PHYSICAL REVIEW VOLUME 87, NUMBER 5 SEPTEMBER 1, 1952

Recoil Correction to Bremsstrahlung Cross Section*

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(Received May 23, 1952)

A correction to the Bethe-Heitler differential cross section for bremsstrahlung (or pair production) due to the recoil of the source of the Coulomb field (proton) is derived. Terms of order v/c, where v is the recoil velocity of the scattering center, are studied. They are shown not to contribute in the analysis of electron-proton scattering given by Schiff in the preceding paper.

I. DISCUSSION AND RESULTS

BETHE and Heitler[1] have calculated differential and total cross sections for the emission of bremsstrahlung by an electron scattered in a static Coulomb field. They have also obtained formulas for the closely related process of pair production by a gamma-ray in a static Coulomb field.

This paper derives a correction to the Bethe-Heitler differential cross sections due to the recoil of the source of the Coulomb field (proton) during the interaction. The recoil terms are reduced relative to the leading terms of the B-H formulas by a factor $q/Mc \cong v/c$, where c is the velocity of light, and M, q, and v are, respectively, the mass, recoil momentum, and recoil velocity of the scattering center. Thus, the recoil correction may be appreciable in the case of radiative scattering of energetic electrons by protons.[2]

This calculation is motivated by an experiment of Panofsky[3] now in progress at Stanford and by an analysis of this experiment given by Schiff in the preceding paper.[4] In this experiment electrons of several hundred Mev energy are scattered, and the recoil protons are detected with photographic plates. The sum of the nonradiative (elastic) and radiative electron-proton scattering cross sections is measured as a function of the recoil proton angular distribution for proton recoils of all energies greater then a minimum value that we take here to be 1.5 Mev.[5] Lower energy recoils are not detected, so that the experiment discriminates in favor of high momentum proton recoils ($>100\ mc$).

Large momentum transfer to the recoil proton requires the scattered electron or the photon, or both, to make an angle with the direction of the incident electron that is large compared with mc^2/E_0, where E_0 is the initial electron energy. Then, as pointed out in S, the photon exhibits a strong preference to emerge in the direction of either the incident or scattered electron. Utilizing the strong directional correlation of the photon to simplify the formulas, Schiff has integrated the B-H differential bremsstrahlung cross section to obtain a correction to the Mott-Rutherford[6] formula for elastic Coulomb scattering. Joining this result with the Schwinger[7] correction (for the emission of soft quanta and the reactive effect of virtual quanta), he thereby obtains a fractional radiative correction to the Mott-Rutherford formula [See Eq. (7) of S]. For incident electrons of 200 Mev and for proton recoils at 45° with the incident direction, the fractional increase of the Mott-Rutherford formula amounts to[4] 6.8 percent. This breaks down into larger individual corrections of +17.7 percent for emission of quanta with energy greater than 10 Mev, and of −10.9 percent for softer and virtual quanta.

We are thus motivated to investigate recoil corrections. Both the B-H formula, which serves as the starting point for Schiff's calculation, and the Schwinger correction are derived for an electron scattered in a static Coulomb potential. The Mott-Rutherford cross section for scattering of electrons of energy E_0 through an angle θ in the Coulomb field of charge Ze is

$$\sigma d\Omega = (Ze^2/2E_0)^2 \cot^2\tfrac{1}{2}\theta \csc^2\tfrac{1}{2}\theta d\Omega. \tag{1}$$

To order $E_0/M(<1)$, this cross section is reduced by a factor

$$(1-2(E_0/M)\sin^2\tfrac{1}{2}\theta), \tag{2}$$

if the proton recoil is included in the calculation of electron-proton elastic scattering.[8] For large momentum transfer (large angle scattering) the above recoil factor amounts to an appreciable reduction in the differential cross section ($\sim$20 percent for $E_0 = 200$ Mev).

We here study the analogous recoil correction to the B-H formula. A complete investigation of the elastic plus radiative electron-proton scattering cross section requires calculation of the recoil correction to the Schwinger formula also. We do not perform that calculation in this work. The motivation to consider

* Assisted by the joint program of the ONR and the AEC.

[1] H. Bethe and W. Heitler, Proc. Roy. Soc. (London) **146**, 83 (1934), referred to here as B-H. See W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition, page 161, for further discussion and for references to earlier literature.

[2] We confine remarks here to the bremsstrahlung process. For application to pair production by high energy gamma-rays one need only transcribe the notation in the manner explicitly given on page 196 of Heitler's book (reference 1).

[3] Private communication from W. K. H. Panofsky.

[4] L. I. Schiff, preceding article, Phys. Rev. **87**, 750 (1952), referred to here as S.

[5] Energy analysis of the recoil proton tracks would require more labor for reasonable statistics than is now feasible.

[6] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1949), second edition, p. 80.

[7] J. Schwinger, Phys. Rev. **75**, 899 (1949).

[8] M. N. Rosenbluth, Phys. Rev. **79**, 615 (1950).

the bremsstrahlung recoil correction primarily and separately is twofold: (1) As seen in S, the radiative is the larger of the two correction terms to the Mott-Rutherford formula, and therefore, a recoil correction to it is expected to be, percentage-wise, more significant in the final result. (2) The matrix elements are of order e^3 for the radiative process, whereas, for the Schwinger correction, they are of order e^4, and correspondingly more difficult to calculate.

To summarize briefly the results of this work, we derive a correction of order (q/Mc) to the B-H differential cross section for bremsstralung in electron-proton scattering. As in Rosenbluth's work,[8,9] magnetic moment contributions to the cross section are reduced in the order of $(q/Mc)^2$. Correction terms of order (q/Mc) result from both dynamical and kinematical considerations. The two Feynman graphs in Fig. 1 represent emission of the bremsstralung gamma-ray by the electron before and after scattering. In the limit of infinite proton mass, these graphs yield the B-H cross section. Terms of order (q/Mc) resulting from these two graphs will be seen in the following to result solely from the kinematical equations of momentum and energy conservation. That is, to order (q/Mc), the proton and electron interact only through their static Coulomb fields, for the processes depicted in Fig. 1. In calculating the analogous two Feynman graphs for proton emission of the bremsstrahlung gamma-ray (Fig. 2), we directly use the kinematics for $M\rightarrow\infty$. This is because the matrix element for proton emission of the gamma-ray is reduced in the order (q/Mc) compared with the corresponding matrix element for electron emission of the gamma-ray. Contributions from these graphs of Fig. 2 comprise the dynamical correction to the B-H formula.

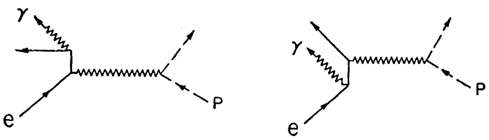

FIG. 1. Diagrams for the inelastic scattering of an electron and a proton with emission of the bremsstrahlung photon by the electron.

In developing the recoil correction to S we use only the dynamical correction to the B-H formula. The kinematical conservation equations for scattering in a fixed Coulomb field must be used in joining the formulas given here with the Schwinger formula,[7] which has been derived for infinitely massive nuclei. The final results obtained here support the qualitative validity of the no recoil analysis given in S. This may be understood as follows. It is shown in S that the photon is emitted within an angle $\sim mc^2/E_0$ of the incident or scattered electron direction. The terms of order (q/Mc), given below, that arise because of the dynamical part of the recoil contribution, correlate the electron and photon directions much less strongly and are effectively masked by the peaked distribution obtained by Schiff.

II. CALCULATION

Calculation of the matrix elements is "straightforward but tedious." The Feynman-Dyson methods are used. We exhibit here just the square of the matrix element to order (q/Mc), summed over final electron spin and photon polarization, and averaged over initial spin. Choosing the proton to be initially at rest in the laboratory frame, and with the notation

$\hbar=c=1$;
$\mathbf{k}$, k = emitted photon momentum, energy;
$\mathbf{p}_0$, E_0 = initial electron momentum, energy;
$\mathbf{p}$, E = scattered electron momentum, energy;
$\mathbf{q}=\mathbf{p}_0-\mathbf{p}-\mathbf{k}$ = momentum transferred to the proton;
Ze = proton charge;

we obtain

$$\begin{aligned}\langle|\mathrm{M.E.}|^2\rangle_{\mathrm{Av}}=(2\pi)^4(Z^2e^6/k^3E_0E)\Bigg[&\frac{1}{q^4}\Bigg\{\frac{(\mathbf{k}\times\mathbf{p})^2}{(kE-\mathbf{k}\cdot\mathbf{p})^2}[4E_0^2-q^2(1+2E_0/M)]+\frac{(\mathbf{k}\times\mathbf{p}_0)^2}{(kE_0-\mathbf{k}\cdot\mathbf{p}_0)^2}[4E^2-q^2(1-2E/M)]\\&+2\frac{k^2(\mathbf{k}\times\mathbf{p})^2+k^2(\mathbf{k}\times\mathbf{p}_0)^2-(4E_0E+2k^2-q^2\{1-k/M\})\mathbf{k}\times\mathbf{p}\cdot\mathbf{k}\times\mathbf{p}_0}{(kE-\mathbf{k}\cdot\mathbf{p})(kE_0-\mathbf{k}\cdot\mathbf{p}_0)}\Bigg\}\\&-\frac{2Z}{Mkq^2[(\mathbf{p}-\mathbf{p}_0)^2-k^2]}\Bigg\{\mathbf{k}\times\mathbf{q}\cdot\mathbf{k}\times\mathbf{p}_0\left(1-\frac{2EE_0-kE+2m^2+2\mathbf{p}\cdot\mathbf{p}_0-\mathbf{k}\cdot\mathbf{p}}{(kE_0-\mathbf{k}\cdot\mathbf{p}_0)}\right)\\&+\mathbf{k}\times\mathbf{q}\cdot\mathbf{k}\times\mathbf{p}\left(1+\frac{2EE_0+kE_0+2m^2+2\mathbf{p}_0\cdot\mathbf{p}+\mathbf{k}\cdot\mathbf{p}_0}{(kE-\mathbf{k}\cdot\mathbf{p})}\right)\\&-\frac{2k}{(kE_0-\mathbf{k}\cdot\mathbf{p}_0)}(2E(m^2k^2-EE_0k^2+\mathbf{k}\cdot\mathbf{p}\mathbf{k}\cdot\mathbf{p}_0)-(k^2E_0\mathbf{p}\cdot\mathbf{q}-E\mathbf{k}\cdot\mathbf{p}_0\mathbf{k}\cdot\mathbf{q}))\\&+\frac{2k}{(kE-\mathbf{k}\cdot\mathbf{p})}(2E_0(m^2k^2-E_0Ek^2+\mathbf{k}\cdot\mathbf{p}_0\mathbf{k}\cdot\mathbf{p})+(k^2E\mathbf{p}_0\cdot\mathbf{q}-E_0\mathbf{k}\cdot\mathbf{p}\mathbf{k}\cdot\mathbf{q}))\Bigg\}\Bigg].\end{aligned}\quad(3)$$

[9] Use of Pauli terms to represent the anomalous magnetic moment of the proton seems justified on the basis of Rosenbluth's calculation (reference 8) for not too high bombarding energies (~200 Mev).

Formula (3) is seen to reduce directly to the B-H formula[1] for $M\rightarrow\infty$. The terms of order E/M, E_0/M and k/M in the first bracket comprise the kinematical correction due to recoil; the dynamical correction is contained in the second bracket. After this calculation was performed, a paper by Rzewuski[10] which contains part of formula (3) was discovered. Rzewuski considers the instantaneous Coulomb interaction between two Dirac electrons and calculates a cross section for their radiative collision with neglect of exclusion principle interference effects. His result contains the kinematical corrections to the B-H formula plus a dynamical correction resulting from the nonretarded part of the particles' interaction. To order (q/Mc), it differs from formula (3) above in two respects: (1) The energy denominator $[(\mathbf{p}-\mathbf{p}_0)^2-k^2]$ for the second set of terms appears without the retardation correction as $(\mathbf{p}-\mathbf{p}_0)^2$. (2) The last two lines of Eq. (3), expressing the possibility for proton pair formation in the intermediate (virtual) state, are not present. It is of interest to speculate that these last two terms might give definite information concerning existence of the negative proton at some future date when precision coincidence experiments on high energy electron-proton bremsstrahlung (or high energy pair production in the field of a proton) become feasible.

In order to convert Eq. (3) to a cross section we need to calculate the transition rate and divide by the incident electron flux. The result is

$$d\sigma=2\pi(E_0/p_0)\langle|\mathrm{M.E.}|^2\rangle_{\mathrm{Av}}\rho_f,$$

where ρ_f denotes the final state sum for the three outgoing particles—the recoil proton, the scattered electron, and the bremsstrahlung photon. It is given by $d^3kd^3pd^3q$, where four of these nine dimensions are collapsed by the four relations of energy-momentum

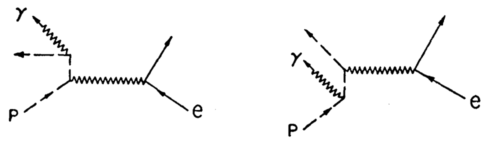

FIG. 2. Diagrams for the inelastic scattering of an electron and a proton with emission of the bremsstrahlung photon by the proton.

conservation:

$$\mathbf{p}_0=\mathbf{p}+\mathbf{q}+\mathbf{k},$$
$$E_{p0}=E_p+(q^2/2M)+k.$$

The density ρ_f may be expressed in variables appropriate to the experimental situation considered. In terms of recoil proton variables and the photon angular distribution, we have, for example,

$$\rho f=k^3E_pq^2dqd\Omega_kd\Omega_q/(kE_p-\mathbf{k}\cdot\mathbf{p}).$$

We consider now the contribution of the dynamical correction to the B-H formula [second set of terms in Eq. (3)] to the calculation given in S. It is simply verified by direct computation[11] that these correction terms do not correlate the direction of the emitted photon very strongly with the direction of the incident or scattered electron. Thus, in the last line of Eq. (3) the multiplier of $1/(kE-\mathbf{k}\cdot\mathbf{p})\approx 2/kp\theta^2$, according to Schiff's approximation in the limit $\theta\rightarrow 0$ (i.e., for the photon emitted in the direction of the scattered electron), vanishes with θ. To leading order, therefore, these recoil terms do not contribute.

To conclude, we calculate a leading order recoil correction to the Bethe-Heitler differential cross section for bremsstrahlung.[2] This correction does not contribute in the analysis of electron-proton scattering (elastic and radiative) given in the preceding paper by Schiff.[4]

[10] J. Rzewuski, Acta Phys. Polon. **IX**, 121 (1947–48).

[11] The kinematics for $M\leftarrow\infty$ are used since the magnitude of the correction terms is already reduced by the factor (q/Mc).