TABLE I. Data regarding narrow angle pairs.

Event	Angle between tracks in degrees	Mean scattering angle in degrees per 100 microns Track C_1	Track C_2	Energy of electrons in Mev C_1	C_2	Total
Pr_1	0.30 ± 0.06	0.78	0.14	32 ± 8	173 ± 25	205 ± 39
Pr_2	$6.3\ \pm0.3$	0.21	0.30	118 ± 24	83 ± 17	201 ± 25
Pr_3	$2.4\ \pm0.5$	1.25	2.78	20 ± 3	9 ± 2	29 ± 3

of 4 microns from the star if the mean life is 10^{-14} sec or longer.[7] Examination of the two collisions described above indicates that Pr_1 occurred less than 1 micron and Pr_2 less than 1/2 micron from the centers of their respective stars, which, however, would not be incompatible with a mean life of the order of 10^{-15} sec.[8] The angle between the electrons would be small since in the above cases the total energy of the neutral pion is about equal to its momentum times velocity of light. Process (ii) occurs in about 3/4 of the pion-proton collisions, and this is again an upper limit for pion collisions in the emulsion. Since Dalitz[6] has calculated that 1 out of 80 neutral pions should decay into two electrons and a gamma-ray, it would be expected that at most about 1 pion collision out of 100 in the emulsion would be associated with a pair of fast electrons.

While electron pairs produced through process (i) would lead to close angular correlation of the tracks, the expected frequency is about 20 times less than measured. On the other hand, although through process (ii) the expected frequency of pairs is of the right order of magnitude, the mean life of the neutral pion would have to be of the order of 10^{-15} sec or less.

We wish to thank Professor H. L. Anderson and the cyclotron group for their assistance in this experiment and Professors E. Fermi and G. Wentzel for very stimulating discussions.

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The Validity of Born Expansions

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RECENTLY the nature of the Born expansions[1] for the case of a nonrelativistic particle scattered by a static potential has been clarified by Jost and Pais.[2] We have supplemented this work by establishing, for central potentials, estimates for the radii of convergence for various energy ranges and any angular momentum.

We consider the radial Schrödinger equation

$$\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+k^2\right)\psi(r)=\lambda V(r)\psi(r), \tag{1}$$

where $\lim_{r\to0}r|V(r)|<\infty$ and $\lim_{r\to\infty}r^2V(r)=0$. The various Born expansions of the solution of (1) differ by the choice of boundary conditions.[3] The following two are commonly used:

$$\psi(0)=0,\ \psi(r)\to\sin\left(kr-\frac{l\pi}{2}\right)+\tan\eta_l\cos\left(kr-\frac{l\pi}{2}\right)\quad\text{for}\quad r\to\infty; \tag{2}$$

$$\psi(0)=0,\ \psi(r)\to\sin\left(kr-\frac{l\pi}{2}\right)+\frac{S_l-1}{2i}\exp i\left(kr-\frac{l\pi}{2}\right)\quad\text{for } r\to\infty. \tag{2'}$$

$S_l-1\equiv e^{2i\eta_l}-1$ occurs in the three-dimensional scattering amplitude; S_l is a scattering matrix element. The iteration of the integral equation equivalent to (1) and (2) leads to a power series in λ for $\tan\eta_l$, and similarly (1) and (2′) yield a series for S_l.

Expansion of $\tan\eta_l$:—For a given $V(r)$ let λ_T be that value of $|\lambda|$ up to which this expansion converges. One can then show from the integral equation that for *all* potentials

$$k=0:\quad \lambda_T\int_0^\infty r|V(r)|dr\geqslant 2l+1; \tag{3}$$

$$\text{All } k:\quad \lambda_T\int_0^\infty r|V(r)|dr\geqslant t_l;$$

l	0	1	2	3		large
t_l	1	2.344	3.339	4.198		$1.157\cdot(2l+1)^{2/3}$

(4)

These estimates are optimal in the sense that the right-hand sides cannot be replaced by larger numbers. The equality signs are approached as $V(r)\to\delta(r-a)$.

For any *fixed* potential (3) and (4) become very conservative for large l. The following asymptotic expression for large l is then useful. Let $r^2|V(r)|$ have its maximum value at r_0. Then

$$\lambda_T\sim\frac{1}{r_0^2|V(r_0)|}\left\{l(l+1)+\left(3-\frac{r_0^2V''(r_0)}{V(r_0)}\right)[l(l+1)]^{1/2}\right\} \tag{5}$$

to within terms of order $<[l(l+1)]^{1/3}$ which contain the energy dependence. At low energies, (5) has an error of only 10–15 percent for the usual potentials, even for $l=1$.

As for the behavior of λ_T at low energies, one can show that

$$l=0:\quad \left.\frac{\partial\lambda_T}{\partial(k^2)}\right|_{k=0}>0, \tag{6}$$

if $V(r)$ does not change sign (otherwise the inequality may go the other way!); and for all potentials

$$l\geqslant1:\quad \left.\frac{\partial\lambda_T}{\partial(k^2)}\right|_{k=0}<0, \tag{7}$$

i.e., λ_T *decreases* as the centrifugal barrier is being overcome. For large l,

$$\left.\frac{\partial\lambda_T}{\partial(k^2)}\right|_{k=0}\to-\frac{1}{|V(r_0)|}. \tag{8}$$

At high energies and for any l

$$\lambda_T\left|\int_0^\infty V(r)dr\right|=\pi k+O(k), \tag{9}$$

provided the integral is neither zero nor infinite; for singular potentials with $\lim_{r\to0}r|V(r)|=\beta$,

$$\lambda_T\beta=\pi k/\log k+O(k/\log k). \tag{10}$$

Expansion of $S_l\equiv e^{2i\eta_l}$:—Calling the radius of convergence λ_S, we find

$$k=0:\quad \lambda_S\int_0^\infty r|V(r)|dr=\lambda_T\int_0^\infty r|V(r)|dr\geqslant 2l+1; \tag{3'}$$

$$\text{All } k:\quad \lambda_S\int_0^\infty r|V(r)|dr\geqslant s_l;$$

l	0	1	2	3		large
s_l	1	2.047	2.783	3.416		$\approx0.86(2l+1)^{2/3}$

(4′)

Again the numbers are optimal. Equation (5) holds also for λ_S, but no inequality corresponding to (6) was found. For

$$l\geqslant1:\quad \left.\frac{\partial\lambda_S}{\partial(k^2)}\right|_{k=0}=\left.\frac{\partial\lambda_T}{\partial(k^2)}\right|_{k=0}, \tag{11}$$

so that (7) and (8) hold also for λ_S.

At high energies

$$\lambda_S/k\to\infty, \tag{9'}$$

provided $|\int_0^\infty V(r)dr|<\infty$.

A number of properties of Born expansions have been derived, some of which have been previously observed.[4]

(1) The expansion for $\tan\eta_l$ converges until the phase shift corresponding to either + or − $|\lambda|V(r)$ becomes + or − $\pi/2$; but, when $ka \ll l+1$ [a = typical dimension of $V(r)$], the smallness of $|\eta_l|$ compared with $\pi/2$ is not a criterion for *rapid* convergence of the Born approximations.

(2) For any l, the existence of bound states implies failure of the Born approximations at zero energy and vice versa. However, even in the absence of bound states the Born series may diverge at some higher energy.

(3) For $l \geqslant 1$, λ_T and λ_S first decrease with increasing energy, before finally increasing. Both increase rapidly with l, like $l(l+1)$.

(4) At high energies, η remains almost proportional to λ up to large values of η, so that it is very effective to expand $\eta = \tan^{-1}(\tan\eta)$.

(5) λ_T and λ_S may differ substantially. Thus, for the $n-p$ ^3S potential, the series for $\tan\eta$ converges at 20 Mev and above, while that for $e^{2i\eta}$ only converges above ≈100 Mev.

(6) For $n-p$ scattering, the failure of the three-dimensional Born expansion at low energies is due entirely to the S-wave, the P-scattering being already convergent.

A detailed account, including illustrative examples, will be published in the near future.

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Entropy and Specific Heat of Liquid He^3

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The formulas for the temperature dependence of viscosity, specific heat, and thermal conductivity of liquid He^3 were given by Singwi and Kothari[1] and later by other investigators,[2,3] on the assumption that the elementary excitations of He^3 are of the Fermi-Dirac type. This viewpoint has been confirmed by the experiments of Weinstock, Osborne, and Abraham[4] on the temperature variation of viscosity of pure liquid He^3. We, here, attempt to explain the entropy of liquid He^3, as calculated by Abraham *et al.*,[5] from their vapor pressure measurements. We have also calculated the specific heat of He^3 for which no experimental data are yet available.

The degeneracy temperature of liquid He^3 is about 5°K. In the temperature range, so far investigated experimentally, He^3 is partially degenerate and, therefore, one has to use the exact rather than the asymptotic formulas for the thermodynamic quantities. For a system of particles, obeying F−D statistics, it can easily be shown that the entropy and the specific heat are,

Table I. Calculated entropy and specific heat as functions of the temperature.

η	Temp T (°K)	Entropy S (cal mole^{-1}, deg^{-1})	Specific heat C_v (cal mole^{-1}, deg^{-1})
20	0.24	0.50	0.48
10	0.48	0.98	0.96
8	0.60	1.20	1.16
7	0.68	1.36	1.28
6	0.79	1.56	1.44
5	0.94	1.82	1.64
4	1.15	2.18	1.86
3	1.47	2.68	2.12
2	2.01	3.38	2.38
1	2.96	4.36	2.68

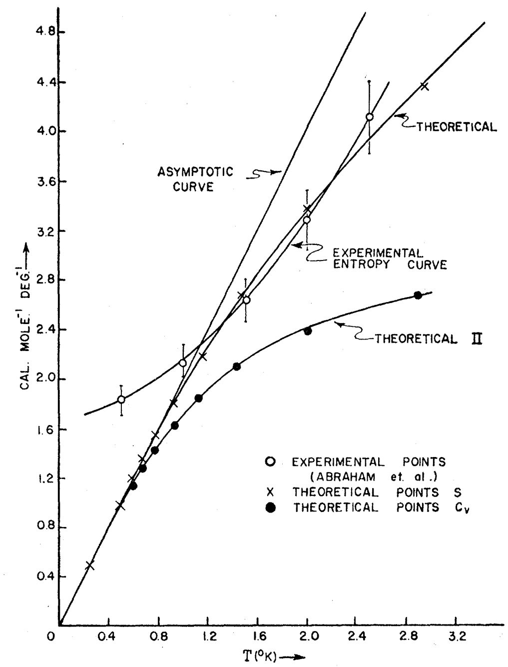

Fig. 1. Curve I: entropy S *versus* temperature T; curve II: specific heat C_v *versus* temperature T.

respectively, given by:

$$\frac{S}{R} = \frac{5}{2}\frac{\frac{2}{3}F_{3/2}(\eta)}{F_{1/2}(\eta)} - \eta, \tag{1a}$$

and

$$\frac{C_v}{R} = \frac{15}{4}\frac{\frac{2}{3}F_{3/2}(\eta)}{F_{1/2}(\eta)} - \frac{9}{4}\frac{F_{1/2}(\eta)}{\frac{d}{d\eta}F_{1/2}(\eta)}, \tag{2a}$$

where $F_k(\eta)$ are the well-known F−D functions and η is the degeneracy parameter. Using the asymptotic expansions of $F_k(\eta)$ for large values of η (i.e., $kT \ll \epsilon_0$), as given by McDougall and Stoner,[6] it can easily be shown that

$$S/R \approx 4.93(kT/\epsilon_0), \tag{1b}$$

and

$$C_v/R \approx 4.93(kT/\epsilon_0), \tag{2b}$$

where ϵ_0, the Fermi energy, is given by

$$\epsilon_0 = (3/\pi)^{2/3}h^2\rho^{2/3}/8m^{5/3}. \tag{3}$$

ρ is the density of liquid He^3 and m is the mass of an He^3 atom.

Using (1a) and (2a) we have calculated the entropy and the specific heat of liquid He^3 for various temperatures and the results are given in Table I and also shown graphically in Fig. 1. The density[7] of liquid He^3 was taken as 0.08 g/cc at 0°K. For a given value of η, the temperature was calculated from the relation

$$\tfrac{2}{3}(\epsilon_0/kT)^{3/2} = F_{1/2}(\eta),$$

[see Eqs. (1–11), reference 6]. We have also plotted in Fig. 1 the entropy values given by Abraham *et al.*,[5] to which the nuclear spin entropy $R\log 2$ has been added. It will be seen from Fig. 1 that the experimental curve for entropy, in the entire temperature range from 1°K to 2.5°K, lies very close to the theoretical curve, calculated on the basis of an ideal F−D gas. It is not surprising that liquid He^3 behaves more like an ideal F−D gas than like a liquid because of its high zero-point energy.

No experimental data are available below 1°K. The experimental curve, however, shows a strong tendency to approach a constant value as $T \to 0$; the extrapolatal value of the entropy is 1.8 cal mole^{-1}, deg^{-1}, at 0°K. This has led Abraham and co-