\sum_{C_1^n}\int_0^\infty q_{n,\,m}{}^{(j)}(E_0;E_1',\cdots,E_{n-1}',U;E_{n+1},\cdots,E_{n+m};x)w^{(1)}(E_n',U-E_n')dU$$

$$+\sum_{C_1^m}\int_0^\infty q_{n+1,\,m-1}{}^{(j)}(E_0;E_1,\cdots,E_n,U;E_{n+1}',\cdots,E_{n+m-1}';x)w^{(1)}(U-E_{n+m}'E_{n+m}')dU$$

$$+\beta\sum_{k=1}^{n}\frac{\partial}{\partial E_k}q_{n,\,m}{}^{(j)}(E_0;E_1,\cdots,E_n;E_{n+1},\cdots,E_{n+m};x). \quad (1)$$

This equation, apart from the last term which expresses a shift in energy due to ionization loss by electrons, is the last-collision diffusion equation for the differential moment function in approximation A.[5] The reader is referred to this paper for definitions of the notation used in (1) and subsequently.

If the Laplace-Mellin transform $Q_{n,\,m}{}^{(j)}$ and the Mellin transform $W^{(j)}$ are defined by

$$Q_{n,\,m}{}^{(j)}(s_1,\cdots,s_n;s_{n+1},\cdots,s_{n+m};\lambda)=\int_0^\infty dE_1\cdots\int_0^\infty dE_{n+m}\int_0^\infty dx(E_1/E_0)^{s_1}\cdots(E_{n+m}/E_0)^{s_{n+m}}e^{-\lambda x}q_{n,\,m}{}^{(j)} \quad (2)$$

and

$$W^{(j)}(s_1,s_2)=\int_0^\infty\left(\frac{E_1}{E_1+E_2}\right)^{s_1}\left(\frac{E_2}{E_1+E_2}\right)^{s_2}w^{(j)}(E_1,E_2)dE_2, \quad (3)$$

then (1) may be transformed to

$$(\lambda+n\alpha^{(1)}+m\alpha^{(2)})Q_{n,\,m}{}^{(j)}-\delta_{n+j,\,2}\delta_{m+1,\,j}$$

$$=\sum_{C_1^n}\sum_{C_1^m}Q_{n,\,m-1}{}^{(j)}(s_1',\cdots,s_{n-1}',s_n'+s_{n+m}';s_{n+1}',\cdots,s_{n+m-1}';\lambda)W^{(1)}(s_n',s_{n+m}')$$

$$+\sum_{C_2^n}Q_{n-2,\,m+1}{}^{(j)}(s_1',\cdots,s_{n-2}';s_{n+1},\cdots,s_{n+m},s_{n-1}'+s_n';\lambda)W^{(2)}(s_{n-1}',s_n')$$

$$+\sum_{C_1^n}Q_{n-1,\,m+1}{}^{(j)}(s_1',\cdots,s_{n-1}';s_{n+1},\cdots,s_{n+m},s_n';\lambda)2W^{(2)}(s_n',0)$$

$$+\sum_{C_1^n}Q_{n,\,m}{}^{(j)}(s_1,\cdots,s_n;s_{n+1},\cdots,s_{n+m};\lambda)W^{(1)}(s_n',0)$$

$$+\sum_{C_1^m}Q_{n+1,\,m-1}{}^{(j)}(s_1,\cdots,s_n,s_{n+m}';s_{n+1}',\cdots,s_{n+m-1}';\lambda)W^{(1)}(0,s_{n+m}')$$

$$-(\beta/E_0)\sum_{C_1^n}s_nQ_{n,\,m}{}^{(j)}(s_1,\cdots,s_{n-1},s_n-1;s_{n+1},\cdots,s_{n+m};\lambda). \quad (4)$$

This equation may be transformed into the matrix equations*

$$[\lambda\mathbf{E}_1+\mathbf{A}_1(s_1)]\mathbf{Q}_1(s_1;\lambda)=\mathbf{E}_1-(\beta/E_0)\mathbf{S}_1(s_1)\mathbf{Q}_1(s_1-1;\lambda) \quad (5)$$

and

$$[\lambda\mathbf{E}_N+\sum_{r=1}^{N}\mathbf{A}_N(s_r)]\mathbf{Q}_N(s_1,\cdots,s_N;\lambda)=\sum_{C_2^N}\mathbf{W}_{N-1}(s_{N-1},s_N)\mathbf{Q}_{N-1}(s_1,\cdots,s_{N-2},s_{N-1}+s_N;\lambda)-(\beta/E_0)\sum_{C_1^N}\mathbf{S}_N(s_N)\mathbf{Q}_N(s_1,\cdots,s_{N-1},s_N-1;\lambda),\quad N>1. \quad (6)$$

* For definitions of $A_n(s_r)$ see Eqs. (22) and (23) of reference 5.

The notation is as used in reference 5 with the addition that $\mathbf{S}_N(s_r)$ is the direct product of N 2×2 matrices:

$$\mathbf{S}_N(s_r)=\mathbf{E}_1\times\cdots\times\underset{r\text{th factor}}{\begin{bmatrix}s_r & 0\\0 & 0\end{bmatrix}}\times\cdots\times\mathbf{E}_1. \quad (7)$$

(b) Series Solution for $q_{n,\,m}{}^{(j)}$

The Eqs. (5) and (6) are matrix recurrence relations, the solution of which may be obtained in the form of a power series in (β/E_0). Although this series itself cannot be used directly because of its slow convergence, it is shown in a later section that it may be transformed to give rapidly convergent results.

If

$$Q_{n,m}{}^{(j)}=\sum_{a=0}^{\infty}(-\beta/E_0)^a Q_{n,m,a}{}^{(j)}, \tag{8}$$

$$\mathbf{Q}_N=\sum_{a=0}^{\infty}(-\beta/E_0)^a\mathbf{Q}_{N,a}, \quad N=n+m, \tag{9}$$

then from (5) and (6)

$$\mathbf{Q}_{1,a}(s_1;\lambda)=[\lambda\mathbf{E}_1+\mathbf{A}_1(s_1)]^{-1}\mathbf{S}_1(s_1)\mathbf{Q}_{1,a-1}(s_1-1;\lambda) \tag{10}$$

and

$$\begin{aligned}\mathbf{Q}_{N,a}&(s_1,\cdots,s_N;\lambda)\\ &=\sum_{C_2{}^N}[\lambda\mathbf{E}_N+\sum_{r=1}^{N}\mathbf{A}_N(s_r)]^{-1}\mathbf{W}_{N-1}(s_{N-1},s_N)\\ &\quad\times\mathbf{Q}_{N-1,a}(s_1,\cdots,s_{N-2},s_{N-1}+s_N;\lambda)\\ &\quad+\sum_{C_1{}^N}[\lambda\mathbf{E}_N+\sum_{r=1}^{N}\mathbf{A}_N(s_r)]^{-1}\mathbf{S}_N(s_N)\\ &\quad\times\mathbf{Q}_{N,a-1}(s_1,\cdots,s_{N-1},s_N-1;\lambda),\quad N>1.\end{aligned} \tag{11}$$

For $a=0$, the solution in approximation A is obtained, namely,[5]

$$\begin{aligned}\mathbf{Q}_{N,0}=\Bigg\{\prod_{d=N-1}^{1}\sum_{C_2{}^{d+1}}&[\lambda\mathbf{E}_{d+1}+\mathbf{A}_{d+1}(s_1)+\cdots\\ &+\mathbf{A}_{d+1}(s_d)+\mathbf{A}_{d+1}(s_{d+1}+\cdots+s_N)]^{-1}\\ &\times\mathbf{W}_d(s_d,s_{d+1},+\cdots+s_N)\Bigg\}\\ &\qquad\times[\lambda\mathbf{E}_1+\mathbf{A}_1(s_1+\cdots+s_N)]^{-1}.\end{aligned} \tag{12}$$

To solve (10) and (11) is by no means an easy task because of the recurrence on N and also on a. The general solution given below may be verified by induction:

$$\mathbf{Q}_{N,a}=\Bigg\{\prod_{e=a-1}^{0}\sum_{b(e)=0}^{b(e+1)}\{\prod_{d(e)=b(e+1)}^{b(e)+1}\Sigma'\,\mathbf{F}_{d(e)}\}\Sigma''\,\mathbf{G}_{b(e)}\Bigg\}\times\mathbf{Q}_{b(0)+1,0}, \tag{13a}$$

where

$$\begin{aligned}\mathbf{F}_{d(e)}=[\lambda\mathbf{E}_{d(e)+1}&+\mathbf{A}_{d(e)+1}(s_1)+\cdots+\mathbf{A}_{d(e)+1}(s_{d(e)})\\ &+\mathbf{A}_{d(e)+1}(s_{d(e)+1}+\cdots+s_N-a+1+e)]^{-1}\\ &\times\mathbf{W}_{d(e)}(s_{d(e)},s_{d(e)+1}+\cdots+s_N-a+1+e);\end{aligned} \tag{13b}$$

$$\begin{aligned}\mathbf{G}_{b(e)}=[\lambda\mathbf{E}_{b(e)+1}&+\mathbf{A}_{b(e)+1}(s_1)+\cdots+A_{b(e)+1}(s_{b(e)})\\ &+\mathbf{A}_{b(e)+1}(s_{b(e)+1}+\cdots+s_N-a+1+e)]^{-1}\\ &\times\mathbf{S}_{b(e)+1}(s_{b(e)+1}+\cdots+s_N-a+1+e);\end{aligned} \tag{13c}$$

$\mathbf{Q}_{b(0)+1,0}=\mathbf{Q}_{b(0)+1,0}(s_1,\cdots,s_{b(0)},s_{b(0)+1}+\cdots+s_N-a)$, which is given by (12);

$\Sigma'=\sum_{C_2{}^{d(e)+1}}$ signifies summation over all choices of $s_{d(e)}, s_{d(e)+1}+\cdots+s_N-a+1+e$ from $s_1,\cdots,s_{d(e)}, s_{d(e)+1}+\cdots+s_N-a+1+e$;

$\Sigma''=\sum_{C_1{}^{b(e)+1}}$ signifies summation over all choices of $s_{b(e)+1}+\cdots+s_N-a+1+e$ from $s_1,\cdots,s_{b(e)}, s_{b(e)+1}+\cdots+s_N-a+1+e$; and $b(a)=N-1$.

The order in which the summations and products in (13a) are to be carried out is as follows: first $\prod_e$, then $\sum_{b(e)}$ from the left, next $\prod_{d(e)}$ and finally $\sum_C$ from the right.† To illustrate the notation and result, a simple case will be discussed in a later section.

Equations (13), (9), and (8) determine the solution for $Q_{n,m}{}^{(j)}$. Taking an inverse Laplace-Mellin transform yields

$$\begin{aligned}q_{n,m}{}^{(j)}&(E_0;E_1,\cdots,E_n;E_{n+1},\cdots,E_{n+m};x)\\ &=I_{n+m}E_0{}^{-(n+m)}(E_0/E_1)^{s_1+1}\cdots\\ &\qquad\times(E_0/E_{n+m})^{s_{n+m}+1}\sum_{a=0}^{\infty}(-\beta/E_0)^a\mathfrak{Q}_{n,m,a}{}^{(j)},\end{aligned} \tag{14}$$

with I_{n+m} defined as the operator

$$I_{n+m}=1/(2\pi i)^{n+m}\int_{u_1-i\infty}^{u_1+i\infty}ds_1\cdots\int_{u_{n+m}-i\infty}^{u_{n+m}+i\infty}ds_{n+m}, \tag{15}$$

and

$$\begin{aligned}\mathfrak{Q}_{n,m,a}{}^{(j)}&(s_1,\cdots,s_n;s_{n+1},\cdots,s_{n+m};x)\\ &=(2\pi i)^{-1}\int_{\lambda_0-i\infty}^{\lambda_0+i\infty}e^{\lambda x}Q_{n,m,a}{}^{(j)}\\ &\qquad\times(s_1,\cdots,s_n;s_{n+1},\cdots,s_{n+m};\lambda)d\lambda.\end{aligned} \tag{16}$$

In carrying out the inverse Laplace transform (16), it is convenient to express the inverse matrices appearing in (13) as partial fractions. This can be simply done by using the following result in the theory of resolvents:[10] If $\mathbf{M}$ is a matrix of order p with nondegenerate eigenvalues λ_r, then

$$[\lambda\mathbf{E}+\mathbf{M}]^{-1}=\sum_{r=1}^{p}\frac{1}{\lambda+\lambda_r}\prod_{k\neq r}\frac{[\lambda_k\mathbf{E}-\mathbf{M}]}{\lambda_k-\lambda_r}. \tag{17}$$

The eigenvalues of the matrices appearing in (13) may be easily obtained. For example, consider the matrix

$$\sum_{r=1}^{N}\mathbf{A}_N(s_r),$$

where $\mathbf{A}_N(s_r)$ is the direct product of N matrices of

† The summation Σ_C refers to Σ' and Σ''. Furthermore, $b(e)$ and $d(e)$ are dummy variables.

[10] Frazer, Duncan, and Collar, *Elementary Matrices* (Cambridge University Press, London, England, 1938).

order 2:[5]

$$\mathbf{A}_N(s_r)=\mathbf{E}_1\times\cdots\times\underset{r\text{th factor}}{\begin{bmatrix}A_1(s_r) & A_2(s_r)\\ A_3(s_r) & A_4(s_r)\end{bmatrix}}\times\cdots\times\mathbf{E}_1. \quad (18)$$

The N matrices $\mathbf{A}_N(s_r)$, $r=1,\cdots,N$, form a set of commuting matrices, and hence the eigen values of their sum are the sums of the eigenvalues of each. If the eigenvalues of the 2×2 matrix

$$\begin{bmatrix}A_1(s_r) & A_2(s_r)\\ A_3(s_r) & A_4(s_r)\end{bmatrix}$$

are $\lambda_1(s_r)$ and $\lambda_2(s_r)$, then the eigenvalues of the direct product $\mathbf{A}_n(s_r)$, being the products of the eigenvalues of the factors, are $\lambda_1(s_r)$ or $\lambda_2(s_r)$ according to whether, in the 2^N binary numbers $s_1s_2\cdots s_N$ with digits 1 and 2, the rth digit is a 1 or 2. Hence the eigenvalues of

$$\sum_{r=1}^{N}\mathbf{A}_N(s_r)$$

are

$$\lambda_1(s_1)+\lambda_1(s_2)+\cdots+\lambda_1(s_{N-1})+\lambda_1(s_N),$$

$$\lambda_1(s_1)+\lambda_1(s_2)+\cdots+\lambda_1(s_{N-1})+\lambda_2(s_N),$$

etc., which may be read off from the binary numbers

$$\begin{matrix}1 & 1 & \cdots & 1 & 1\\ 1 & 1 & \cdots & 1 & 2,\end{matrix}$$

etc. A table of numerical values for $\lambda_1(s)$ and $\lambda_2(s)$ has already been given by Janossy and Messel.[11]

The solution for the (n,m)th factorial moments, derived from the $q_{n,m}{}^{(j)}$, will be given after the differential moment function for the proton-neutron cascade has been determined.

3. THE PROTON-NEUTRON CASCADE IN A FINITE ABSORBER

(a) Preliminaries

The analytical solution for the moments of the nucleon cascade in approximation A has been given by Messel and Potts.[3] Some of their results required for the present work are quoted below.

If $y_N(E_0;E_1,\cdots,E_N;\theta)$ is the differential moment function for nucleons, then the Laplace-Mellin transform of the diffusion equation for y_N is

$$\{\lambda+h(s_1)\}Y_1(s_1;\lambda)=1, \quad (19)$$

$$\{\lambda+\sum_{r=1}^{N}h(s_r)\}Y_N(s_1,\cdots,s_N;\lambda)$$
$$=\sum_{k=2}^{N}\sum_{(k)}B_k(s_1',\cdots,s_k')Y_{N-k+1}$$
$$\times(s_{k+1}',\cdots,s_N',s_1'+\cdots+s_k';\lambda),\quad N>1, \quad (20)$$

[11] L. Janossy and H. Messel, Proc. Roy. Irish Acad. **A54**, 245 (1951).

where

(a) Y_N is the Laplace-Mellin transform of y_N as in Eq. (2).

(b) $B_N(s_1,\cdots,s_N)$ is the N-fold Mellin transform of the distribution function $b_N(E_0;E_1,\cdots,E_N)$ giving the probability that a primary nucleon of energy E_0 collides with a nucleus giving rise to N nucleons with energies E_k, dE_k and any number of nucleons with arbitrary energies.

(c) $h(s)=1-B_1(s)$.

(d) $\sum_{(k)}$ signifies summation over all compositions of $s_1,\cdots,s_N$ into the two groups $s_1',\cdots,s_k'$ and $s_{k+1}',\cdots,s_N'$.

The solution of (19) and (20) is

$$Y_N(s_1,\cdots,s_N;\lambda)$$
$$=\sum_{c(N-1)}\{\prod_{d=t}^{1}\sum{}'(\lambda+h(s_1)+\cdots+h(s_{q(d)})$$
$$+h(s_{q(d)+1}+\cdots+s_N))^{-1}B_{c(d)+1}$$
$$\times(s_{q(d-1)+1},\cdots,s_{q(d)},s_{q(d)+1}+\cdots+s_N))\}$$
$$\times\{\lambda+h(s_1+\cdots+s_N)\}^{-1}, \quad (21)$$

where $\sum_{c(N-1)}$ signifies summation over the 2^{N-2} compositions of $N-1$, c being the composition $c(1)$, $c(2)$, $\cdots$, $c(t)$, with $q(d)=c(1)+\cdots+c(d)$; and $\sum'$ signifies summation over all combinations of the $q(d)+1$ symbols $s_1,\cdots,s_{q(d)},s_{q(d)+1}+\cdots+s_N$ taken $c(d)+1$ at a time.

(b) The Diffusion Equations in Approximation B

The diffusion equations for the differential moment functions appearing in a proton-neutron cascade theory are derived in a manner analogous to that used for the electron-photon case. Let $y_{n,m}{}^{(j)}(E_0;E_1,\cdots,E_n;E_{n+1},\cdots,E_{n+m};\theta)$ be the differential moment function expressing the probability that after a depth θ (measured in interaction mean free paths) in dispersed matter a primary (j) of energy E_0 has given rise to n protons in the energy ranges E_k, dE_k, $k=1,\cdots,n$ in any order, to m neutrons with energies in the ranges E_{n+l}, dE_{n+l}, $l=1,\cdots,m$ in any order, and to any numbers of protons and neutrons with arbitrary energies. For $j=1$, the primary is a proton, for $j=2$, a neutron. Furthermore, we take $b_{n,m}{}^{(j)}(E_0;E_1,\cdots,E_n;E_{n+1},\cdots,E_{n+m})$ as the corresponding differential moment function for nucleon-nucleus collisions. Assuming that protons suffer a constant energy loss β by ionization, the last-collision diffusion equation satisfied by $y_{n,m}{}^{(j)}$ is

$$(\partial/\partial\theta+n+m)y_{n,m}^{(j)}(E_0;E_1,\cdots,E_n;E_{n+1},\cdots,E_{n+m};\theta)$$

$$=\sum_{\substack{k=0\\k+l>0}}^{n}\sum_{l=0}^{m}\sum_{(k)}\sum_{(l)}\int_0^\infty\{b_{k,l}^{(1)}(U;E_1',\cdots,E_k';E_{n+1}',\cdots,E_{n+l}')y_{n-k+1,\,m-l}^{(j)}$$

$$\times(E_0;E_{k+1}',\cdots,E_n',U;E_{n+l+1}',\cdots,E_{n+m}';\theta)+b_{k,l}^{(2)}(U;E_1',\cdots,E_k';E_{n+1}',\cdots,E_{n+l}')y_{n-k,\,m-l+1}^{(j)}$$

$$\times(E_0;E_{k+1}',\cdots,E_n';E_{n+l+1}',\cdots,E_{n+m}',U;\theta)\}dU+\beta\sum_{k=1}^{n}\frac{\partial}{\partial E_k}y_{n,m}^{(j)}(E_0;E_1,\cdots,E_n;E_{n+1},\cdots,E_{n+m};\theta).\quad(22)$$

If we define the Laplace-Mellin transforms of $y_{n,m}^{(j)}$ and $b_{n,m}^{(j)}$ as

$$Y_{n,m}^{(j)}(s_1,\cdots,s_n;s_{n+1},\cdots,s_{n+m};\lambda)=\int_0^\infty dE_1\cdots\int_0^\infty dE_{n+m}\int_0^\infty d\theta(E_1/E_0)^{s_1}\cdots$$

$$\times(E_{n+m}/E_0)^{s_{n+m}}e^{-\lambda\theta}y_{n,m}^{(j)}B_{n,m}^{(j)}(s_1,\cdots,s_n;s_{n+1},\cdots,s_{n+m})\quad(23)$$

$$=\int_0^\infty dE_1\cdots\int_0^\infty dE_{n+m}(E_1/E_0)^{s_1}\cdots(E_{n+m}/E_0)^{s_{n+m}}b_{n,m}^{(j)},\quad(24)$$

then the Laplace-Mellin transform of (22) is

$$(\lambda+n+m)Y_{n,m}^{(j)}(s_1,\cdots,s_n;s_{n+1},\cdots,s_{n+m};\lambda)-\delta_{n+j,}\,\delta_{n+j,2}$$

$$=\sum_{k=0}^{n}\sum_{l=0}^{m}\sum_{(k)}\sum_{(l)}\{B_{k,l}^{(1)}(s_1',\cdots,s_k';s_{n+1}',\cdots,s_{n+l}')Y_{n-k+1,\,m-l}^{(j)}(s_{k+1}',\cdots,s_n',s_1'+\cdots+s_k'$$

$$+s_{n+1}'+\cdots+s_{n+l}';s_{n+l+1}',\cdots,s_{n+m}';\lambda)+B_{k,l}^{(2)}(s_1',\cdots,s_k';s_{n+1}',\cdots,s_{n+l}')$$

$$\times Y_{n-k,\,m-l+1}^{(j)}(s_{k+1}',\cdots,s_n';s_{n+l+1}',\cdots,s_{n+m}',s_1'+\cdots+s_k'+s_{n+1}'+\cdots+s_{n+l}';\lambda)\}$$

$$-(\beta/E_0)\sum_{C_1^n}s_nY_{n,m}^{(j)}(s_1,\cdots,s_{n-1},s_n-1;s_{n+1},\cdots,s_{n+m};\lambda).\quad(25)$$

Just as the transformed diffusion equation for the electron-photon cascade was written in the matrix form, (5) and (6), so (25) may be written

$$[\lambda\mathbf{E}_1+\mathbf{h}_1(s_1)]\mathbf{Y}_1(s_1;\lambda)=\mathbf{E}_1-(\beta/E_0)\mathbf{S}_1(s_1)\mathbf{Y}_1(s_1-1;\lambda),\quad(26)$$

$$[\lambda\mathbf{E}_N+\sum_{r=1}^{N}\mathbf{h}_N(s_r)]\mathbf{Y}_N(s_1,\cdots,s_N;\lambda)=\sum_{k=2}^{N}\sum_{(k)}\mathbf{B}_{N,k}(s_1',\cdots,s_k')\mathbf{Y}_{N-k+1}(s_{k+1}',\cdots,s_N',s_1'+\cdots+s_k';\lambda)$$

$$-(\beta/E_0)\sum_{C_1^N}\mathbf{S}_N(s_N)\mathbf{Y}_N(s_1,\cdots,s_{N-1},s_N-1;\lambda),\quad N>1.\quad(27)$$

The matrix $\mathbf{h}_N(s_r)$ is the direct product of N 2×2 matrices:

$$\mathbf{h}_N(s_r)=\mathbf{E}_1\times\cdots\times\underset{r\text{th factor}}{\mathbf{h}(s_r)}\times\cdots\times\mathbf{E}_1,\quad(28)$$

where $\mathbf{h}(s)$ is defined as

$$\mathbf{h}(s)=\begin{bmatrix}1-B_{1,0}^{(1)}(s) & -B_{1,0}^{(2)}(s)\\ -B_{0,1}^{(1)}(s) & 1-B_{0,1}^{(2)}(s)\end{bmatrix}.\quad(29)$$

The matrix $\mathbf{Y}_N$ is a $2^N\times2$ matrix the columns of which correspond to $Y^{(1)}$ and $Y^{(2)}$, and the rows are ordered by the binary numbers $s_1\cdots s_N$ with digits 1, 2 in the same way as $\mathbf{Q}_N$ was formed for the electron-photon cascade.[5] The matrix $\mathbf{B}_{N,k}(s_1,\cdots,s_k)$ is a $2^N\times2^{N-k+1}$ matrix in which the rows are ordered by the binary numbers $s_1\cdots s_N$. The nonzero elements are arranged according to the following rule: if in the binary number $s_1\cdots s_N$ it occurs that $s_1'=s_2'=\cdots=s_r'=1$, and $s_{r+1}'=s_{r+2}'=\cdots=s_k'=2$, then all the elements of the row are zero except for the terms $B_{r,k-r}^{(1)}(s_1',\cdots,s_r';s_{r+1}',\cdots,s_k')$ and $B_{r,k-r}^{(2)}(s_1',\cdots,s_r';s_{r+1}',\cdots,s_k')$, which are placed in the first odd-numbered and first even-numbered columns, respectively, in which these terms have not already appeared. For example,

$$\mathbf{B}_{2,2}(s_1,s_2)=\begin{Bmatrix}B_{2,0}^{(1)}(s_1,s_2) & B_{2,0}^{(2)}(s_1,s_2)\\ B_{1,1}^{(1)}(s_1;s_2) & B_{1,1}^{(2)}(s_1;s_2)\\ B_{1,1}^{(1)}(s_2;s_1) & B_{1,1}^{(2)}(s_2;s_1)\\ B_{0,2}^{(1)}(s_1,s_2) & B_{0,2}^{(2)}(s_1,s_2)\end{Bmatrix},\quad(30a)$$

$$\mathbf{B}_{3,2}(s_1,s_2)=\begin{bmatrix} B_{2,0}{}^{(1)}(s_1,s_2) & B_{2,0}{}^{(2)}(s_1,s_2) & 0 & 0 \\ 0 & 0 & B_{2,0}{}^{(1)}(s_1,s_2) & B_{2,0}{}^{(2)}(s_1,s_2) \\ B_{1,1}{}^{(1)}(s_1;s_2) & B_{1,1}{}^{(2)}(s_1;s_2) & 0 & 0 \\ 0 & 0 & B_{1,1}{}^{(1)}(s_1;s_2) & B_{1,1}{}^{(2)}(s_1;s_2) \\ B_{1,1}{}^{(1)}(s_2;s_1) & B_{1,1}{}^{(2)}(s_2;s_1) & 0 & 0 \\ 0 & 0 & B_{1,1}{}^{(1)}(s_2;s_1) & B_{1,1}{}^{(2)}(s_2;s_1) \\ B_{0,2}{}^{(1)}(s_1,s_2) & B_{0,2}{}^{(2)}(s_1,s_2) & 0 & 0 \\ 0 & 0 & B_{0,2}{}^{(1)}(s_1,s_2) & B_{0,2}{}^{(2)}(s_1,s_2) \end{bmatrix}. \tag{30b}$$

(c) Solution for $y_{n,m}{}^{(j)}$, $\beta=0$

If we set $\beta=0$ in Eqs. (26) and (27) and write $\mathbf{Y}_{N,0}$ for $\mathbf{Y}_N$ in this case, we get

$$[\lambda\mathbf{E}_1+\mathbf{h}_1(s_1)]\mathbf{Y}_{1,0}(s_1;\lambda)=\mathbf{E}_1, \tag{31}$$

$$[\lambda\mathbf{E}_N+\sum_{r=1}^{N}\mathbf{h}_N(s_r)]\mathbf{Y}_{N,0}(s_1,\cdots,s_N;\lambda)$$
$$=\sum_{k=2}^{N}\sum_{(k)}\mathbf{B}_{N,k}(s_1',\cdots,s_k')\mathbf{Y}_{N,0}(s_{k+1}',\cdots,$$
$$\times s_N',s_1'+\cdots+s_k';\lambda),\quad N>1. \tag{32}$$

These are the transforms of the last-collision diffusion equation for the proton-neutron cascade in approximation A. Their solution is

$$\mathbf{Y}_{N,0}(s_1,\cdots,s_N;\lambda)$$
$$=\sum_{c(N-1)}\{\prod_{d=t}^{1}\sum{}'[\lambda\mathbf{E}_{q(d)+1}+\mathbf{h}_{q(d)+1}(s_1)+\cdots$$
$$+\mathbf{h}_{q(d)+1}(s_{q(d)})+\mathbf{h}_{q(d)+1}(s_{q(d)+1}+\cdots+s_N)]^{-1}$$
$$\times\mathbf{B}_{q(d)+1,c(d)+1}(s_{q(d-1)+1},\cdots,s_{q(d)},s_{q(d)+1}$$
$$+\cdots+s_N)\}[\lambda\mathbf{E}_1+\mathbf{h}_1(s_1+\cdots+s_N)]^{-1}, \tag{33}$$

with the same notation as in (21). The solution for $y_{n,m}{}^{(j)}$, $\beta=0$, is

$$y_{n,m,0}{}^{(j)}=I_{n+m}E_0{}^{-(n+m)}(E_0/E_1)^{s_1+1}\cdots$$
$$\times(E_0/E_{n+m})^{s_{n+m}+1}\mathfrak{Y}_{n,m,0}{}^{(j)}, \tag{34}$$

where $\mathfrak{Y}_{n,m,0}{}^{(j)}$ is the inverse Laplace transform of $Y_{n,m,0}{}^{(j)}$ as given by (33).

The matrix Eqs. (31), (32), and their solution (33) for the proton-neutron cascade in approximation A are very similar in form to Eqs. (19), (20), and their solution (21) for the nucleon cascade in which the protons are not distinguished from the neutrons. It is the power of the matrix method that the matrices alone keep the protons separated from the neutrons. Once the equations for the proton-neutron cascade have been written in matrix form the difference between the protons and neutrons is hidden. Even the "trees" used to describe the solution for the nucleon cascade[3] can be used as they stand for the proton-neutron cascade in approximation A.

This matrix method is a general one and has already been applied by us[5] in electron-photon cascade theory in approximation A. Here again the matrices alone take account of the differences between the electrons and photons and the different cross sections for *bremsstrahlung* and pair production. In matrix notation the equations and solutions become analogous to those for the proton-neutron cascade in homogeneous nuclear matter, in which just one type of particle cascades by collisions with single particles of the same type.

(d) Series Solution for $y_{n,m}{}^{(j)}$, $\beta\neq0$

Equations (26) and (27) are matrix recurrence relations, the solution of which may be obtained in a manner similar to that used in Sec. 2(b) for $q_{n,m}{}^{(j)}$. Set

$$Y_{n,m}{}^{(j)}=\sum_{a=0}^{\infty}(-\beta/E_0)^aY_{n,m,a}{}^{(j)}, \tag{35}$$

and hence

$$\mathbf{Y}_N=\sum_{a=0}^{\infty}(-\beta/E_0)^a\mathbf{Y}_{N,a}. \tag{36}$$

From (26) and (27),

$$\mathbf{Y}_{1,a}(s_1;\lambda)=[\lambda\mathbf{E}_1+\mathbf{h}_1(s_1)]^{-1}\mathbf{S}_1(s_1)\mathbf{Y}_{1,a-1}(s_1-1;\lambda) \tag{37}$$

and

$$\mathbf{Y}_{N,a}(s_1,\cdots,s_N;\lambda)$$
$$=\sum_{k=2}^{N}\sum_{(k)}[\lambda\mathbf{E}_N+\sum_{r=1}^{N}\mathbf{h}_N(s_r)]^{-1}\mathbf{B}_{N,k}(s_1',\cdots,s_k')$$
$$\times\mathbf{Y}_{N-k+1,a}(s_{k+1}',\cdots,s_N',s_1'+\cdots+s_k';\lambda)$$
$$+\sum_{C_1{}^N}[\lambda\mathbf{E}_N+\sum_{r=1}^{N}\mathbf{h}_N(s_r)]^{-1}\mathbf{S}_N(s_N)\mathbf{Y}_{N,a-1}$$
$$\times(s_1,\cdots,s_{N-1},s_N-1;\lambda),\quad N>1. \tag{38}$$

For $a=0$, we obtain the solution for the proton-neutron cascade in approximation A as given by (33). The general solution for $\mathbf{Y}_{N,a}$ is [compare Eq. (13)],

$$\mathbf{Y}_{N,a}=\{\prod_{e=a-1}^{0}\sum_{b(e)=0}^{b(e+1)}\{\sum{}'\prod_{d=t}^{1}\sum{}''\,\mathbf{H}_d\}\sum{}'''\,\mathbf{G}_{b(e)}\}\mathbf{Y}_{b(0)+1,0}, \tag{39}$$

where

$$\mathbf{H}_d=[\lambda\mathbf{E}_{q(d)+b(e)+1}+\mathbf{h}_{q(d)+b(e)+1}(s_1)+\cdots$$
$$+\mathbf{h}_{q(d)+b(e)+1}(s_{q(d)+b(e)})+h_{q(d)+b(e)+1}$$
$$\times(s_{q(d)+b(e)+1}+\cdots+s_N-a+1+e)]^{-1}$$
$$\times\mathbf{B}_{q(d)+b(e)+1,c(d)+1}(s_{q(d-1)+b(e)+1},\cdots,$$
$$\times s_{q(d)+b(e)},s_{q(d)+b(e)+1}+\cdots+s_N-a+1+e). \tag{40}$$

$\mathbf{G}_{b(e)}$ is as given by (13b) with the matrix $\mathbf{A}_k(s_r)$ replaced by $\mathbf{h}_k(s_r)$.

$\mathbf{Y}_{b(0)+1,0}=\mathbf{Y}_{b(0)+1,0}(s_1,\cdots,s_{b(0)},s_{b(0)+1}+\cdots+s_N-a;\lambda)$ as given by (33).

$\sum'=\sum_c\{b(e+1)-b(e)\}$ signifies summation over all compositions of $\{b(e+1)-b(e)\}$, c being the composition $c(1), c(2), \cdots, c(t)$ with $q(d)=c(1)+\cdots+c(d)$.

$\sum''=\sum_{C_{c(d)+1}q(d)+b(e)+1}$ signifies summation over all choices of $s_{q(d)+b(e)}, s_{q(d)+b(e)+1}+\cdots+s_N-a+1+e$ from $s_1, \cdots, s_{q(d)+b(e)}, s_{q(d)+b(e)+1}+\cdots+s_N-a+1+e$, and

$\sum'''=_{C_1 b(e)+1}$ signifies summation over all choices of $s_{b(e)+1}+\cdots+s_N-a+1+e$ from $s_1, \cdots, s_{b(e)}, s_{b(e)+1}+\cdots+s_N-a+1+e$.

The order in which the summations and products in (39) are to be carried out is as follows: first $\prod_e$, then $\sum_{b(e)}$ from the left, next $\sum'$, then $\prod_d$ and $\sum_C$ from the right.

Inverting the Mellin transform we obtain the solution for $y_{n,m}^{(j)}$ in the form

$$y_{n,m}^{(j)}(E_0;E_1,\cdots,E_n;E_{n+1},\cdots,E_{n+m};\theta)$$
$$=I_{n+m}E_0^{-(n+m)}(E_0/E_1)^{s_1+1}\cdots(E_0/E_{n+m})^{s_{n+m}+1}$$
$$\times\sum_{a=0}^{\infty}(-\beta/E_0)^a\mathfrak{Y}_{n,m,a}^{(j)}, \quad (41)$$

where

$$\mathfrak{Y}_{n,m,a}^{(j)}(\theta)=\frac{1}{2\pi i}\int_{\lambda_0-i\infty}^{\lambda_0+i\infty}e^{\lambda\theta}Y_{n,m,a}^{(j)}(\lambda)d\lambda.$$

The same method as described in Sec. 2(b) may be used to perform this inverse Laplace transformation.

4. SOLUTIONS FOR THE (n, m)th FACTORIAL MOMENTS $T_{n,m}^{(j)}$

If $T_{n,m}^{(j)}(E_0;E,\beta;x)$ is the (n,m)th factorial moment of the distribution function $\varphi_{n,m}^{(j)}$ giving the probability of finding specified numbers of electrons and photons above a given energy E and an arbitrary number of electrons and photons with energies less than E, then[4,5]

$$T_{n,m}^{(j)}(E_0;E,\beta;x)=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{(n+a)!}{a!}\frac{(m+b)!}{b!}$$
$$\times\varphi_{n+a,m+b}^{(j)}(E_0;E,\beta;x), \quad (42)$$

by definition. The solution for $T_{n,m}^{(j)}$ is obtained immediately by making use of the relation[4]

$$T_{n,m}^{(j)}(E_0;E,\beta;x)=\int_E^\infty dE_1\cdots\int_E^\infty dE_{n+m}q_{n,m}^{(j)}; \quad (43)$$

hence from (13),

$$T_{n,m}^{(j)}(E_0;E,\beta;x)$$
$$=I_{n+m}(E_0/E)^{s_1+\cdots+s_{n+m}}\sum_{a=0}^{\infty}(-\beta/E_0)^a$$
$$\times\mathfrak{Q}_{n,m,a}^{(j)}(s_1,\cdots,s_n;s_{n+1},\cdots,$$
$$\times s_{n+m};x)/s_1\cdots s_{n+m}. \quad (44)$$

An identical set of relations holds between the (n,m)th factorial moments and $y_{n,m}^{(j)}$ for the proton-neutron cascade. Thus,

$$T_{n,m}^{(j)}(E_0;E,\beta;\theta)=\int_E^\infty dE_1\cdots\int_E^\infty dE_{n+m}y_{n,m}^{(j)}(\theta) \quad (45)$$

and, using (39),

$$T_{n,m}^{(j)}(E_0;E,\beta;\theta)$$
$$=I_{n+m}(E_0/E)^{s_1+\cdots+s_{n+m}}\sum_{a=0}^{\infty}(-\beta/E_0)^a$$
$$\times\mathfrak{Y}_{n,m,a}(s_1,\cdots,s_n;s_{n+1},\cdots,$$
$$\times s_{n+m};\theta)/s_1\cdots s_{n+m}. \quad (46)$$

The solutions (44) and (46) for the (n,m)th factorial moments are of little value for computational purposes because of their slow convergence. They may, however, be transformed into a series which is rapidly converging. The method is a generalization of that used by Bhabha and Chakrabarty[6,7] and Messel[8,9] for the electron-photon and proton-neutron cascades, respectively. The Bhabha-Chakrabarty method has been severely criticised,[12,13] but Messel[14] has pointed out that these criticisms were inapplicable and that the Bhabha-Chakrabarty method of solving for the first moments in approximation B is the best available and gives the most reliable results.

The following development will be carried through for just the electron-photon cascade; an identical treatment may be given for the proton-neutron case.

By writing $s_k+a/(n+m)$ for s_k in (44) and by suitably changing the contour of integration, we obtain

$$T_{n,m}^{(j)}(E_0;E,\beta;x)$$
$$=I_{n+m}\sum_{a=0}^{\infty}(E_0/E)^{s_1+\cdots+s_{n+m}+a}(-\beta/E_0)^a$$
$$\times\left\{\left(s_1+\frac{a}{n+m}\right)\cdots\left(s_{n+m}+\frac{a}{n+m}\right)\right\}^{-1}$$
$$\times\mathfrak{Q}_{n,m,a}\left(s_1+\frac{a}{n+m},\cdots,s_n+\frac{a}{n+m};\right.$$
$$\left.\times s_{n+1}+\frac{a}{n+m},\cdots,s_{n+m}+\frac{a}{n+m};x\right). \quad (47)$$

For E we write $\{(E+\beta g)-\beta g\}$, where

$$g=g_{n,m}^{(j)}(s_1,\cdots,s_n;s_{n+1},\cdots,s_{n+m};x) \quad (47a)$$

is as yet an arbitrary function, to be determined at a later stage. Thus

$$(1/E)^{s_1+\cdots+s_{n+m}+a}=\sum_{b=0}^{\infty}\frac{\Gamma(s_1+\cdots+s_{n+m}+a+b+1)}{b!\Gamma(s_1+\cdots+s_{n+m}+a+1)}$$
$$\times(\beta g)^b(E+\beta g)^{-(s_1+\cdots+s_{n+m}+a+b)}. \quad (48)$$

[12] I. E. Tamm and S. Belenky, Phys. Rev. **70**, 660 (1946).
[13] H. S. Snyder, Phys. Rev. **76**, 1563 (1949).
[14] H. Messel, Phys. Rev. **82**, 259 (1951).

By introducing (48) into (47) and setting $b=c-a$ we get

$$T_{n,m}^{(i)}(E_0;E,\beta;x)=I_{n+m}\sum_{c=0}^{\infty}\left(\frac{E_0}{E+\beta g}\right)^{s_1+\cdots+s_{n+m}}\left(\frac{\beta}{E+\beta g}\right)^{c}\times\frac{\Gamma(s_1+\cdots+s_{n+m}+c+1)}{\Gamma(s_1+\cdots+s_{n+m}+1)}f_{n,m,c}^{(i)},\quad(49)$$

where

$$f_{n,m,c}^{(i)}(s_1,\cdots,s_n;s_{n+1},\cdots,s_{n+m};x)=\sum_{a=0}^{c}\frac{(-1)^a g^{c-a}}{(c-a)!}\frac{\Gamma(s_1+\cdots+s_{n+m}+1)}{\Gamma(s_1+\cdots+s_{n+m}+a+1)}\cdot\left\{\left(s_1+\frac{a}{n+m}\right)\cdots\left(s_{n+m}+\frac{a}{n+m}\right)\right\}^{-1}\times\mathfrak{Q}_{n,m,a}^{(i)}\left(s_1+\frac{a}{n+m},\cdots;\cdots,\times s_{n+m}+\frac{a}{n+m};x\right).\quad(50)$$

In particular,

$$f_{n,m,0}^{(i)}=\mathfrak{Q}_{n,m}^{(i)}(s_1,\cdots,s_n;\times s_{n+1},\cdots,s_{n+m};x)/s_1\cdots s_{n+m}.\quad(51)$$

The function g is so chosen that $f_{n,m,1}^{(i)}=0$, i.e.,

$$g_{n,m}^{(i)}(s_1,\cdots,s_n;s_{n+1},\cdots,s_{n+m};x)=\{s_1s_2\cdots s_{n+m}\}(s_1+\cdots+s_{n+m})^{-1}\times\{s_1+(n+m)^{-1}\}^{-1}\cdots\{s_{n+m}+(n+m)^{-1}\}^{-1}\times\mathfrak{Q}_{n,m,1}^{(i)}\left(s_1+\frac{1}{n+m},\cdots;\cdots,s_{n+m}+\frac{1}{n+m};x\right)\times\{\mathfrak{Q}_{n,m,0}^{(i)}(s_1,\cdots,s_n;s_{n+1},\cdots,s_{n+m};x)\}^{-1}.\quad(52)$$

The general series solution (49) for the factorial moments may now be used for numerical work in the case of small n and m. For $E\approx\beta$ and $E_0\gg\beta$ the following formula gives approximate results for the moments

$$T_{n,m}^{(i)}(E_0;E,\beta;x)\approx[E/(E+\beta g)]^{s_1+\cdots+s_{n+m}}T_{n,m}^{(i)}(E_0;E,\beta=0;x),\quad(53)$$

where $[E/(E+\beta g)]^{s_1+\cdots+s_{n+m}}$ is a correction factor multiplying $T_{n,m}^{(i)}(\beta=0)$, the solution for the moments in approximation A.[6–9]

This completes the solution for the (n,m)th factorial moments in approximation B, both for the electron-photon and proton-neutron cascades. Together with the results presented,[1–5] it constitutes a complete analytical solution of the fluctuation problems arising in electron-photon and proton-neutron cascade theory. It is true that the solutions obtained in approximation A were exceedingly complex and that those just presented were more complex still; however, this is not surprising when one appreciates the complexity of the problems solved. It should be recalled that the solution of the cascade fluctuation problem demands a mathematical representation of every possible event, with suitable weight factors, which may take place in building up the cascade. When one further realizes that in many large air showers one often deals with millions of particles, the results are not unduly complicated. One may wonder what possibility there is of using the solutions presented for computational purposes. Numerical results for the 1st moments in both approximation A and B are easily obtained and have been given for instance by Janossy and Messel,[11] Bhabha and Chakrabarty,[6,7] and Messel.[8,9] The second moments in approximation A have also been calculated, both for the electron-photon and nucleon cascades (see Janossy and Messel[15] and Messel[16]); in approximation B results using (53) will be given in a subsequent publication. With the aid of an electronic brain it is proposed to calculate the third moments as well and then to use the first three moments to reconstruct the appropriate distribution functions (see Green and Messel[17]). This at present appears to be a more satisfactory method of attack than attempting to evaluate directly the analytical solutions given for the distribution functions.

In the next section we will give a simple example of the matrix method used in this paper.

5. THE FIRST MOMENTS OF THE ELECTRON-PHOTON CASCADE

To give a simple example of the Laplace transform and matrix method used in this paper, the g functions and expressions for the first moments of the electron-photon cascade will be derived.

To obtain the g function defined by (52) we require $\mathbf{Q}_{1,0}$ and $\mathbf{Q}_{1,1}$. From (12),

$$\mathbf{Q}_{1,0}(s;\lambda)=[\lambda\mathbf{E}_1+\mathbf{A}_1(s)]^{-1},\quad(54)$$

where

$$\mathbf{A}_1(s)=\begin{bmatrix}A_1(s) & A_2(s)\\ A_3(s) & A_4\end{bmatrix}.\quad(55)$$

Hence

$$\mathbf{Q}_{1,0}(s;\lambda)=\{(\lambda+\lambda_1(s))(\lambda_2(s)-\lambda_1(s))\}^{-1}\times[\lambda_2(s)\mathbf{E}_1-\mathbf{A}_1(s)]+\{(\lambda+\lambda_2(s))(\lambda_1(s)-\lambda_2(s))\}^{-1}\times[\lambda_1(s)\mathbf{E}_1-A_1(s)],\quad(56)$$

where $\lambda_1(s)$ and $\lambda_2(s)$ are the eigenvalues of $\mathbf{A}_1(s)$.

[15] L. Janossy and H. Messel, Proc. Phys. Soc. (London) **A63**, 1101 (1950).
[16] H. Messel, *Progress in Cosmic Ray Physics* (North Holland Publishing Company, Amsterdam, 1952), Vol. 2.
[17] H. S. Green and H. Messel, Proc. Cambridge Phil. Soc. (to be published).

An inverse Laplace transform of (56) yields

$$\mathfrak{Q}_{1,0}(s;x)=\begin{bmatrix}\mathfrak{Q}_{1,0,0}^{(1)}(s;x) & \mathfrak{Q}_{1,0,0}^{(2)}(s;x)\\ \mathfrak{Q}_{0,1,0}^{(1)}(s;x) & \mathfrak{Q}_{0,1,0}^{(2)}(s;x)\end{bmatrix}$$
$$=\frac{e^{-\lambda_1(s)x}}{\lambda_2(s)-\lambda_1(s)}\begin{bmatrix}\lambda_2(s)-A_1(s) & -A_2(s)\\ -A_3(s) & \lambda_2(s)-A_4\end{bmatrix}+\frac{e^{-\lambda_2(s)x}}{\lambda_1(s)-\lambda_2(s)}\begin{bmatrix}\lambda_1(s)-A_1(s) & -A_2(s)\\ -A_3(s) & \lambda_1(s)-A_4\end{bmatrix}. \quad (57)$$

From (13),

$$\mathbf{Q}_{1,1}(s;\lambda)=[\lambda\mathbf{E}_1+\mathbf{A}_1(s)]^{-1}\mathbf{S}_1(s)[\lambda\mathbf{E}_1+\mathbf{A}_1(s-1)]^{-1}, \quad (58)$$

$$\mathbf{Q}_{1,1}(s;\lambda)=s\{(\lambda+\lambda_1(s))(\lambda+\lambda_2(s))\times(\lambda+\lambda_1(s-1))(\lambda+\lambda_2(s-1))\}^{-1}\times\begin{bmatrix}(\lambda+A_4)^2 & -A_2(s-1)(\lambda+A_4)\\ -A_3(s)(\lambda+A_4) & A_3(s)A_2(s-1)\end{bmatrix}. \quad (59)$$

Taking the inverse Laplace transform of (59) we find

$$\mathfrak{Q}_{1,1}(s;x)=\begin{bmatrix}\mathfrak{Q}_{1,0,1}^{(1)}(s;x) & \mathfrak{Q}_{1,0,1}^{(2)}(s;x)\\ \mathfrak{Q}_{0,1,1}^{(1)}(s;x) & \mathfrak{Q}_{0,1,1}^{(2)}(s;x)\end{bmatrix}$$
$$=\frac{se^{-\lambda_1(s)x}}{\{\lambda_2(s)-\lambda_1(s)\}\{\lambda_1(s-1)-\lambda_1(s)\}\{\lambda_2(s-1)-\lambda_1(s)\}}\begin{bmatrix}\{\lambda_1(s)-A_4\}^2 & A_2(s-1)\{\lambda_1(s)-A_4\}\\ A_3(s)\{\lambda_1(s)-A_4\} & A_3(s)A_2(s-1)\end{bmatrix}$$
$$+\frac{se^{-\lambda_2(s)x}}{\{\lambda_1(s)-\lambda_2(s)\}\{\lambda_1(s-1)-\lambda_2(s)\}\{\lambda_2(s-1)-\lambda_2(s)\}}\begin{bmatrix}\{\lambda_2(s)-A_4\}^2 & A_2(s-1)\{\lambda_2(s)-A_4\}\\ A_3(s)\{\lambda_2(s)-A_4\} & A_3(s)A_2(s-1)\end{bmatrix}$$
$$+\frac{se^{-\lambda_1(s-1)x}}{\{\lambda_1(s)-\lambda_1(s-1)\}\{\lambda_2(s)-\lambda_1(s-1)\}\{\lambda_2(s-1)-\lambda_1(s-1)\}}\times\begin{bmatrix}\{\lambda_1(s-1)-A_4\}^2 & A_2(s-1)\{\lambda_1(s-1)-A_4\}\\ A_3(s)\{\lambda_1(s-1)-A_4\} & A_3(s)A_2(s-1)\end{bmatrix}$$
$$+\frac{se^{-\lambda_2(s-1)x}}{\{\lambda_1(s)-\lambda_2(s-1)\}\{\lambda_2(s)-\lambda_2(s-1)\}\{\lambda_1(s-1)-\lambda_2(s-1)\}}\times\begin{bmatrix}\{\lambda_2(s-1)-A_4\}^2 & A_2(s-1)\{\lambda_2(s-1)-A_4\}\\ A_3(s)\{\lambda_2(s-1)-A_4\} & A_3(s)A_2(s-1)\end{bmatrix}. \quad (60)$$

Formula (52) now gives the values of $g_{1,0}^{(1)}$, $g_{1,0}^{(2)}$, $g_{0,1}^{(1)}$, and $g_{0,1}^{(2)}$. The result for $g_{1,0}^{(1)}$ agrees with that obtained previously by Bhabha and Chakrabarty[6] in a different manner. The result (60) enables one to calculate $T_{n,m}^{(i)}(E_0;E,\beta;x)$ using (53). To obtain a more accurate value for $T_{n,m}^{(i)}$ one must use (49) and (50); for instance, to carry the approximation (53) one stage further it is necessary to evaluate $f_{n,m,2}^{(i)}$ and hence $\mathfrak{Q}_{1,2}(s;x)$. From (13),

$$\mathbf{Q}_{1,2}(s;\lambda)=[\lambda\mathbf{E}_1+A_1(s)]^{-1}\mathbf{S}_1(s)[\lambda\mathbf{E}_1+\mathbf{A}_1(s-1)]^{-1}\times\mathbf{S}_1(s-1)[\lambda\mathbf{E}_1+\mathbf{A}_1(s-2)], \quad (61)$$

and hence, taking the inverse Laplace transform, we find

$$\mathfrak{Q}_{1,2}(s;x)=\begin{bmatrix}\mathfrak{Q}_{1,0,2}^{(1)}(s;x) & \mathfrak{Q}_{1,0,2}^{(2)}(s;x)\\ \mathfrak{Q}_{0,1,2}^{(1)}(s;x) & \mathfrak{Q}_{0,1,2}^{(2)}(s;x)\end{bmatrix}$$
$$=s(s-1)\sum_{i=1}^{6}\{A_4-\gamma_i(s)\}e^{-\gamma_i(s)x}\times\{\prod_{i\neq k}(\gamma_k(s)-\gamma_i(s))\}^{-1}\times\begin{bmatrix}(\gamma_i(s)-A_4)^2 & A_2(s-2)(\gamma_i(s)-A_4)\\ A_3(s)(\gamma_i(s)-A_4) & A_3(s)A_2(s-2)\end{bmatrix}, \quad (62)$$

where

$$\gamma_1(s)=\lambda_1(s),\quad \gamma_2(s)=\lambda_1(s-1),\quad \gamma_3(s)=\lambda_1(s-2),$$
$$\gamma_4(s)=\lambda_2(s),\quad \gamma_5(s)=\lambda_2(s-2)\text{ and }\gamma_6(s)=\lambda_2(s-2).$$

As mentioned previously, the second moments with numerical calculations will be discussed in a separate paper.