Small Angle Coherent Scattering of Gammas by Bound Electrons*

J. S. LEVINGER
Louisiana State University, Baton Rouge, Louisiana
(Received March 28, 1952)

Franz has shown that for nonrelativistic changes of photon momentum (q) the coherent scattering of gammas by bound electrons reduces to the form factor calculation. We consider the near-relativistic region, by an expansion in q/mc. The corrections to the form factor calculation are very small for small atomic numbers, the order of magnitude being $(Z/137)(q/mc)$. Numerical results for scattering by K electrons of tin give an increase of the amplitude of 25 percent above the form factor value for $q/mc=1.5$ and a smaller change for smaller values of q. Our results are incomplete since we neglect the effects of binding in the intermediate state.

I. INTRODUCTION

THE coherent scattering of radiation by electrons in an atom is usually calculated as the atomic form factor

$$F(q)=-m^{-1}\int \bar{u}e^{iq\cdot r}u d^3r, \tag{1}$$

where $\bar{u}u$ is the charge density, and q is the change of momentum of the photon: $q=2(\hbar\omega/c)\sin\theta/2$, where θ is the scattering angle. (We should multiply by e^2/c^2 and by $[\frac{1}{2}(1+\cos^2\theta)]^{\frac{1}{2}}$ to find the scattering amplitude.) Franz[1] has written down the expression for the scattering of a high energy photon by a bound electron, using second-order perturbation theory in coordinate space. He shows that his expression reduces to the usual form factor calculation if he neglects binding in the intermediate state and makes the nonrelativistic approximation:

$$q\ll mc. \tag{2}$$

Halpern and Hall[2] have shown that for nonrelativistic change of photon momentum the coherent scattering calculation gives very similar results for their scheme I and scheme II. We shall make calculations by the usual one-electron theory (scheme I) in the near-relativistic region.

Knowledge of the amplitude for elastic scattering of gamma-rays by bound electrons is essential for the analysis of Wilson's experiment[3] on potential, or Delbruck scattering. He has measured elastic scattering in the range 40° to 135° of Co^{60} gammas (1.33 Mev) and ThC″ gammas (2.62 Mev) by $_{50}$Sn, and by $_{82}$Pb and neighboring elements.

Several of us[4] have calculated the coherent scattering for a rather special relativistic value of q (Co^{60} gammas scattered through 180°). In this paper the present author will estimate the relativistic corrections to the scattering amplitude by making an expansion in the parameter q/mc. This expansion in q/mc works well for tin and scatterers of smaller atomic number since the coefficient of q/mc in the expansion contains the small factor $Z/137$. We shall regard our expansion as basically an expansion in just the parameter q/mc; and we shall keep small terms of order $Z/137$ or its powers wherever they do not contain q/mc.

We shall continue the approximation of a free intermediate state. Preliminary work by Peierls, Woodward, and Brown[5] indicates that the corrections to the scattering amplitude resulting from binding in the intermediate state are of the same order of magnitude as the relativistic corrections calculated in this paper. However, the results of this paper are still needed as one part of the complete calculation.

Our correction will be expressed as a factor multiplying the amplitude given by the form factor calculation. We should note that Franz[1] in calculating the form factor used the Thomas-Fermi electronic charge distribution. While this distribution is appropriate for very low values of q, Bethe[6] has shown that for values of q larger than the characteristic momentum of a K electron ($mcZ/137$) one should calculate the form factor using the Dirac wave function for the K electrons. Rohrlich and Rosenzweig[7] have extended Bethe's result by calculating the increase in the form factor due to the L electrons.

In the next section we give the general expression for coherent scattering by bound electrons and find an approximation for the relativistic corrections to the scattering amplitude. In Sec. III we find numerical results for this correction for the case of low Z and for tin. In Secs. IV and V we consider the absorptive contributions to the scattering and the different cases of photon polarization. We shall not compare our calculations with the experiments of Wilson[3] or of Moon and Storruste[8] at this time for three reasons: (1) our calculations appear incomplete because of the approximation of a free intermediate state; (2) the experimental measurements include the effects of potential

* Part of this work was done while the author was a research associate at Cornell University under an ONR contract.
[1] W. Franz, Z. Physik **95**, 652 (1935); **98**, 314 (1936).
[2] O. Halpern and H. Hall, Phys. Rev. **84**, 997 (1951).
[3] R. R. Wilson, Phys. Rev. **82**, 295 (1951).
[4] Greifinger, Levinger, and Rohrlich (to be published).
[5] R. Peierls, private communication.
[6] H. A. Bethe, private communication.
[7] F. Rohrlich, private communication.
[8] P. B. Moon, Proc. Phys. Soc. (London) **A63**, 1189 (1950); A. Storruste, Proc. Phys. Soc. (London) **A63**, 1197 (1950).

scattering, and this has been calculated only for very small angles;[9] (3) only the $\theta=40°$ point of Wilson's measurements falls in the region where our expansion is valid.

II. GENERAL EXPRESSION FOR COHERENT SCATTERING

Following Bethe, we draw the Feynman diagrams and write down the matrix elemtns for coherent scattering by bound electrons. We use the notation symmetrical between initial and final electron states:

$$\mathbf{Q}=\tfrac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)=\tfrac{1}{2}\mathbf{q}, \tag{3}$$

$$\mathbf{K}=\tfrac{1}{2}(\mathbf{k}_1+\mathbf{k}_2). \tag{4}$$

$\mathbf{k}_1$ and $\mathbf{k}_2$ are initial and final photon momenta. The electron momenta are $\mathbf{p}-\mathbf{Q}$ for the initial state and $\mathbf{p}+\mathbf{Q}$ for the final state. The matrix elements for scattering a photon of polarization state μ into state ν are given below: M' for the case where the photon is first absorbed; M'' for the case where the photon is first emitted. (See Appendix A for the derivation.) The principal value gives the dispersive scattering; the contribution at the pole gives the absorptive scattering.

$$M_{\mu\nu}'=\int d^3p\,\bar{\varphi}(p+Q)\{-\gamma_\nu\gamma_\mu[i\boldsymbol{\gamma}\cdot(\mathbf{p}+\mathbf{K}) -\gamma_4(E+\omega)+m]+2i\gamma_\nu(p_\mu-Q_\mu)\}\varphi(p-Q)/(b-a). \tag{5}$$

$$M_{\mu\nu}''=\int d^3p\,\bar{\varphi}(p+Q)\{-\gamma_\mu\gamma_\nu[i\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{K}) -\gamma_4(E-\omega)+m]+2i\gamma_\mu(p_\nu-Q_\nu)\} \times\varphi(p-Q)/(b+a). \tag{6}$$

$E=mc^2-\epsilon$ is the energy of the bound electron; ϵ is its binding energy; the γ's are the Dirac matrices in Pauli's notation; φ is the electronic wave function in momentum space,

$$b=p^2-Q^2+2m\epsilon-\epsilon^2, \tag{7}$$

$$a=2\omega(m-\epsilon)-2\mathbf{p}\cdot\mathbf{K}. \tag{8}$$

Let us first consider the case $\mu=\nu=y$; $\gamma_\mu\gamma_\nu=1$; the photon propagation $\mathbf{K}$ in the x-direction, and $\mathbf{Q}$ in the z-direction. That is, there is no change of polarization, and the polarization is perpendicular to the scattering plane which contains $\mathbf{K}$ and $\mathbf{Q}$. We shall show below that, for $\mathbf{Q}<mc$, the case where the polarizations are both in the scattering plane gives a very similar result, except for the usual factor of $\cos\theta$. There is a much smaller amplitude for change of photon polarization. The scattering amplitude $M_{yy}=M_{yy}'+M_{yy}''$ equals

$$M_{yy}=\int d^3p\,\bar{\varphi}(p+Q)\{2b[-i\boldsymbol{\gamma}\cdot\mathbf{p}+\gamma_4E+m+2i\gamma_yp_y] +2a(-i\gamma_xK+\gamma_4\omega)\}\varphi(p-Q)/(b^2-a^2). \tag{9}$$

[9] F. Rohrlich and R. L. Gluckstern, Phys. Rev. **86**, 1 (1952); H. A. Bethe and F. Rohrlich, Phys. Rev. **86**, 10 (1952).

In the nonrelativistic case, $Q^2=0$, $p=0$ inside the brackets, $b=0$, $a=2m\omega$, and only the γ_4 term need be considered. We then keep only $2a\omega/(-a^2)=-1/m$ for the fraction.

Then the amplitude

$$M_{yy}\cong\int d^3p\,\bar{\varphi}(p+Q)(-1/m)\varphi(p-Q). \tag{10}$$

Equation (10) is just the form factor written in momentum space; it is equivalent to the coordinate space expression of Eq. (1), so we have confirmed the results of Franz. (This result was shown by H. A. Bethe.)

Using Eq. (8) for a, we have the more exact expression

$$M_{yy}\cong\int d^3p\,\bar{\varphi}(p+Q) \times\{-\omega/[(m-\epsilon)\omega-\mathbf{p}\cdot\mathbf{K}]\}\varphi(p-Q). \tag{11}$$

In Eq. (10) we used the product of the fourth components of the four-vectors for the photon $(\omega, \mathbf{K})$ and the electron $(E, \mathbf{p})$. In Eq. (11) we use the invariant expression for the scalar product of the two four-vectors.

We show in the appendix that Eq. (11) is a good approximation for Eq. (9) provided we have (*a*) $Q\leq mc$; (*b*) not too large atomic number; and (*c*) small angle scattering. Condition (*b*) is needed so that the small components of the Dirac wave function can be neglected. This greatly simplifies the numerator since we can then neglect the terms with γ_x, γ_y, or γ_z. Assumptions (*a*) and (*c*) are needed so that we can neglect b [Eq. (7)] compared with a [Eq. (8)].

With $\mathbf{K}$ along the x-axis, and of magnitude $\omega\cos\theta/2$, Eq. (8) becomes

$$a=2\omega[m-\epsilon-p_x\cos\theta/2]=2\omega(m-\epsilon)(1-s); \tag{12}$$

$$s=p_x\cos\tfrac{1}{2}\theta/(m-\epsilon). \tag{13}$$

The amplitude

$$M_{yy}=-(m-\epsilon)^{-1}\int d^3p\,\bar{\varphi}(p+Q)(1-s)^{-1}\varphi(p-Q). \tag{14}$$

Since the wave functions for a bound electron in a Coulomb field are more tractable in coordinate space than in momentum space, we wish to find a way to get from the latter, as in Eq. (14), to the former. We shall do this in an approximate manner by expanding the fraction in powers of s; shortly thereafter we correct for the fact that the integration over d^3p goes to infinity, while our expansion holds only for $|s|<1$. The expansion gives

$$M=-(m-\epsilon)^{-1}\int d^3p\,\bar{\varphi}(p+Q)(1+s+s^2+\cdots)\varphi(p-Q) =m(m-\epsilon)^{-1}[F(Q)+M_1]; \tag{15}$$

$$M_1=\int d^3p\,\bar{\varphi}(p+Q)s^2\varphi(p-Q). \tag{16}$$

The term 1 in the parentheses gives just the form factor result $F(Q)$ multiplied by the factor of $m/(m-\epsilon)$ which is close to unity. The term s gives zero on angular integrations. The main correction is the term M_1, which is evaluated by a Fourier transformation to coordinate space (see Appendix B). M_1 is positive leading to an increase of the scattering amplitude:

$$M_1=\cos^2\tfrac{1}{2}\theta(m-\epsilon)^{-2}\int d^3r\,\exp(2i\mathbf{Q}\cdot\mathbf{r}) \times u^*(r)(-\partial^2/\partial x^2)u(r). \quad (17)$$

We must justify our expansion of $(1-s)^{-1}$. For very large p, $\varphi(p)$ decreases about as p^{-3}, so the integral with p_x^2 does converge, but the integral with p_x^4 and higher powers diverge. Let us divide the integration into 2 regions and use the expansions

$$\begin{aligned}&\text{region } I,\ |s|<1:\quad (1-s)^{-1}=1+s+s^2+\cdots,\\ &\text{region } II,\ |s|>1:\quad (1-s)^{-1}=-(s^{-1}+s^{-2}+\cdots).\end{aligned} \quad (18)$$

In Eq. (15) we have integrated the terms 1 and s^2 over both regions, rather than just over region I. We should have $M_{VV}=[F(Q)+M_1-M_2]m/(m-\epsilon)$, where M_1 is given by Eq. (16) integrating over all p, and for M_2 we integrate over region II:

$$M_2=\int_{II} d^3p\,\bar{\varphi}(p+Q)(1+s+s^2+\cdots +s^{-1}+s^{-2}+\cdots)\varphi(p-Q). \quad (19)$$

The advantage of this formulation is that for $Q<mc$ the term M_1 is greater than M_2. We calculate M_1 exactly by making a Fourier transformation into coordinate space, which is only possible for an integration over all momentum space. We shall calculate M_2 approximately using an asymptotic expression for the momentum space wave functions valid for large momenta, i.e., in region II.

Since we terminate the series in M_1 at s^2, we terminate the M_2 series at s^2 for positive powers of s and terminate the series for negative powers of s at s^{-2}.

The discussion above applies to the calculation of the principal value of the integral of Eq. (14), which we call the amplitude for dispersive scattering. The amplitude for absorptive scattering, represented by the residue at the pole is quite small, as shown by the calculation in Sec. IV.

III. EVALUATION OF RELATIVISTIC CORRECTION

It is convenient to work in atomic units of a_0/Z for position, and of $\hbar/(a_0/Z)=mcZ/137$ for momentum; a_0 is the Bohr radius. We use $\hbar=1$. In atomic momentum units

$$mc=137/Z, \quad (20)$$

$$Q=(137/Z)(\hbar\omega/mc^2)\sin\tfrac{1}{2}\theta. \quad (21)$$

The evaluation of Eq. (17) for M_1 and Eq. (19) for M_2 is easier if we use Schrödinger wave functions for $u(r)$ and $\varphi(p)$ for the K electrons. We normalize so that an integration over r^2dr gives 2 electronic charges. Schrödinger functions are given in Eqs. (22) and (23):

$$u_S(r)=2^{\frac{3}{2}}e^{-r}, \quad (22)$$

$$\varphi_S(p)=(2\pi)^{-\frac{1}{2}}2^{5/2}(1+p^2)^{-2}. \quad (23)$$

Dirac functions for the large component are

$$u_D(r)=2^{1+\gamma}r^{\gamma-1}e^{-r}[(2\gamma)!]^{-\frac{1}{2}}, \quad (24)$$

$$\varphi_D(p)=\frac{2^{1+\gamma}(\gamma+1)!}{(2\pi)^{\frac{1}{2}}[(2\gamma)!]^{\frac{1}{2}}}\frac{\sin[(\gamma+1)\tan^{-1}p]}{p(1+p^2)^{(\gamma+1)/2}}, \quad (25)$$

$$\gamma=(1-\alpha^2Z^2)^{\frac{1}{2}}\cong 1-\tfrac{1}{2}\alpha^2Z^2=1-\epsilon. \quad (26)$$

$\alpha=1/137$; ϵ is the binding energy in units of mc^2. For $\pi\epsilon/2\ll 1$, we can expand Eq. (25) for the Dirac wave function in momentum space as

$$\varphi_D(p)\cong\varphi_S(p)[1+(p^2-1)\epsilon\tan^{-1}p/2p](1+p^2)^{\epsilon/2}. \quad (27)$$

We have omitted the small difference in the normalization constants.

Equation (27) shows that the Schrödinger function $\varphi_S(p)$ is a good approximation to the Dirac function for $\epsilon p\ll 1$, since for $\epsilon p\ll 1$ and $p\gg 1$ we have

$$\varphi_D(p)\cong\varphi_S(p)(1+\pi\epsilon p/4)p^{\epsilon}. \quad (28)$$

We are interested in values of Q, and consequently of p, of order of magnitude $mc=1/Z\alpha$. Then

$$\pi\epsilon p/4=(\pi/4)(\tfrac{1}{2}\alpha^2Z^2)(1/Z\alpha)=\pi\alpha Z/8. \quad (29)$$

The Schrödinger wave function is a good approximation for the Dirac wave function (large component) for $Z\leq 30$ and $p\leq mc$, and does not give too bad results for $Z=50$, where we shall make a numerical comparison. Also for $Z\leq 30$, the small Dirac components can be neglected, owing to the small normalization constant $(1-\gamma)^{\frac{1}{2}}(1+\gamma)^{-\frac{1}{2}}=\frac{1}{2}\alpha Z$. (See Appendix B.)

Using Schrödinger wave functions for K electrons to evaluate the form factor $F(Q)$ in Eq. (1) or Eq. (10) we have the well-known result

$$F(Q)=2/(1+Q^2)^2. \quad (30)$$

The first relativistic correction M_1 [Eq. (17)] is

$$M_1=\cos^2\tfrac{1}{2}\theta(m-\epsilon)^{-2}[\tan^{-1}Q/Q^3-1/Q^2(1+Q^2)]. \quad (31)$$

We see that M_1 decreases more slowly than $F(Q)$ for $Q\gg 1$ (as Q^{-3} instead of as Q^{-4}).

The next relativistic correction M_2 [Eq. (18)] can be approximated writing $(1+p^2)^{-2}\cong p^{-4}$ in Eq. (23).

TABLE I. Coherent scattering by K-electrons of tin using Dirac wave functions.

Q (atomic units)	$F(Q)$	M_1	M_2	$\frac{M_{vv}}{F(Q)}$	$[M_{vv}/F(Q)]_{corr}$
0	2.0	0.098	0.015	1.10	1.08
1	0.53	0.028	0.011	1.10	1.11
2	0.105	0.019	0.0062	1.19	1.28
3	0.0304	0.0092	0.0031	1.27	1.49

We take $\cos^2\frac{1}{2}\theta=1$, and $\epsilon\ll m$.

$$M_2=\frac{64}{\pi}\int_{II}\frac{(p_x^2/m^2c^2+1+m^2c^2/p_x^2)d^3p}{[(p^2+Q^2)^2-4p_z^2Q^2]^2}$$

$$\cong\frac{8}{6\pi}\Bigg[\frac{15m^6c^6+22m^4c^4Q^2+2m^2c^2Q^4+Q^6}{Q^6mc(m^2c^2+Q^2)^2}+\frac{Q^4+3m^2c^2Q^2-15m^4c^4}{m^2c^2Q^2}\cot^{-1}\left(\frac{mc}{Q}\right)\Bigg]$$

$$\cong 2.3(mc)^{-5}(1-2.1Q^2/m^2c^2\cdots). \quad (32)$$

The last expression is convenient for $Q\ll mc$. In evaluating the integral over region II (i.e., $|p|\geq mc$ or $1/Z\alpha$ in atomic units) we have underestimated the integral somewhat by neglecting the $4p_z^2Q^2$ in the denominator.

Similar calculations can be made using the Dirac large component wave functions given in Eq. (24) and (25). The form factor is calculated by Bethe as

$$F(Q)=\sin(2\gamma\tan^{-1}Q)/\gamma Q(1+Q^2)^\gamma. \quad (33)$$

The main correction M_1 is

$$M_1=-\tfrac{1}{2}\cos^2\tfrac{1}{2}\theta(m-\epsilon)^{-2}2^{2+2\gamma}[(2\gamma)!]^{-1}\times[s(2\gamma)-(2\gamma-1)s(2\gamma-1)+(\gamma-1)^2s(2\gamma-2)-t(2\gamma)+(2\gamma-3)t(2\gamma-1)+(\gamma-1)(3-\gamma)t(2\gamma-2)], \quad (34)$$

where

$$s(n)=(n-1)!\sin(n\tan^{-1}Q)/Q2^{n+1}(1+Q^2)^{n/2}, \quad (35)$$

$$t(n)=(n-3)!\frac{[n(n-1)Q^2-2]\sin(n\tan^{-1}Q)+2Qn\cos(n\tan^{-1}Q)}{Q^32^{n+1}(1+Q^2)^{n/2}}. \quad (36)$$

The second correction M_2 is calculated for Dirac wave functions using the Schrödinger wave function as an approximation, thus obtaining the results given in Eq. (32).

Numerical results for coherent scattering by K electrons of tin are given in Tables I and II. In Table I, $F(Q)$, M_1 and M_2 are given using the large component of the Dirac wave function. In the last two columns we give the ratio scattering amplitude to form factor $=M_{vv}/F(Q)$. The next to last column is found using $M_{vv}=[m/(m-\epsilon)][F(Q)+M_1-M_2]$. The last column gives this result making a correction for the small Dirac components discussed in Appendix B. For tin $m/(m-\epsilon)=1.06$. We have taken $\cos^2\frac{1}{2}\theta=1.0$. Note that Q is given in atomic units of momentum [Eq. (21)] and that amplitudes are in units of $r_0=e^2/mc^2$. The row $Q=0$ is for illustrative purposes only, as for $Q\ll 1$ the L, $M\cdots$ electrons are of great importance, and our calculation holds only for K electrons.

TABLE II. Coherent scattering by K-electrons of tin using Schrödinger wave functions.

Q	$F(Q)$	$M_{vv}/F(Q)$
0	2.0	1.10
1	0.50	1.15
2	0.080	1.14
3	0.020	1.14

For comparison we give in Table II the results for the form factor $F(Q)$ and the ratio $M_{vv}/F(Q)$ using Schrödinger wave functions. Comparing the two tables we see that the Schrödinger wave functions give good results for small Q, but underestimate both the form factor and the ratio $M_{vv}/F(Q)$ for large Q. The values given for $Q=3$ must be regarded as dubious since the expansion parameter Q/mc becomes greater than 1. (Still, the corrections M_1 and M_2 in Table I are reasonably small compared to $F(Q)$ for $Q=3$; i.e., $Q=1.1\ mc$, or scattering of Co^{60} 1.332-Mev gammas through an angle of 48°. The small values of the corrections M_1 and M_2 justify the approximations made in their calculation. We also note that the correction in Appendix B (last column, Table I) for the effect of the small Dirac components does not change the ratio $M_{vv}/F(Q)$ greatly, except for the dubious case $Q=3$.

As an illustration of the use of Schrödinger wave functions let us consider scattering by K electrons in an atom for the case $mc\gg Q\gg 1$, where all quantities are expressed in atomic units. Equation (30) gives $F(Q)\cong Q^{-4}$; Eq. (31) gives $M_1\cong(\pi/2)(mc)^{-2}Q^{-3}$; Eq. (32) gives $M_2\cong 2.3(mc)^{-5}$. We see that the relativistic corrections M_1 and M_2 are extremely small relative to form factor due to the extra powers of the mass in the denominator. For very small atomic numbers the relativistic corrections are very small even for Q comparable to mc, since Q and mc are comparable large numbers, when expressed in atomic units.

We conclude that the relativistic corrections considered here cause some increase (10 to 30 percent) in the scattering amplitude for coherent small angle scattering by K electrons of tin and that the corrections increase with increasing Q. The simpler equations using Schrödinger electronic wave functions are useful for rough evaluation of the amplitude/form factor ratio, up to atomic number 50. As noted above, the scattering amplitude may be changed appreciably due to corrections for binding in the intermediate state.

IV. ABSORPTIVE SCATTERING

The amplitude for absorptive scattering is calculated as π times the residues at the poles on the real axis. In Eq. (14) for M_{yy} we have a pole at $s=p_x\cos\frac{1}{2}\theta/(m-\epsilon)=1$: i.e., $p_x=mc$. Then, as in the calculation of M_2 above, we can use the Schrödinger form (Eq. (23)) for an approximate momentum wave function for large momenta. The integral over p_y and p_z is done as before neglecting the term $4p_z^2Q^2$ in the denominator in Eq. (32), giving

$$M_{yy}=-\int\frac{dp_x}{p_x-mc}\int\int_{-\infty}^{\infty}dp_ydp_z\bar{\varphi}(p+Q)\varphi(p-Q). \quad (37)$$

The amplitude M_p for absorptive scattering is given in units of r_0 by

$$M_p=(8\pi m/3)(m^2c^2+Q^2)^{-3}. \quad (38)$$

As above, mc and Q are in atomic units; $mc=137/Z=2.7$ for tin. The absorptive scattering should be compared with the dispersive scattering given approximately by Bethe's form factor [Eq. (33)]. For $Q\ll1$, $M_p=(8\pi/3)(mc)^{-5}$ while the form factor$=2$, so the amplitude for absorptive scattering is negligible, being only 2 percent of the form factor for the case of tin. It is even smaller for smaller Z. (Note that for combining the absorptive scattering and form factor we add the squared amplitudes, since they are out of phase by 90°. In combining various terms for the dispersive scattering we add the amplitudes, and subsequently square to find the cross section.) The absorptive scattering is small even in the most favorable case of $Q=mc$. Eq. (33) gives a form factor of $0.8(mc)^{-2.36}$, while Eq. (38) gives an absorptive scattering amplitude of $(\pi/2)(mc)^{-5}$. The ratio $1.3(mc)^{-2.1}=0.2$ for tin.

V. OTHER POLARIZATION CASES

The case of photon polarization in the scattering plane, both before and after scattering, is quite similar to the case of perpendicular polarization discussed in Sec. III. For the limiting case of zero scattering angle, the two results must be identical, from the symmetry of the problem. In the general case, we combine Eq. (5) and Eq. (6) to give a result analogous to our Eq. (9) for the special case treated above. For our present case the combination $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\cos\theta$, where θ is the scattering angle. This combination gives a result just the same as in Sec. III except for the extra factor of $\cos\theta$; which occurs here in just the same way as in the Thomson calculation of coherent electronic scattering by free electrons. The terms with

$$\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu=2\gamma_\mu\gamma_\lambda\sin\theta$$

give zero using only large Dirac components since they contain two different space-like Dirac operators, multiplying expressions containing either one or no Dirac operators [see Eq. (9)]. Here λ is in the scattering plane, perpendicular to μ. For the case of Sec. III this term was identically zero. For atoms of low atomic number, where we neglect the small Dirac components we obtain then just $\cos\theta$ times the result of Sec. III. For unpolarized gammas this gives a cross section $\frac{1}{2}(1+\cos^2\theta)[r_0F(Q)]^2$.

The effects of the small Dirac components are estimated as in Appendix B. They are quite small for the case of tin, as here we are interested only in the difference between the values of this correction term for the two different polarization cases.

In the nonrelativistic form factor calculation there is zero amplitude for change of direction of photon polarization by 90°, i.e., either (1) polarized perpendicular scattering plane before, polarized parallel after; or (2) polarized parallel before, and perpendicular after. We find that the amplitude for this process is quite small in our case. We note that for this case, as for the case of absorptive scattering, the amplitude combines with the form factor amplitude by first squaring and then adding.

In combining Eqs. (5) and (6) we now use $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=0$ and $\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu=2\gamma_\mu\gamma_\nu$. By the same argument as above, the large Dirac components give zero amplitude for this scattering process. The main term, using one large Dirac component and one small Dirac component, is estimated in a manner similar to that in Appendix B, giving

$$M\cong\tfrac{1}{3}[(1-\gamma)/(1+\gamma)]^{\frac{1}{2}}\sin\tfrac{1}{2}\theta F(Q). \quad (39)$$

Even for $Q=mc$, the ratio $M/F(Q)$ equals only 0.02 for scattering of Co^{60} gammas by tin ($\gamma=0.932$) and should be neglected.

VI. DISCUSSION

We have found that coherent scattering by bound electrons for photon change of momentum in the near-relativistic region gives a scattering amplitude somewhat larger than that of the form factor calculation. The numerical results have been calculated for scattering by K electrons of tin. It is straightforward to obtain numerical results for smaller atomic number. For larger atomic number more careful calculations are needed for terms that are small in the case considered here, so that they could be estimated crudely in the appendix. These calculations could be made by methods like those of this paper.

For the case of tin, for near-relativistic values of photon change of momentum, the scattering amplitude has the following properties: (1) The amplitude is a function of the photon change of momentum q but not appreciably of the photon energy. (2) The dependence on photon polarization is very nearly the same as for the form factor calculation; (*a*) the case polarizations in the scattering plane gives the same scattering amplitude as polarizations perpendicular to the scattering plane, except for the usual factor $\cos\theta$; (*b*) the amplitude for change of photon polarization by 90° is very small.

(3) The absorptive scattering amplitude is much smaller than the amplitude for dispersive scattering.

We are grateful to H. A. Bethe, P. Greifinger, O. Halpern, G. Jaffe, R. Peierls, F. Rohrlich, and R. R. Wilson for discussions of this problem.

APPENDIX A

Here we shall derive the amplitudes for coherent scattering of radiation by a bound electron. This calculation neglects the effects of binding in the intermediate state, since it uses the Feynman propagators for a free electron. For this type of calculation the Feynman method is equivalent to usual second-order perturbation theory. Greifinger[10] has shown explicitly that Franz' expression[1] reduces to our present result if a free intermediate state is assumed. The derivation given here was done originally by H. A. Bethe and was extended to a more general case by F. Rohrlich and the present author.

The Feynman diagrams for $M_{\mu\nu}'$ (photon of polarization μ absorbed; photon of polarization ν subsequently emitted) and $M_{\mu\nu}''$ (ν emitted, then μ absorbed) are shown in Fig. 1. We use the symmetrical notation of Eqs. (3) and (4) of the text. We shall use Feynman's notation for four-vectors, and for the Dirac γ matrices.[9] The amplitude $M_{\mu\nu}'$ is

$$M_{\mu\nu}'=\int\bar{\varphi}(p+Q)i\gamma_\nu[i(\not{p}+\not{K})+m]^{-1}i\gamma_\mu\varphi(p-Q)d^3p$$

$$=\int\bar{\varphi}(p+Q)\gamma_\nu\frac{i(\not{p}+\not{K})-m}{(\not{p}+\not{K})^2+m^2}\gamma_\mu\varphi(p-Q)d^3p, \quad \text{(A1)}$$

where $\not{p}\equiv\sum_\mu\gamma_\mu p_\mu$.

Note that the integration over p_4 (energy) has already been done, to give the case of elastic scattering. The energy denominator is evaluated using the relations

$$(\not{p})^2=p^2-E^2=p^2-(m-\epsilon)^2,$$

and

$$2\not{p}\cdot\not{K}=2\mathbf{p}\cdot\mathbf{K}-2(m-\epsilon)\omega.$$

From Eqs. (3) and (4) we have $(\not{K})^2=-Q^2$, the square of the length of the three-vector. The numerator is evaluated using $\gamma_\nu\not{p}\gamma_\mu=-\gamma_\nu\gamma_\mu\not{p}+2\gamma_\nu p_\mu$. We use an analogous expression for $\gamma_\nu\not{K}\gamma_\mu$, and then use $K_\mu=-Q_\mu$.

Following this procedure, Eq. (A1) gives us the result for the amplitude $M_{\mu\nu}'$ given as Eq. (5) of the text. $M_{\mu\nu}''$, the amplitude for emission first, is treated in a completely analogous manner using the Feynman diagram of Fig. 1(B), and gives Eq. (6) of the text.

APPENDIX B

The Fourier transformation leading from Eq. (16) for M_1 as an integral in momentum space to Eq. (17) for M_1 as an integral in coordinate space is based on the orthogonality of the vectors $\mathbf{K}$ and $\mathbf{Q}$. We write

$$\varphi(p+Q)=\int\exp[-i(\mathbf{p}+\mathbf{Q})\cdot\mathbf{R}]u(R)d^3R, \quad \text{(A2)}$$

$$\varphi(p-Q)=\int\exp[-i(\mathbf{p}-\mathbf{Q})\cdot\mathbf{r}]u(r)d^3r. \quad \text{(A3)}$$

Since $s=\mathbf{p}\cdot\mathbf{K}/(m-\epsilon)\omega$ and $\mathbf{K}$ is perpendicular to $\mathbf{Q}$, we can write

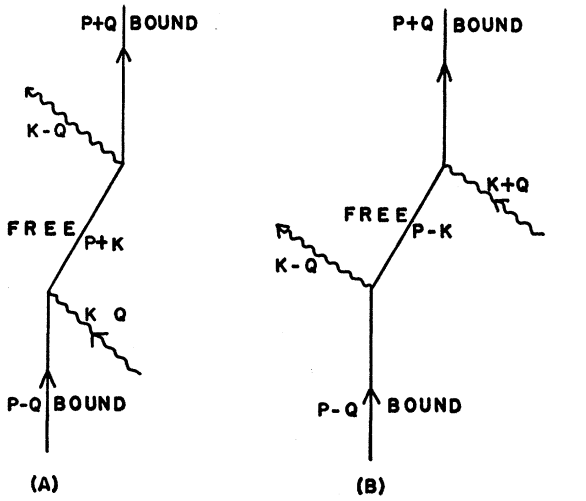

FIG. 1. Feynman diagrams for scattering of radiation: 1(A), for the case of absorption first; 1(B), for the case of emission first. "Bound" means use the momentum space wave function of a bound electron; "free" means use the (approximate) propagator for a free electron.

$$s^2\varphi(p-Q)=\cos^2\tfrac{1}{2}\theta(m-\epsilon)^{-2}$$

$$\times\int(-\partial^2/\partial x^2)\exp[-i(\mathbf{p}-\mathbf{Q})\cdot\mathbf{r}]u(r)d^3r. \quad \text{(A4)}$$

Using expressions (A2) and (A4) in Eq. (16), performing two integrations by parts so that the $-\partial^2/\partial x^2$ operates on the wave function $u(r)$, and integrating over d^3R and d^3p, we have Eq. (17). This proof was suggested by J. Goldstein.

APPENDIX C

We discuss here two different types of terms that were neglected in the calculation of the coherent scattering amplitudes for polarization perpendicular to the scattering plane. In going from Eq. (9) to Eq. (11) we neglected b in comparison with a [see Eqs. (7) and (8)]; we also neglected the contribution of the small Dirac components.

We neglected b both in the numerator and in the denominator of Eq. (9). The b term in the numerator

[10] P. Greifinger, private communication.

contributes:

$$M_b=\int d^3p\,\bar{\varphi}(p+Q)(2b\epsilon/a^2)\varphi(p-Q)$$

$$\leq\tfrac{1}{4}(Z\alpha)^2\sin^2\tfrac{1}{2}\theta F(Q)\leq 0.006F(Q). \quad \text{(A5)}$$

We find the second expression on the right by taking $b\cong -Q^2$ (an overestimate); $a=2m\omega$. The numerical result applies for 48° scattering of Co^{60} gamma-rays from tin ($Q=3$ atomic units$=1.1\ mc$).

The effect of b in the denominator of Eq. (9) can be estimated by an expansion in b^2/a^2:

$$M_b'=\int d^3p\,\varphi(p+Q)(-1/m)(b^2/a^2+\cdots)\varphi(p-Q)$$

$$\leq\tfrac{1}{4}(Q/mc)^2\sin^2\tfrac{1}{2}\theta F(Q)\leq 0.04F(Q). \quad \text{(A6)}$$

We have made the same approximations as before, and the numerical result applies to the value of Q given above.

We shall neglect both these terms in our present calculation. The first term becomes larger for high Z, the second term for high Q. Both become larger with scattering angle θ.

The small Dirac components have a normalization factor smaller by a factor $[(1-\gamma)/(1+\gamma)]^{\frac{1}{2}}=0.18$ from that of the large components. (The numerical result applies to tin, $\gamma=0.932$.) The term in Eq. (9) using small components for both $\varphi(p+Q)$ and $\varphi(p-Q)$ will then be about 3 percent of that using the large components for the case of tin, so we shall neglect any relativistic changes in this small term. (The term itself is included in Bethe's form factor calculation.)

If we use a large Dirac component for $\varphi(p+Q)$ and a small component for $\varphi(p-Q)$ (or vice versa), we get an appreciable correction to our result. Neglecting b in comparison with a, we have the matrix element from Eq. (9)

$$M_{LS}=-m^{-1}\int d^3p\,\bar{\varphi}(p+Q)(-i\gamma_x-i\gamma_x s\cdots)\varphi(p-Q)$$

$$=m^{-1}\int d^3p[u_1^*(p+Q)(p_x/m)u_4(p-Q)\sin\theta e^{i\varphi}$$

$$+u_\mu^*(p+Q)\sin\theta e^{-i\varphi}(p_x/m)u_1(p-Q)]. \quad \text{(A7)}$$

The term $-i\gamma_x$ in the first integral goes out on angular integrations, leaving the term $-i\gamma_x s$ as the leading term, written as the second integral. The second integral is expressed as an integral in coordinate space:

$$M_{LS}=-2m^{-1}\int f(r)\sin\theta e^{i\varphi}$$

$$\times\exp(2i\mathbf{Q}\cdot\mathbf{r})m^{-1}(\partial/\partial x)g(r)d^3r. \quad \text{(A8)}$$

(There should, of course, be a further correction term like M_2 of the text.) Here $f(r)$ is the radial function for the small Dirac components. For K electrons it is equal to the large component $g(r)$ [Eq. (24)] except for the term $[(1-\gamma)/(1+\gamma)]^{\frac{1}{2}}$ in the normalization factor. Using the Schrödinger wave function of Eq. (22) as a convenient approximation, we have

$$M_{LS}\cong -m^{-1}[-2m^{-1}(1-\gamma)^{\frac{1}{2}}(1+\gamma)^{-\frac{1}{2}}Q^{-2}]$$

$$\times\left[\frac{1+2Q^2}{(1+Q^2)^2}-\frac{\tan^{-1}Q}{Q}\right]. \quad \text{(A9)}$$

This correction term is calculated for tin and used for the last column of Table I to give $[M_{vu}/F(Q)]_{\text{corr}}$. The term M_{LS} is very small relatively for small Q, and increases to about 15 percent of the form factor for $Q=mc$.