The Angular and Lateral Distribution Functions for the Nucleon Component of the Cosmic Radiation

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A method is developed, whereby the angular and radial distribution functions for the nucleon component of the cosmic radiation in the atmosphere may be obtained in terms of the moments of the distributions. Essentially the method consists of an expansion of the unknown distribution functions in a series of derivatives of the Dirac delta-function, whose coefficients are identified with the angular and radial moments. The method gives results identical with those obtained by a more tedious procedure, and is of general applicability in reconstructing distribution functions when only their moments are known.

The angular and radial distribution functions are found for various initial conditions and numerical results given for the case of an integral primary proton power law spectrum with exponent $\gamma=1.1$. Variation of atmospheric density with height is taken into account. Using a form of the differential cross section for nucleon-nucleon collisions, $R+S'\cos^2\bar{\theta}$ in the center-of-mass frame of reference, predicted by most field theoretic treatments, it is shown that the calculated results are in disagreement by an order of magnitude with experimental data. The various assumptions on which our theory is based (power law, homogeneous form of total cross section, etc.) are then critically examined. It is concluded that the theory can be reconciled with experiment only when the number of scattered particles decreases exponentially from the direction of motion of the incident particle in the laboratory frame of reference. Using this result, the new angular and radial distribution functions are then calculated and found to be given at sea level essentially by $\exp\{-166U(1-C)\}$ and $\exp\{-3.72U^{\frac{1}{2}}r\}$, respectively, for particles of energy greater than U Bev, where C is the cosine of the angle with the shower axis and r is the distance in kilometers from the shower axis. The half-value of the radial distribution occurs at a distance of 92 meters from the shower axis.

I. INTRODUCTION

A SHORT time ago the authors[1,2] made a theoretical determination of the mean square angular deviation and mean square distance of the nucleons in extensive air showers from the shower axis. The numerical results were of considerable interest in showing the horizontal and vertical development of air showers in consequence of a given differential cross section for nucleon-nucleon collisions and a given energy spectrum for the primary particles. It might be hoped further that comparison with the experimental data would provide valuable information on the nuclear differential cross sections at very high energies.

From the quantitative point of view, the lateral spread derived from a general type of cross section was surprisingly large, indicating that either the corresponding distribution function must possess a long "tail" or that some unknown physical factor comes into play. We were not able at the time to confirm the existence of a tail quantitatively and were thus led to consider the problem of the determination of the angular and lateral distribution functions for the nucleon component of the cosmic radiation.

So far as we are aware, this problem has not been considered before, although Molière[3–5] has considered the corresponding problem for the soft component. His result were later verified by Cocconi *et al.*[6] within the experimental error and within 200 meters of the shower axis. This was remarkable, in view of the physical hypotheses on which Molière's theory was based. For, at the time, it was not known that the soft component was secondary to the nucleon component of the cosmic radiation, being produced continuously throughout the atmosphere by decay of the neutral π-mesons evolved in nuclear collisions. Furthermore, Molière's results were calculated only for the cascade maximum in an atmosphere of constant density; and it would be very coincidental if this procedure should yield even approximately correct results. For our treatment of the corresponding nucleon problem has shown that the development of showers in media of constant and variable density has very different characteristics.

Molière's calculations were based on Landau's equation,[7] as were also those of other authors[8–10] who attempted to evaluate the higher moments of the angular and radial distributions for the soft component. Landau's equation, however, neglects the fourth and higher even moments of the angular distribution for the individual processes concerned; it will therefore give correctly only the mean square angular and lateral deviation of the shower particles from the trajectory of the generating primary. It might be hoped that the error for the higher moments would be small; but our experience with the corresponding nucleon problem lends no support to this hope. Molière was evidently aware of the difficulty when he recently[11] made a detailed calculation with the limited objective of obtaining the mean square angular and radial deviations.

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Experimental work on the subject has been performed by Cocconi *et al.*,[6] Greisen *et al.*,[12] and also by the Russian school,[13–15] which extended measurements to distances of the order of 1 km. Although the latter obtained a radial distribution varying as $r^{-2.6}$ with the distance r from the shower axis, which would imply an infinite mean square distance from it, their distribution agrees roughly with the earlier determinations in having an apparent half-width of about 150 meters.

Since the method which we shall adopt requires a detailed mathematical analysis, a mathematical introduction follows. The calculations are embodied in the second half of the paper, and a nonmathematical discussion of the results obtained will be found at the end.

II. MATHEMATICAL INTRODUCTION

The use of the δ-function has been well established in quantum mechanics[16] and pulse theory; however, the rigor of the mathematical procedures in which it is used has sometimes been questioned, and we therefore state at the outset the unambiguous meaning of an equation of the type

$$f(x)=w(x)\sum a_k(x)\delta^{(k)}(x), \tag{1}$$

where x is an arbitrary variable and the superscript represents the number of differentiations with respect to x. The functions $a_k(x)$ must be regular at $x=0$, but $w(x)$ may have any kind of singularity. If $q(x)$ is an arbitrary function, regular at $x=0$, then (1) is equivalent to the assertion

$$\int_{-a}^{b} dxq(x)f(x)/w(x)=\sum(-1)^k q_k^{(k)}(0),\quad q_k(x)=q(x)a_k(x), \tag{2}$$

where a and b are any positive numbers less than the radius of convergence of the power series

$$q(x)=\sum_{k=0}^{\infty} q^{(k)}(0)x^k/k!.$$

Now

$$\int_{-a}^{b} q(x)f(x)dx=\sum_{k=0}^{\infty} q^{(k)}(0)\int_{-a}^{b} x^k f(x)dx/k!. \tag{3}$$

Hence, using the equivalence of (1) and (2) for the special case $w(x)=1$, one has

$$f(x)=\sum_{k=0}^{\infty}(-1)^k f_{(k)}\delta^{(k)}(x)/k!, \tag{4}$$

$$f_{(k)}=\int_{-a}^{b} x^k f(x)dx. \tag{5}$$

Thus any integrable function $f(x)$ can be expanded as a series of derivatives of the δ-function, with coefficients which are proportional to the moments (5) of the function $f(x)$, within a certain range of the variable.

This result is easily extended to functions of any number of variables. For example,

$$f(r_1, r_2, C)=\sum_{l,m,n=0}^{\infty}\frac{(-1)^{l+m+n}2^{-n}}{l!m!n!}f_{(l,m,n)}\times\delta^{(l)}(r_1)\delta^{(m)}(r_2)\delta^{(n)}(1-C), \tag{6}$$

where

$$f_{(l,m,n)}=2^n\int\int\int f(r_1, r_2, C)r_1^l r_2^m(1-C)^n dr_1 dr_2 dC. \tag{7}$$

We shall require the value of

$$\int_0^{2\pi} f(r\cos\Psi, r\sin\Psi, C)d\Psi=f(r, C), \tag{8}$$

where the function f is defined by (6). It is clear from (8) that $f(r, C)$ must be an even function of r; hence, by an analog of (4),

$$f(r, C)=\sum_{l,n=0}^{\infty}\frac{(-1)^{l+n}2^{-n}}{l!n!}f_{(l,n)}\delta^{(l)}(r^2)\delta^{(n)}(1-C), \tag{9}$$

where

$$\begin{aligned}f_{(l,n)}&=2^n\int\int\int f(r\cos\Psi, r\sin\Psi, C)\times r^{2l}(1-C)^n d(r^2)dCd\Psi\\&=2^{n+1}\int\int\int f(r_1, r_2, C)(r_1^2+r_2^2)^l\times(1-C)^n dr_1 dr_2 dC\\&=2\sum_{k=0}^{l}\binom{l}{k}f_{(2k,\,2l-2k,\,n)},\end{aligned} \tag{10}$$

according to (7). Substituting (10) into (9), one has

$$f(r, C)=2\sum_{l,m,n=0}^{\infty}\frac{(-1)^{l+m+n}2^{-n}}{l!m!n!}f_{(2l,2m,n)}\times\delta^{(l+m)}(r^2)\delta^{(n)}(1-C). \tag{11}$$

Returning to (4) we examine the question of reconstructing the function $f(x)$ from a knowledge of the moments, if $f_{(k)}$ has a singularity at $x=0$, let $w(x)$ be a "weight function" with a singularity of the same type; and let

$$f(x)=w(x)\sum f_n S_n(x), \tag{12}$$

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[16] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), third edition.

where $S_n(x)$ is a set of polynomials defined by

$$S_n(x)=\begin{vmatrix} w_{(0)} & w_{(1)} & \cdots & w_{(n)} \\ w_{(1)} & w_{(2)} & \cdots & w_{(n+1)} \\ \cdot & \cdot & \cdot & \cdot \\ w_{(n-1)} & w_{(n)} & \cdots & w_{(2n-1)} \\ 1 & x & \cdots & x^n \end{vmatrix}, \quad (13)$$

and

$$w_{(k)}=\int w(x)x^k dx. \quad (14)$$

The polynomials satisfy the orthogonality relations

$$\int w(x)S_m(x)S_n(x)dx=N_m\delta_{m,n}, \quad (15)$$

where the normality constants are given by

$$N_n=\Delta_n\Delta_{n-1},$$

$$\Delta_n=\begin{vmatrix} w_{(0)} & w_{(1)} & \cdots & w_{(n)} \\ w_{(1)} & w_{(2)} & \cdots & w_{(n+1)} \\ \cdot & \cdot & \cdot & \cdot \\ w_{(n)} & w_{(n+1)} & \cdots & w_{(2n)} \end{vmatrix}. \quad (16)$$

Then the f_n in (12) are given by

$$N_nf_n=\int f(x)S_n(x)dx = \begin{vmatrix} w_{(0)} & w_{(1)} & \cdots & w_{(n)} \\ w_{(1)} & w_{(2)} & \cdots & w_{(n+1)} \\ \cdot & \cdot & \cdot & \cdot \\ w_{(n-1)} & w_{(n)} & \cdots & w_{(2n-1)} \\ f_{(0)} & f_{(1)} & \cdots & f_{(n)} \end{vmatrix}. \quad (17)$$

Thus the function $f(x)$ has been expanded in a series of orthogonal polynomials, the coefficients of which are linear combinations of its moments.

This development may be used to verify the formula (4). For, replacing $f(x)$ by $\delta^{(k)}(x)$ in the above argument, one has

$$\delta^{(k)}(x)=w(x)(-1)^k k!\sum_{l=0}^{k}\Delta_{l(k)}S_l(x)/N_l, \quad (18)$$

where $\Delta_{l(k)}$ is the minor of $w(k+l)$ in the last row or column of the determinant Δ_l. Substituting this result in (4), one obtains again (12) with the coefficients f_n given by (17).

We now state a series of formulas involving δ-functions which will be required in the subsequent work. Let O and Q represent the differential operators

$$O\equiv\frac{d}{dC}(1-C^2)\frac{d}{dC},\quad Q\equiv\frac{\partial}{\partial C}(1-C^2)\frac{\partial}{\partial C}+\frac{1}{1-C^2}\frac{\partial^2}{\partial\beta^2}, \quad (19)$$

of the Legendre and associated Legendre functions, respectively. If $b(C)$ is a regular function of C, then

$$b(C)\delta^{(k)}(1-C)=\sum_{l=0}^{k}\binom{k}{l}b^{(l)}(1)\delta^{(k-l)}(1-C), \quad (20)$$

$$O\delta^{(k)}(1-C)=-k(k+1)\delta^{(k)}(1-C)-2(k+1)\delta^{(k+1)}(1-C), \quad (21)$$

$$O(O+1.2)(O+2.3)\cdots\{O+l(l-1)\}\delta(1-C)=(-2)^l l!\delta^{(l)}(1-C). \quad (22)$$

Also, if $S=(1-C^2)^{\frac{1}{2}}$,

$$Q\{(S\cos\beta)\delta^{(1)}(1-C)\}=-2S\cos\beta\delta^{(2)}(1-C), \quad (23)$$

$$Q\{\cos^2\beta\delta(1-C)\}=-\delta^{(1)}(1-C)-\tfrac{1}{2}\delta(1-C)\cos2\beta. \quad (24)$$

These formulas may all be tested by the rules of integration given at the beginning of this section.

III. HIGHER ANGULAR MOMENTS FOR NUCLEON-NUCLEON COLLISIONS

The differential probability that a nucleon of energy U, dU should be scattered inelastically at an angle of cosine c, dc to the direction of motion of the incident nucleon of energy U_0 is taken to be

$$W(U_0,U,c)dUdc=UF(U_0,U)y\{U(1-c)\}dUdc$$
$$\{c_0\approx1-U^{-1},\ F(U_0,U)=20U_0^{-5}U(U-U_0)^3\} \quad (25)$$

for $c\geqq c_0$. For relativistic reasons it is impossible for a nucleon to be scattered through an angle greater than $\cos^{-1}c_0$ in a nucleon-nucleon collision.

If we adopt the form of $W(U_0, U, c)$ determined in our previous paper[17] by assuming that in the center-of-mass coordinates the differential cross section is of the form $R(\bar{U})+S'(\bar{U})\cos^2\bar{\theta}$, then $y(x)=6x(1-x)$. We shall see, however, at a later stage in our work, that this choice is incompatible with experimental data.

The nth angular moment of $W(U_0, U, c)$ defined by (25) is

$$W_{(n)}=2^n\int_{c_0}^{1}W(U_0,U,c)(1-c)^n dc$$
$$=(2/U)^n\int_0^1 F(U_0,U)y(x)x^n dx$$
$$=(2/U)^n F(U_0,U)y_{(n)}. \quad (26)$$

IV. ANGULAR MOMENTS FOR NUCLEON-NUCLEUS COLLISIONS

In a previous paper[18,10] the authors derived the mean square angle of emission of nucleons resulting from nucleon-nucleus collisions, for both light and heavy ele-

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ments. The method is to consider the development of a nucleon cascade in homogeneous nuclear matter, finally averaging over all possible paths through the nucleus, assumed to be spherical in shape. At the energies considered interference effects are negligible.

For our present purpose, it will be necessary to determine the higher moments of the angular distribution of particles emitted from the nucleus. The probability $\nu(U, C, z)dUdC$ of finding a nucleon at depth z (measured in the direction of motion of the incident particle) in homogeneous nuclear matter, with energy U, dU and direction of motion making an angle of cosine C, dC with that of the incident particle satisfies the integro-differential equation

$$C\frac{\partial\nu}{\partial z}(U,C)+\nu(U,C)=2\int_U^\infty dU'\int_{c0}^1 dc\int_0^{2\pi}\frac{d\omega}{2\pi}\times\nu(U',C')W(U',U,c),\quad(27)$$

$$C'=Cc+Ss\cos\omega,\quad S=(1-C^2)^{\frac{1}{2}},\quad s=(1-c^2)^{\frac{1}{2}}.$$

The integrand can be developed as a series of differentia coefficients of $\nu(U', C)$, in the following way. Expand $\nu(U', C')$ as a series of Legendre polynomials, thus:

$$\nu(U',C')=\sum_{k=0}^\infty(k+\tfrac{1}{2})\nu_k(U')P_k(C').\quad(28)$$

Then making use of the addition theorem for Legendre polynomials, one has

$$\int_{c0}^1 dc\int_0^{2\pi}\frac{d\omega}{2\pi}\nu(U',C')W(U',U,c)$$
$$=\sum_{k=0}^\infty(k+\tfrac{1}{2})\nu_k(U')P_k(C)\int_{c0}^1 P_k(c)W(U',U,c)dc$$
$$=\sum_{k=0}^\infty(k+\tfrac{1}{2})\nu_k(U')P_k(C)\sum_{l=0}^k\frac{(k+l)!}{(k-l)!}\frac{(-4)^{-l}}{(l!)^2}W_{(l)}\quad(29)$$

by expanding $P_k(c)$ in powers of $(1-c)$ and substituting from (26). Interchanging the order of the summations and using the property $k(k+1)P_k(C)=OP_k(C)$, where O is the operator defined in (19), one obtains

$$\int_{c0}^1 dc\int_0^{2\pi}\frac{d\omega}{2\pi}\nu(U',C')W(U',U,c)$$
$$=\sum_{l=0}^\infty O(O+1.2)\cdots\{O+l(l-1)\}\nu(U',C)4^{-l}(l!)^{-2}W_{(l)}.\quad(30)$$

Substituting for $W_{(l)}$ from (26) and inserting in (30) one has

$$C\frac{\partial\nu}{\partial z}(U,C)+\nu(U,C)=2\int_U^\infty dU'F(U',U)\sum_{l=0}^\infty\frac{(2U)^{-l}}{(l!)^2}y_{(l)}\times O(O+2)\cdots\{O+l(l-1)\}\nu(U',C).\quad(31)$$

Thus the equation satisfied by the Mellin transform

$$\nu(v,C)=\int_0^\infty dU U^v\nu(U,C)\quad(32)$$

is then

$$C\frac{\partial\nu(v)}{\partial z}+\alpha(v)\nu(v)=\sum_{l=1}^\infty\frac{2^{-l}}{(l!)^2}y_{(l)}O(O+2)\cdots\times\{O+l(l-1)\}\nu(v-l)W(v-l),\quad(33)$$

where

$$\alpha(v)=1-W(v),$$
$$W(v)=2\int_0^\infty dU(U/U')^vF(U',U)\quad(34)$$
$$=240\{(v+2)(v+3)(v+4)(v+5)\}^{-1}.$$

The initial conditions to be applied to these equations are

$$\nu(U,C,z=0)=\delta(U-U_0)\delta(1-C),$$
$$\nu(v,C,z=0)=U_0{}^v\delta(1-C).\quad(35)$$

Equation (33) can be conveniently handled by taking the Laplace transform with respect to z:

$$\bar\nu(v,C,\lambda)=\int_0^\infty e^{-\lambda z}\nu(v,C,z)dz.\quad(36)$$

Equation (33) then becomes

$$\{\lambda C+\alpha(v)\}\bar\nu(v)=U_0{}^v\delta(1-C)+\sum_{l=1}^\infty 2^{-l}(l!)^{-2}y_{(l)}O(O+2)\cdots\times\{O+l(l-1)\}\bar\nu(v-l)W(v-l).\quad(37)$$

The solution of (37), in the form of a series of derivatives of $\delta(1-C)$, can now be obtained by an iterative procedure:

$$\bar\nu(v)=\sum_{l=0}^\infty(-2)^l l!A_v{}^l\delta^{(l)}(1-C),\quad(38)$$

where

$$A_v{}^l=U_0{}^{v-l}\{\lambda+\alpha(v-l)\}^{-1}\sum{}'\prod_{i=1}^l\alpha_{v-(a_1+\cdots+a_{i-1})}^{a_i},$$
$$\alpha_v{}^l=y_{(l)}W(v-l)2^{-l}(l!)^{-2}\{\lambda+\alpha(v)\}^{-1},\quad l=1,2,\text{ etc.}\quad(39)$$

The formulas (21) and (22) have been used to simplify the result (38), terms differing from those given by a factor U_0^{-1} having been neglected in the high energy approximation. In (39) the summation $\sum'$ is applied to all different products for which

$$a_1+a_2+\cdots+a_l=1. \tag{40}$$

The a's may take positive integral values, including zero and $\alpha^0=1$.

Using (4) with $x=2(1-C)$, $a=b=1$, the value of the transform of the lth angular moment may be immediately inferred from (38):

$$\bar{\nu}_{(l)}(v,\lambda)=4^l(l!)^2A_v{}^l. \tag{41}$$

To obtain the lth moment itself, for the distribution of particles within the nucleus, one simply takes an inverse Laplace-Mellin transform.

The distribution of particles emitted by the nucleus is given by

$$n(U,C)=2D^{-2}\int_0^D \nu(U,C,z)zdz, \tag{42}$$

where D is the average number of collisions suffered by a nucleon in a diametrical passage through the nucleus; and the moments of this distribution are therefore

$$n_{(l)}(U)=2D^{-2}\int_0^D zdz\frac{1}{2\pi i}\int_{v_0-i\infty}^{v_0+i\infty}U^{-(v+1)}dv \cdot\frac{1}{2\pi i}\int_{\lambda_0-i\infty}^{\lambda_0+i\infty}e^{\lambda z}d\lambda\bar{\nu}_{(l)}(v,\lambda). \tag{43}$$

For large values of l, this expression is of the order $2^l y_{(l)}$.

V. THE ANGULAR AND LATERAL SPREAD OF THE ATMOSPHERIC NUCLEON CASCADE

We have previously[1,2] developed a method for obtaining the mean square angle made with, and the mean square distance from, the shower axis of the nucleon component of extensive air showers. We set up a vector equation for the probability $f(\mathbf{p},\mathbf{r},t)\bar{d}\mathbf{p}d\mathbf{r}$ of finding a nucleon with momentum $\mathbf{p}$, $\bar{d}\mathbf{p}(=dp_1dp_2dp_3/2\pi p^2)$, at height t, and with a radius vector $\mathbf{r}$, $d\mathbf{r}$ measured normal to the shower axis. The height t and the two components of $\mathbf{r}$ are measured in cm, the three components (normal to and along the shower axis) of $\mathbf{p}$ in units of Mc, where M is the proton mass and c the velocity of light. Then, at ultrarelativistic energies, the scalar magnitude of $\mathbf{p}$ is the energy U, measured in proton mass units.

Let $\delta(t)$ be the density of the atmosphere at height t; then the probability that a nucleon will traverse the layer between height τ and height $t<\tau$, without colliding with an air nucleus, is

$$\exp-\{\theta(t)-\theta(\tau)\}/C_1,$$

where

$$\theta(t)=\frac{1}{75}\int_t^\infty \delta(\tau)d\tau, \tag{44}$$

and C_1 is the angle between the vector $\mathbf{p}$ and the vertical. The phase-space distribution function f satisfies the partial integro-differential equation

$$-\frac{1}{\theta'(t)}\left\{\frac{p_3}{U}\frac{\partial f}{\partial u}+\frac{p_1}{U}\frac{\partial f}{\partial r_1}+\frac{p_2}{U}\frac{\partial f}{\partial r_2}\right\}+f =\int f(\mathbf{p}',\mathbf{r},t)n(\mathbf{p}',\mathbf{p})\bar{d}\mathbf{p}', \tag{45}$$

where u is the distance measured along the shower axis, so $t=uC_2$ where C_2 is the angle made by the shower axis with the vertical. The arguments $\mathbf{p}'$, $\mathbf{p}$ of $n(\mathbf{p}',\mathbf{p})$ are equivalent to those

$$[U=|\mathbf{p}|,\ U'=|\mathbf{p}'|,\ C=\mathbf{p}\cdot\mathbf{p}'/UU']$$

of the previous section, and $\bar{d}\mathbf{p}'=dp_1'dp_2'dp_3'/2\pi p'^2$.

Henceforth C will represent the cosine of the angle between the vector $\mathbf{p}$ and the shower axis, and Ψ the angle between the component of $\mathbf{p}$ normal to the shower axis and the radius vector $\mathbf{r}$. Then, replacing the variable $\mathbf{p}$ by U, C; $\mathbf{p}'$ by U', C'; and $\mathbf{r}$ by r, Ψ, (45) becomes

$$\frac{1}{\theta'(t)}\left\{C\frac{\partial f}{\partial u}-S\cos\Psi\frac{\partial f}{\partial r}+\frac{S\sin\Psi}{r}\frac{\partial f}{\partial\Psi}\right\}+f =\int_U^\infty dU'\int_{-1}^1 dC'\int_0^{2\pi}\frac{d\Psi'}{2\pi}f(U',C',\Psi',r,t) \times n\{U',U,CC'+SS'\cos(\Psi-\Psi')\}, \tag{46}$$

where $C=-P_3/U$, $S=(1-C^2)^{\frac{1}{2}}$, and $S'=(1-C'^2)^{\frac{1}{2}}$.

The solution of this equation will now be obtained in the form (6) of Sec. 2, which, with the aid of (12) will enable us to construct the angular and radial distribution functions for the nucleon component of extensive air showers.

The integral on the right-hand side of (46) is first developed in a series of differential coefficients of $f(U',C,\Psi)$ with respect to C and Ψ, in a manner precisely analogous to that of the previous section. If P is the associated Legendre operator, defined by

$$P\equiv\frac{\partial}{\partial C}S^2\frac{\partial}{\partial C}+\frac{1}{S^2}\frac{\partial^2}{\partial\Psi^2}, \tag{47}$$

one has

$$\int_{-1}^1 dC'\int_0^{2\pi}\frac{d\Psi'}{2\pi}f(U',C',\Psi') \times n\{U',U,CC'+SS'\cos(\Psi-\Psi')\} =\sum_{l=0}^\infty 4^{-l}(l!)^{-2}P(P+1.2)\cdots\{P+l(l-1)\} \times f(U',C,\Psi)n_{(l)}(U',U), \tag{48}$$

where

$$n_{(l)}(U', U) = 2^l \int_{-1}^{1} n(U', U, c)(1-c)^l dc \tag{49}$$

is the same function as in (43). Defining the Mellin transform $f(v, C, \Psi)$ of $f(U, C, \Psi)$ by

$$f(v, C, \Psi) = \int_0^\infty U^v dU f(U, C, \Psi), \tag{50}$$

(46) reduces to

$$C_2 C \frac{\partial f(v)}{\partial \theta} - \frac{S}{\theta'(t)} \cos\Psi \frac{\partial f(v)}{\partial r} + \frac{S}{\theta'(t)} \frac{\sin\Psi}{r} \frac{\partial f(v)}{\partial \Psi} + h(v) f(v)$$
$$= \sum_{l=1}^{\infty} 4^{-l}(l!)^{-2} n_{(l)}(v) P(P+2) \cdots \times\{P + l(l-1)\} f(v-l), \tag{51}$$

where

$$n_{(l)}(v) = U'^l \int_0^\infty (U/U')^v n_{(l)}(U', U) dU, \tag{52}$$

and

$$h(v) = 1 - n_{(0)}(v). \tag{53}$$

Equation (51) may be simplified by introducing the further transform

$$g(v, K, \beta) = \int_0^{2\pi} d\Psi \int_0^\infty r dr \times \exp\{iKr\cos(\Psi-\beta)\} f(v, r, \Psi), \tag{54}$$

of which the inverse is

$$f(v, r, \Psi) = (2\pi)^{-2} \int_0^{2\pi} d\beta \int_0^\infty K dK \times \exp\{-iKr\cos(\Psi-\beta)\} g(v, K, \beta). \tag{55}$$

Then one has

$$C_2 C \frac{\partial g(v)}{\partial \theta} - iKS\cos\beta\, g(v)/\theta'(t) + h(v) g(v)$$
$$= \sum_{l=1}^{\infty} 4^{-l}(l!)^{-2} n_{(l)}(v) Q(Q+2) \cdots \times\{Q + l(l-1)\} g(v-l), \tag{56}$$

where Q is the operator defined in (19). The initial conditions of the preceding equations are

$$\begin{aligned} f(U, C, r, \theta=0) &= \delta(U-U_0)\delta(1-C)\delta(r)/2\pi r, \\ f(v, C, r, \theta=0) &= U_0{}^v \delta(1-C)\delta(r)/2\pi r, \\ g(v, \theta=0) &= U_0{}^v \delta(1-C). \end{aligned} \tag{57}$$

On the left-hand side of (56) we may set $C=1$, with the neglect of terms differing by an order U_0^{-1} in comparison with those retained.

In an isothermal atmosphere with surface pressure p_0 and surface density δ_0,

$$\begin{aligned} \theta(t) &= (p_0/75g)\exp(-g\delta_0 t/p_0), \\ \theta'(t) &= -g\delta_0 \theta(t)/p_0, \end{aligned} \tag{58}$$

where g is the acceleration due to gravity. The ratio p_0/δ_0, according to the gas laws, is directly proportional to the absolute temperature. Since the denominator θ' in (56) depends on θ, this equation, unlike (33), cannot be solved by Laplace transforms. We therefore adopt the following almost equivalent iteration procedure.

An initial value $g_0(v)$ of $g(v)$ is derived by neglecting the right-hand side of (56) and solving the resultant equation with the help of (57):

$$g_0(v) = U_0{}^v \exp\{-h(v)\theta/C_2\}\delta(1-C). \tag{59}$$

This value is now substituted in the right-hand side of (56) and the resultant equation solved in order to determine the second iteration value $g_1(v)$ of $g(v)$:

$$g_1(v) = g_0(v) + \exp[-h(v)\theta/C_2] \times \sum_{l=1}^{\infty} U_0{}^{v-l}(-2)^{-l}(l!)^{-1} n_{(l)}(v) \delta^{(l)}(1-C) \times \int_0^\theta \exp[\{h(v)-h(v-l)\}\lambda/C_2 + iKS\cos\beta\{l(\lambda)-l(\theta)\}] d\lambda, \tag{60}$$

with the help of (22) and setting $l(\theta) = -\int^\theta d\Theta(\lambda)/\theta'(\lambda)$. In the integrand of (60) one now expands the factor

$$\exp[iKS\cos\beta\{l(\lambda)-l(\theta)\}] = \sum_{m=0}^{\infty} (m!)^{-1}[iKS\cos\beta\{l(\lambda)-l(\theta)\}]^m. \tag{61}$$

Since $S^{2l+1}\delta^{(l)}(1-C)=0$, terms with $m>2l$ will vanish in (60). The terms with $m=1, 2, 3, \cdots, (2l-1)$ correspond to mixed angular-radial moments of no interest for our present purpose; we therefore consider only the terms with $m=0$ and $m=2l$, which correspond to pure angular and radial moments, respectively. Thus

$$g_1(v) = g_0(v) + \exp\{-h(v)\theta/C_2\} \times \sum_{l=1}^{\infty} U_0{}^{v-l} n_{(l)}(v)(l!)^{-1} 2^{-l} \Bigg[(-1)^l \delta^{(l)}(1-C) \times \int_0^\theta \exp[\{h(v)-h(v-l)\}\lambda/C_2] d\lambda$$
$$+ (KS\cos\beta)^{2l}(2l!)^{-1}\delta^{(l)}(1-C) \int_0^\theta \exp[\{h(v)-h(v-l)\}\lambda/C_2] \times\{l(\lambda)-l(\theta)\}^{2l} d\lambda + \text{other terms}\Bigg]. \tag{62}$$

By repeating the iteration procedure one can obtain a better approximation for $g(v)$; we have verified numerically that the corrections so obtained are negligible, for the radial moments, though in the case of the angular moments they may amount to 50 percent of the value of terms so far obtained. We shall not reproduce the elementary but tedious calculations here. Substituting

$$S^{2l}\delta^{(l)}(1-C)=(-2)^l l!P_l(1)=(-2)^l l!, \quad (63)$$

and taking the inverse transform defined by (55), we find

$$f(v,r,\Psi)=U_0^v\exp\{-h(v)\theta/C_2\}\delta(1-C)\delta(r_1)\delta(r_2)$$
$$+\exp\{-h(v)\theta/C_2\}\sum_{l=1}^{\infty}U_0^{v-l}n_{(l)}(v)\Big[(-2)^{-l}(l!)^{-1}\delta^{(l)}(1-C)\delta(r_1)\delta(r_2)\int_0^\theta\exp[\{h(v)-h(v-l)\}\lambda/C_2]d\lambda$$
$$+(2l!)^{-1}\delta(1-C)\delta^{(2l)}(r_1)\delta(r_2)\times\int_0^\theta\exp[\{h(v)-h(v-l)\}\lambda/C_2]\times\{l(\lambda)-l(\theta)\}^{2l}d\lambda+\text{other terms}\Big], \quad (64)$$

where $r_1=r\cos\Psi$ and $r_2=r\sin\Psi$. By comparing (64) with (6) one infers the following values for $f_{(l,m,n)}(r)$:

$$f_{(0,0,l)}(v)=U_0^{v-l}n_{(l)}(v)\exp\{-h(v)\theta/C_2\}\times\int_0^\theta\exp[\{h(v)-h(v-l)\}\lambda/C_2]d\lambda, \quad (65)$$

$$f_{(2l,0,0)}(v)=U_0^{v-l}n_{(l)}(v)\exp\{-h(v)\theta/C_2\}\times\int_0^\theta\exp[\{h(v)-h(v-l)\}\lambda/C_2]\{l(\lambda)-l(\theta)\}^{2l}d\lambda$$
$$=U_0^{v-l}(2l)!\left(\frac{g\delta_0}{p_0}\right)^{-2l}\frac{n_{(l)}(v)\exp\{-h(v)\theta/C_2\}}{h(v)-h(v-l)}\times\sum_{k=1}^{\infty}\frac{\{h(v)-h(v-l)\}^k}{k^{2l}k!}\left(\frac{\theta}{C_2}\right)^k. \quad (66)$$

For a single particle of energy U_0, incident at an angle with the vertical whose cosine is C_2 at the top of the atmosphere, the angular and radial moments relative to the shower axis are given by

$$f_{(l,m,n)}(U_0,U)=\frac{1}{2\pi i}\int_{v_0-i\infty}^{v_0+i\infty}f_{(l,m,n)}(v)U^{-(v+1)}dv. \quad (67)$$

For a differential power-law spectrum whose exponent is $\gamma+1$, the corresponding moments are

$$\gamma U_c^{-1}\int_{U_c}^{\infty}(U_c/U_0)^{\gamma+1}f_{(l,m,n)}(U_0,U)dU_0.$$

For the corresponding integral power-law spectrum, one has

$$F_{(l,m,n)}(U)=\gamma U_c^{-1}\int_U^\infty dU\int_{U_c}^\infty dU'(U_c/U')^{\gamma+1}\times f_{(l,m,n)}(U',U). \quad (68)$$

On substituting (65), (66), and (67) one obtains

$$F_{(0,0,l)}(U)=\frac{U^{-l}}{2\pi i}\int_{v_0-i\infty}^{v_0+i\infty}\left(\frac{U_c}{U}\right)^v\times\frac{n_{(l)}(v+l)[\exp\{-h(v)\theta/C_2\}-\exp\{-h(v+l)\theta/C_2\}]}{\{h(v+l)-h(v)\}}\times\frac{\gamma dv}{(\gamma-v)(v+l)} \quad (69)$$

apart from corrections arising from the advanced stages of the iteration procedure, and

$$F_{(2l,0,0)}(U)=\frac{U^{-l}}{2\pi i}\int_{v_0-i\infty}^{v_0+i\infty}\left(\frac{U_c}{U}\right)^v(2l)!\left(\frac{g\delta_0}{p_0}\right)^{-2l}\times\frac{n_{(l)}(v+l)\exp\{-h(v+l)\theta/C_2\}}{h(v+l)-h(v)}\times\sum_{k=1}^{\infty}\frac{\{h(v+l)-h(v)\}^k}{(C_2k)^{2l}k!}\left(\frac{\theta}{C_2}\right)^k\frac{\gamma dv}{(\gamma-v)(v+l)}. \quad (70)$$

The correction terms for (70) have been evaluated and amount to only 2 percent at small atmospheric depths, rising to 7 percent at sea level.

For energies greater than the geomagnetic cut-off energy U_c, the complex integrals of (69) and (70) are easily evaluated by the method of residues, the integrands having a simple pole at $v=\gamma$ in the right-hand half-plane. Thus

$$F_{(0,0,l)}(U)=\frac{\gamma U^{-l}}{\gamma+l}(U_c/U)^\gamma n_{(l)}(\gamma+l)\times\frac{[\exp\{-h(\gamma)\theta/C_2\}-\exp\{-h(\gamma+l)\theta/C_2\}]}{\{h(\gamma+l)-h(\gamma)\}}, \quad (71)$$

and

$$F_{(2l,0,0)}(U)=\frac{\gamma U^{-l}}{\gamma+l}(U_c/U)^\gamma n_{(l)}(\gamma+l)(2l)!\times\left(\frac{g\delta_0}{p_0}\right)^{-2l}\frac{\exp\{-h(\gamma+l)\theta/C_2\}}{h(\gamma+l)-h(\gamma)}\times\sum_{k=1}^{\infty}\frac{\{h(\gamma+l)-h(\gamma)\}^k}{(C_2k)^{2l}k!}\left(\frac{\theta}{C_2}\right)^k, \quad (72)$$

apart from the correction terms.

TABLE I. The first few moments of the angular and radial distribution functions relative to the shower axis. $C_2=1$ and $\gamma=1.1$ θ is measured in units of the interaction free path.

	$\theta=2$	$\theta=6$	$\theta=13.7$
$F_{(0,0,0)}$	0.303	2.79×10^{-2}	2.83×10^{-4}
$F_{(0,0,1)}$	$0.186y_{(1)}$	$3.64\times10^{-2}y_{(1)}$	$4.92\times10^{-4}y_{(1)}$
$F_{(0,0,2)}$	$0.221y_{(2)}+0.243y_{(1)}^2$	$[4.11y_{(2)}+6.64y_{(1)}^2]\times10^{-2}$	$[4.25y_{(2)}+11.4y_{(1)}^2]\times10^{-4}$
$F_{(2,0,0)}$	$20.0y_{(1)}$	$2.74y_{(1)}$	$1.70\times10^{-2}y_{(1)}$
$F_{(4,0,0)}$	$[2.76y_{(2)}+2.17y_{(1)}^2\times10^4$	$[3.15y_{(2)}+2.47y_{(1)}^2]\times10^3$	$13.6y_{(2)}+10.7y_{(1)}^2$
$F_{(6,0,0)}$	$[4.17y_{(3)}+19.3y_{(1)}y_{(2)}]\times10^7$	$[4.15y_{(3)}+19.2y_{(1)}y_{(2)}]\times10^6$	$[1.35y_{(3)}+6.24y_{(1)}y_{(2)}]\times10^4$

VI. RECONSTRUCTION OF THE DISTRIBUTION FUNCTIONS

It is convenient to reconstruct the radial and angular distribution functions before averaging over all angles of inclination to the shower axis. This requires only a correct application of the theory given in Sec. 2.

The most important step in this procedure is the identification of the proper weight-function $w(x)$ in (12). For this purpose, a knowledge of only the lower moments is useless; and an arbitrary choice will most likely give a divergent series. Indeed, if the function $f(x)$ has a singularity at $x=0$, and $w(x)$ has not a singularity of the same type, the series (12) is inevitably divergent. To identify $w(x)$ correctly without knowledge of the type of singularity involved, one must know the asymptotic behavior of the moment $f_{(n)}$ as $n\rightarrow\infty$. This may then be compared with the corresponding behavior of the moments of the function

$$w(x)=\exp(-Bx^A),$$

which gives

$$w_{(l)}=A^{-1}B^{-(l+1)/A}\Gamma\{(l+1)/A\}.$$

Now the right-hand side of (71) varies asymptotically as $U^{-l}n_{(l)}(\gamma+l)$, or $(2/U)^l y_{(l)}$, as defined in (26); also the right-hand side of (72) varies as $(2/U)^l y_{(l)} \times(g\delta_0/p_0)^{-2l}(2l)!$. For a rational form of $y(x)$, such as $6x(1-x)$ (which we used in our previous work[1,2]) or $(\epsilon+\frac{1}{2})\{(\epsilon+x)(1+\epsilon-x)\ln(1+\epsilon)/\epsilon\}^{-1}$ (which would correspond to a Fermi-type[20] distribution), the factor $y_{(l)}$ is a rational function of l and may be omitted. Then the angular distribution has no singularity, and the radial distribution function, regarded as a function of r^2, has a branch point of the type $\exp(-Br)$, at $r^2=0$. For $y(x)=e^{-\beta x}$, $y_{(l)}$ varies as β^{-l}, so the type of singularity is unchanged, though the value of B is altered. When the numerical values of a limited number (n) of lower moments are available, the most practical procedure is to terminate the series (12) at the $(n-1)$th term and fix B to give correctly the value of the nth moment. (This requires the numerical solution of an algebraic equation of the nth degree.)

We have evaluated the first few moments of both the angular and radial distribution functions relative to the shower axis, given by (71) and (72), together with corrections arising from advanced stages of the iteration procedure. For the power law corresponding to $\gamma=1.1$ and for $C_2=1$ we obtained the results given in Table I, where θ is measured in units of the interaction mean free path. We have used these values to reconstruct the angular and radial distribution functions, but find that, once the weight function has been determined in the manner described above, the series (12) converges so rapidly for small values of the argument that for practical purposes it suffices to consider only the first term. Thus the value of B may be obtained sufficiently accurately from the numerical values of the mean squares. The angular distribution function is

$$\Theta(U,C,\theta/C_2)=2[F^2_{(0,0,0)}/F_{(0,0,1)}] \times\exp\{-2F_{(0,0,0)}(1-C)/F_{(0,0,1)}\}, \quad (73)$$

and the radial distribution function is

$$R(U,r,\theta/C_2)=6[F^2_{(0,0,0)}/F_{(2,0,0)}] \times\exp\{-[6F_{(0,0,0)}/F_{(2,0,0)}]^{\frac{1}{2}}r\}. \quad (74)$$

The numerical values of the $F_{(l,m,n)}$ given above are for a spectrum of particles incident vertically at the top of the atmosphere. Actually, the incident angular distribution is almost isotropic. If $\Theta(U,C,\theta)$ is the angular distribution function,

$$\Theta(U,C,\theta)=\int_U^\infty dU\int\int f(U,C,\Psi,r,\theta)rdrd\Psi \quad (75)$$

calculated for an incident power law spectrum, and $C_2=1$, the corresponding distribution function for other values of C_2 is $\Theta(U,C,\theta/C_2)$, where C is still the angle made with the shower axis. If C_1 is the angle which the track of a particle makes with the vertical, one has

$$C_1=CC_2+SS_2\cos\chi, \quad (76)$$

where χ is the polar angle with respect to the shower axis. Thus the $2l$th moment of the angular distribution with respect to the vertical is

$$2^l\int_0^{2\pi}\frac{d\chi}{2\pi}\int_{-1}^1 dC\Theta(U,C,\theta/C_2)\{1-(CC_2+SS_2\cos\chi)\}^l$$

for a shower axis making an angle C_2 with the vertical, and

$$2^l\int_0^1 dC_2\int_0^{2\pi}\frac{d\chi}{2\pi}\int_{-1}^1 dC\Theta(U,C,\theta/C_2) \times\{1-(CC_2+SS_2\cos\chi)\}^l \quad (77)$$

[20] E. Fermi, Phys. Rev. 81, 683 (1951).

for an incident isotropic distribution. The average number of particles with energy greater than $U > U_c$ at depth θ, averaged over all incident particles, is obtained by setting $l=0$:

$$\int_0^1 F_{(0,0,0)}(U,\theta;C_2)dC_2 = \left(\frac{U_c}{U}\right)^\gamma \int_0^1 e^{-h(\gamma)\theta/C_2}dC_2$$
$$\approx \left(\frac{U_c}{U}\right)^\gamma \frac{\exp\{-h(\gamma)\theta\}}{h(\gamma)\theta} \qquad (78)$$

for all except small atmospheric depths. Similarly the second moment is

$$\int_0^1 \{F_{(0,0,1)}(U,\theta;C_2)+2(1-C_2)F_{(0,0,0)}(U,\theta;C_2)\}dC_2$$

with the omission of a factor $C \approx 1$ in the second term, which reduces to

$$\left(\frac{U_c}{U}\right)^\gamma \int_0^1 \left[\frac{2y_{(1)}W(\gamma)}{U\{\alpha(\gamma+1)-\alpha(\gamma)\}}\frac{\gamma}{(\gamma+1)} \times\{e^{-h(\gamma)\theta/C_2}-e^{-h(\gamma+1)\theta/C_2}\}+2(1-C_2)e^{-h(\gamma)\theta/C_2}\right]dC_2. \qquad (79)$$

The first term does not differ much from

$$\left(\frac{U_c}{U}\right)^\gamma \frac{2y_{(1)}W(\gamma)\gamma}{U\{\alpha(\gamma+1)-\alpha(\gamma)\}(\gamma+1)}\frac{\exp\{-h(\gamma)\theta\}}{h(\gamma)\theta} \times\left\{1-\frac{h(\gamma)\exp\{h(\gamma)-h(\gamma+1)\}\theta}{h(\gamma+1)}\right\};$$

since $h(\gamma)/h(\gamma+1)=0.749$, with $\gamma=1.1$, it gives a contribution to the mean square angle very nearly the same as for vertically incident particles at the top of the atmosphere. The second term gives an energy-independent contribution $2\{\theta h(\gamma)\}^{-1}$ to the mean square angle, owing to the persistence of the initial incident distribution.

To obtain the actual radial distribution from that —$R(U,r,\theta)$ say—calculated for $C_2=1$, one replaces r by $r(\cos^2\chi+C_2^2\sin^2\chi)^{\frac{1}{2}}$ and θ by θ/C_2 and integrates over C_2 and χ. The radial moments of this distribution function do not differ radically from those already calculated, owing to the fact that few incident particles inclined at more than a few degrees to the vertical generate showers which penetrate to a great depth in the atmosphere.

VII. DISCUSSION

The angular distribution function, relative to the shower axis, of the nucleon component of the cosmic radiation is given very nearly by

$$\Theta(U,C,\theta/C_2)=F_{(0,0,0)}B_c\exp\{-B_c(1-C)\}, \qquad (80)$$

which expresses the differential probability of finding a particle with energy greater than U (measured in Bev) at an angle with the shower axis whose cosine is C; θ is the atmospheric depth in units of the interaction mean free path, and C_2 is the angle made by the shower axis with the vertical. Values of B_c for an integral proton primary power-law spectrum of exponent $\gamma=1.1$, and for $C_2=1$ are given below in terms of the mean square angular deviation $y_{(1)}$ is a nucleon-nucleon collision.

The corresponding radial distribution function is given by

$$R(U,r,\theta/C_2)=F_{(0,0,0)}B_r^2\exp(-B_r r), \qquad (81)$$

where r is the normal distance from the shower axis. We find the following values of B_c and B_r:

	$\theta=2$	$\theta=6$	$\theta=13.7$	
B_c	$3.26U/y_{(1)}$	$1.53U/y_{(1)}$	$1.15U/y_{(1)}$	(82)
B_r	$0.301U^{\frac{1}{2}}/y_{(1)}^{\frac{1}{2}}$	$0.246U^{\frac{1}{2}}/y_{(1)}^{\frac{1}{2}}$	$0.310U^{\frac{1}{2}}/y_{(1)}^{\frac{1}{2}}$.	

We have shown[17,18] that, if the cross section for nucleon-nucleon collisions is a quadratic function of the cosine of the scattering angle, in the center-of-mass system of reference, $y_{(1)}=\frac{1}{2}$; this was assumed in our previous calculation of the mean square values of $\{2(1-C)\}^{\frac{1}{2}}$ and r.[1,2] It will be seen from (82) that the value of B_r varies with the depth and has a minimum value at an atmospheric depth of about 450 g/cm^2. When B_r has this minimum value, the spread of a shower attains its maximum extent, as we concluded from our evaluation of the mean square distance from the shower axis, referred to above. On the other hand, the large quantitative values for the mean square distance corresponding to $y_{(1)}=\frac{1}{2}$ could be reconciled with the experimental indications[6,12–15] only on the hypothesis that the distribution possessed a long "tail." The function $\exp(-B_r)$ has a tail, but actually one would require a radial dependence at least like $\exp(-B_r'r^{\frac{1}{2}})$ to account numerically for the experimental results. Thus there remained a discrepancy of an order of magnitude affecting the value of $y_{(1)}$ or one of the physical assumptions on which the theory was based.

Setting aside for the moment the assumed form of differential cross section for nucleon-nucleon collisions, the assumptions on which our results are based will now be examined. It has been supposed throughout that the energies of the particles considered are in the ultra-relativistic range, at a lowest estimate greater than 3 Bev. The results for an "atmosphere" of constant density would be very different from those for an isothermal atmosphere which we have obtained. In the isothermal atmosphere, the density decreases exponentially with height; it is the nearest practicable realization of the physical reality.

We have assumed that the exponent γ of the primary power-law spectrum has the value 1.1, and the numerical results depend quite sensitively on this assumption. Indeed, the results can be brought into agreement with experiment, but only by assuming a value as low as 0.7 for γ. The existing experimental evidence[21] for the very high energies which are important for extensive air showers, however, suggests that, if anything, the value of γ should exceed that which we have assumed. For a given differential cross section, the only way to decrease the mean lateral spread is to increase the number of highly energetic particles throughout the atmosphere; and this conclusion is independent of the type of incident primary spectrum adopted. It is therefore possible, but unlikely, that this particular physical assumption made is the seat of the difficulty, and the reason for the discrepancy must be sought for in the form of cross section adopted.

The results obtained are not much affected by the form of the total cross section $F(U_0, U)$ adopted. They may be improved only by increasing the proportion of energy taken up by the nucleon component at the expense of the mesons. Our choice of $F(U_0, U)$, given in (25), allows, on the average, an energy loss of 33 percent to the meson or mesons in a nucleon-nucleon collision; but even if this loss were completely eliminated, the mean spread would not be sufficiently decreased. Here again, the conclusion is not affected by the functional form, homogeneous or inhomogeneous, of $F(U_0, U)$ adopted. We therefore conclude that the differential cross section adopted previously was at fault.

For relativistic reasons, the particles scattered in a nucleon-nucleon collision must be contained within a cone of semivertical angle $(2U^{-1})^{\frac{1}{2}}$; the particles near the periphery of this cone correspond to backward-scattered particles in the center-of-mass frame of reference, and have low energies in the laboratory frame which contribute little to the propagation of extensive air showers. It is therefore the distribution of particles in the neighborhood of the axis of the cone, i.e., in the forward direction, which is important. Any slowly varying function, such as that adopted by Fermi[20] or ourselves,[17,18] cannot be correct. We therefore arrive at the somewhat surprising conclusion that a cross section of the form $R+S'\cos^2\bar{\theta}$ in the center-of-mass frame—as predicted by most field theories—cannot be correct and that there must be an exponential decrease in the number of scattered particles from the forward direction. We therefore substitute in (25) the function

$$y\{U(1-C)\}=\beta(1-e^{-\beta})^{-1}\exp\{-\beta U(1-C)\}, \quad (83)$$

which gives

$$y_{(l)}=\beta^{-l}(1-e^{-\beta})^{-1}\int_0^{\beta} e^{-x}x^l dx. \quad (84)$$

To reconcile results with experiment, it is necessary to take $\beta\approx 144$. Then

$$y_{(1)}=\beta^{-1}=1/144. \quad (85)$$

At sea level, this gives a half-value distance of 92 m, for the radial distribution of particles with energy greater than 4 Bev as found by Cocconi.[6] At a distance of 1 km the density is approximately 5×10^{-4} times that at the shower axis, which is in sufficient agreement with the Russian factor of 4×10^{-4}.

Recently the angular distribution of penetrating particles with respect to the shower axis at an atmospheric depth of 700 g/cm^2 has been measured by Branch;[22] the experimental half-value is at $2\frac{1}{2}\pm\frac{1}{2}$ degrees, compared with our calculated value of 2°32′. We conclude that all the experimental data are adequately explained by assuming a cross section of the form (83) with $\beta=144$.

A method for the determination of the absorption mean free path[23,24,25] depends on the measurement of the angular distribution of high energy penetrating particles with the vertical, neglecting the scattering due to any collisions which they may have suffered in their passage through the atmosphere. This amounts to neglecting the first of the two terms in Eq. (79). If the value of β were as small as one might expect *a priori*, this procedure would not be justified. However, with the value of β given by (83), the angular deviation from the shower axis due to scattering is negligible compared with the energy independent angular deviation from the vertical which results from the persistence of the incident distribution at the top of the atmosphere.

The preceding results are for an absorber with varying density. A similar treatment may, however, be applied to an absorber with constant density. We shall in the near future present results in this case as well.

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