Radiative Correction to the Angular Distribution of Nuclear Recoils from Electron Scattering*

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A calculation is made of the cross section for the recoil of nuclei from electron scattering when the angle of recoil is specified but the recoil energy is not. The influence of radiation is taken into account. Use is made of the calculation by Drell in the following paper to gauge the accuracy of the computed cross section for light nuclei; the fractional error is believed to be substantially smaller than the ratio of the recoil nucleus velocity to the velocity of light.

EXPERIMENTS on the scattering of high energy electrons from nuclei can provide information from which the distribution of electric charge within the nuclei can be determined. The interpretation of such experimental data is simplest when a separation into elastic and inelastic (radiative) processes can be made. This is the case, for example, when the experiment is performed by measuring the distribution of the recoil nuclei with respect to angle and energy. In a particular experimental arrangement now being employed in this laboratory,[1] the recoil nuclei are to be recorded by a photographic emulsion. It is then convenient to meassure the angle of each nuclear recoil; however, it is difficult to get reasonable statistics on the distribution of recoil energies, except that to be measured they must exceed a minimum value that is determined by the geometrical arrangement of the emulsions and their wrappings. The theoretical results on elastic electron-nuclear scattering can be compared with such experimental data only if a correction is applied that takes into account the emission of photons.

This paper presents an estimate of the radiative correction that is required. The following three approximations are made:

(1) The Bethe-Heitler differential bremsstrahlung cross section[2] is used as the starting point (Born approximation).

(2) The dominant contribution to the bremsstrahlung cross section arises from events in which both the scattered electron and the photon emerge within an angle of order mc^2/E_0 of the incident electron direction, where m is the electron mass and E_0 its initial energy. But in order for the recoil nucleus to be observed as such, its momentum must be large in comparison with mc (about 100 times as large for a proton that recoils with 1.5-Mev kinetic energy).[3] Thus, either the scattered electron or the photon, or both, must make an angle with the incident electron direction that is large in comparison with mc^2/E_0, for those processes of interest here. This means that screening of the nucleus by atomic electrons can be ignored; on the other hand, the nuclear form factor is important, and information concerning the nuclear charge distribution can be obtained from such an experiment. An important simplification in carrying through the integrations over photon energy and angle can then be achieved by noticing that in such cases the photon has a marked tendency to emerge with direction close to that of either the incident electron or the scattered electron. This makes it easy to find the leading term in the radiative correction, which is in error only by a factor in a logarithm. This factor can then be determined by "calibrating" the resulting formula against Schwinger's expression for nearly elastic scattering with emission of very low energy photons.[4]

(3) The effect of the finite recoil velocity of the nucleus on the cross section is neglected. Thus, what is found is the momentum distribution of the recoil nuclei under the assumption that they are infinitely massive. In the following paper, Drell[5] calculates the effect of nuclear recoil on the bremsstrahlung cross section through terms of first order in the recoil velocity. This calculation shows that the tendency of the photon to emerge in line with the incident or the scattered electron is not nearly as marked in the correction terms as in the Bethe-Heitler formula. Because of this, only the argument, and not the coefficient, of the logarithm mentioned in the preceding paragraph is affected. Unfortunately, this uncertainty cannot be "calibrated" out, since Schwinger's formula[4] also fails to take account of recoil.

The Bethe-Heitler differential bremsstrahlung cross section is[2]

$$\frac{Z^2e^4}{2\pi 137}\frac{dk}{k}\frac{p}{p_0}\frac{\sin\theta_0 d\theta_0 \sin\theta d\theta d\phi}{q^4}\times\left\{\frac{p_0^2\sin^2\theta_0(4E^2-q^2)}{(E_0-p_0\cos\theta_0)^2}+\frac{p^2\sin^2\theta(4E_0^2-q^2)}{(E-p\cos\theta)^2}-\frac{2p_0p\sin\theta_0\sin\theta\cos\phi(4E_0E-q^2+2k^2)}{(E_0-p_0\cos\theta_0)(E-p\cos\theta)}+\frac{2k^2(p_0^2\sin^2\theta_0+p^2\sin^2\theta)}{(E_0-p_0\cos\theta_0)(E-p\cos\theta)}\right\}, \quad (1)$$

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[1] Private communication from W. K. H. Panofsky.

[2] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition, p. 164.

[3] For this reason, the numerical results of Jost, Luttinger, and Slotnick, Phys. Rev. **80**, 189 (1950), on nuclear recoils from pair production are not useful in the present connection.

[4] J. Schwinger, Phys. Rev. **75**, 899 (1949).

[5] S. D. Drell, following article, Phys. Rev. **87**, 753 (1952).

where E_0, p_0 are the energy and momentum (expressed in energy units) of the incident electron, E, p those of the scattered electron, k the energy of the emitted photon, θ_0 the angle between photon and incident electron, θ the angle between photon and scattered electron, ϕ the angle between the planes in which θ_0 and θ lie, and q is the momentum (expressed in energy units) transferred to the nucleus of charge Ze. If the nucleus has a finite size, Eq. (1) and all the other expression for cross sections given below must be multiplied by a nuclear form factor that is a function of q. The momentum transfer is given by

$$q^2=p_0^2+p^2+k^2-2p_0k\cos\theta_0+2pk\cos\theta \\ -2p_0p(\cos\theta_0\cos\theta+\sin\theta_0\sin\theta\cos\phi).$$

In principle, we wish to integrate Eq. (1) over all the domains of the four variables θ_0, θ, ϕ, and k except for that one combination that corresponds to a particular value of the angle between the momentum transfer vector and the incident electron direction. In practice, we notice that since we are only interested in the case $q\gg\mu\equiv mc^2$, the differential cross section is largest when one or the other of the quantities $(E_0-p_0\cos\theta)$, $(E-p\cos\theta)$ is small, that is, when either θ_0 or θ is small; they cannot both be small at once since then q would also be small. The approximation is then made of selecting just two regions from the whole domain of multiple integration and neglecting the rest. In one of these, θ_0 is assumed to be small, and the integration is carried over it with all other variables taking the values they would have if θ_0 were zero; in the other, θ is substituted for θ_0, and the same procedure is followed. The leading terms contain multiplicative factors of order $\ln(E_0/\mu)$ or $\ln(E/\mu)$, and only these terms are retained (extreme relativistic region). Since some of the neglected terms would alter the arguments of the logarithms, no attempt is made to obtain this logarithmic factor correctly. Instead, the quantity $\ln(E_0/\mu)$ is used symbolically to denote the result of any integration that would diverge logarithmically if μ were made to vanish, and an estimate of its value is made later. The integration over k extends down to some minimum value k_m, which also appears as the maximum photon energy in the correction owing to radiation of very low energy photons.[4] Finally, the experimental arrangement is such that q must exceed a minimum value q_m, which is assumed here to be independent of recoil angle, although this is not an important restriction.

Inspection of Eq. (1) shows that the leading terms arise from integration of the first term in curly brackets for small θ_0, the second term for small θ, and the first and second parts of the fourth term for small θ and θ_0, respectively. For the small θ_0 contributions, we can put $q\cong 2E\sin\frac{1}{2}\theta$ and $\gamma\cong\frac{1}{2}(\pi-\theta)$, where γ is the nuclear recoil angle. For the small θ contributions, we can put $q\cong 2E_0\sin\frac{1}{2}\theta_0$ and $\gamma\cong\frac{1}{2}(\pi-\theta_0)$. Integration over θ_0 and ϕ of those terms that are important for small θ_0 yields

$$\frac{Z^2e^4}{137}\frac{dk}{k}\frac{E}{E_0}\frac{\sin\theta d\theta}{q^4}\left\{4(4E^2-q^2)+\frac{4k^2E\sin^2\theta}{E_0(1-\cos\theta)}\right\}\ln\left(\frac{E_0}{\mu}\right) \\ =\frac{2Z^2e^4}{137}\frac{dk}{k}\tan^3\gamma d\gamma\left(\frac{E_0^2+E^2}{E_0^2E^2}\right)\ln\left(\frac{E_0}{\mu}\right), \quad (2)$$

where γ has been substituted for θ and q, and μ has been neglected everywhere except in the logarithm. Similarly, integration over θ and ϕ of those terms that are important for small θ yields

$$\frac{2Z^2e^4}{137}\frac{dk}{k}\tan^3\gamma d\gamma\left(\frac{E_0^2+E^2}{E_0^4}\right)\ln\left(\frac{E_0}{\mu}\right). \quad (3)$$

Since q must be greater than q_m, Eq. (2) is valid only for $\gamma<\gamma_m$, where $\cos\gamma_m=q_m/2E=q_m/2(E_0-k)$, and Eq. (3) is valid only for $\gamma<\gamma_m$, where now $\cos\gamma_m=q_m/2E_0$.

The integration of Eq. (2) over k extends from k_m only up to $E_0-(q_m/2\cos\gamma)$, since for larger values of k, q is less than q_m. Equation (3) is to be integrated over k from k_m to E_0. The result from Eq. (2) is

$$\frac{2Z^2e^4}{137}\ln\left(\frac{E_0}{\mu}\right)\tan^3\gamma d\gamma\left\{2\ln\left(\frac{E_0}{k_m}\right)\right. \\ \left.+\ln\left(x+\frac{1}{x}\right)+x-1\right\},\quad x\equiv 2E_0\cos\gamma/q_m,$$

where terms that vanish as k_m becomes zero are neglected. Similarly, from Eq. (3) we get

$$\frac{2Z^2e^4}{137}\ln\left(\frac{E_0}{\mu}\right)\tan^3\gamma d\gamma\left\{2\ln\left(\frac{E_0}{k_m}\right)-\tfrac{3}{2}\right\}.$$

The sum of these represents the cross section for scattering in which the nucleus recoils into $d\gamma$ and a photon of energy greater than k_m is emitted:

$$\frac{8Z^2e^4}{137}\ln\left(\frac{E_0}{\mu}\right)\tan^3\gamma d\gamma\left\{\ln\left(\frac{E_0}{k_m}\right)\right. \\ \left.+\tfrac{1}{4}\ln\left(x+\frac{1}{x}-2\right)+\frac{x}{4}-\tfrac{5}{8}\right\}. \quad (4)$$

In the same notation, the Mott-Rutherford formula for elastic Coulomb scattering is[6]

$$(2\pi Z^2e^4/E_0^2)\tan^3\gamma d\gamma. \quad (5)$$

The correction to this, owing to emission of very low energy photons, consists in multiplication by the factor

[6] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1949), second edition, p. 80.

TABLE I. Fractional radiative correction Δ to the Mott-Rutherford formula.

E_0 (Mev)	γ	$-\delta$ (%)	Eq. (4)/Eq. (5) (%)	Δ (%)
100	0°	−10.7	18.4	7.7
100	45°	− 9.7	15.0	5.3
200	0°	−11.9	27.9	16.0
200	45°	−10.9	17.7	6.8

$(1-\delta)$, where[4]

$$\delta=\frac{4}{137\pi}\left\{\left[\ln\left(\frac{2E_0\cos\gamma}{\mu}\right)-\frac{1}{2}\right]\left[\ln\left(\frac{E_0}{k_m}\right)-\frac{13}{12}\right]+\frac{17}{72}+\phi(\pi-2\gamma)\right\}. \quad (6)$$

Here the function $\phi(\theta)$ is given by Eqs. (10) and (11) of reference 4; for $\gamma=0$, $\phi(\pi)=\pi^2/24$, and for $\gamma=\pi/4$, $\phi(\pi/2)=0.292$. The fractional correction to the Mott-Rutherford formula, owing to emission of both low and high energy photons, should now be Eq. (4) divided by Eq. (5), minus δ as given by Eq. (6). When this combination is formed, the quantity k_m, which is an artificial boundary between the very low energy photons and those of higher energy, should cancel out. It is apparent that this cancellation occurs if the symbolic factor $\ln(E_0/\mu)$ in Eq. (4) is set equal to $\{\ln[(2E_0\cos\gamma)/\mu]-\frac{1}{2}\}$, and this is a plausible substitution to make. We therefore adopt this "calibration" factor and obtain finally for the fractional radiative correction to the Mott-Rutherford formula

$$\Delta=\frac{4}{137\pi}\left\{\left[\ln\left(\frac{2E_0\cos\gamma}{\mu}\right)-\frac{1}{2}\right]\times\left[\frac{1}{4}\ln\left(x+\frac{1}{x}-2\right)+\frac{x}{4}+\frac{11}{24}\right]-\frac{17}{72}-\phi(\pi-2\gamma)\right\},$$

$$x\equiv 2E_0\cos\gamma/q_m. \quad (7)$$

For heavy nuclei, the most important approximation made in this calculation is the use of the Born approximation, and little can be done about that at the present time.[7] For light nuclei, the most important approximation is the third of those mentioned near the beginning of this paper. Now Drell's calculation[5] shows that the recoil corrections are of order β times those terms in Eq. (1) that are neglected in arriving at the first logarithm in Eq. (4), where β is the ratio of the recoil nucleus velocity to the velocity of light. This means that the error in Eq. (4) itself is substantially smaller than β. On the other hand, Rosenbluth[8] has shown that the recoil correction to the Mott-Rutherford formula (5) is of order β. Thus, if we assume, as appears reasonable, that the Schwinger correction to the cross section [product of Eqs. (5) and (6)] has the same accuracy as Eq. (4), then the net radiative correction to the cross section [product of Eqs. (5) and (7)] should also have the same superior accuracy as Eq. (4). This means that the net radiative correction to the cross section may be added to Rosenbluth's elastic scattering cross section to obtain a result that can be compared with the experimental cross section. The comparison will be expected to yield a nuclear form factor that is in error by a factor considerably smaller than β.

In order to get an idea of the order of magnitude of the radiative correction factor, values of Δ (in percent) are given in Table I for four combinations of values of E_0 and γ, with $q_m=100\ mc^2$, which corresponds to a proton recoiling with a minimum of 1.5 Mev kinetic energy. The table also lists the breakdown of the Δ values into $-\delta$ and the ratio of Eq. (4) to Eq. (5), when k_m is arbitrarily chosen to be $E_0/20$; the values of Δ given by Eq. (7) are, of course, independent of the choice of k_m. As might be expected, the emission of all photons makes the observed cross section in each case larger than would be calculated on the basis of elastic scattering.

The correlation of the emitted photon direction with that of either the incident or the scattered electron when the momentum transfer is large might be subject to direct experimental verification. The analogous result in pair production by gamma-rays is that either the electron or the positron tends to emerge close to the direction of the incident photon when the momentum transfer is large.

[7] *Note added in proof:*—Recent calculations of L. C. Maximon and H. A. Bethe [Phys. Rev. (to be published)] extend Eq. (1) beyond the Born approximation.

[8] M. N. Rosenbluth, Phys. Rev. **79**, 615 (1950).