Quantization of Einstein's Gravitational Field Equations. II*

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Explicit expressions are obtained for all the constraints which appear in the canonical formulation of the field equations of general relativity and of electrodynamics.

A. GRAVITATIONAL FIELD

IN a previous paper[1] an explicit expression was obtained for the Hamiltonian of the gravitational field of general relativity. Anderson, Bergmann, and Penfield[2] have shown that constraints are introduced by the equations of motion of the field and have given a full discussion for general types of covariant field theories. In this note we shall obtain these constraints explicitly for the gravitational and electromagnetic fields. Our expressions should, of course, be equivalent to [AB, Eq. (7.4)].

The notation of [PS] will be used in a simplified parameterless formalism[3] in which the parameters u^s, t are chosen to match the space-time coordinates x^s, x^4. The following table indicates the correspondence between various quantities in the two formulations and it enables us to carry over directly the results of [PS]:

Parameter formalism [PS]		Parameterless formalism	
$u^s,\ t$	$\longrightarrow$	$x^s,\ t\equiv x^4$	
$g_{\mu\nu};\ g_{\mu\nu\|s}$	$\longrightarrow$	$g_{\mu\nu};\ g_{\mu\nu,s}$	
$\dot{x}^\sigma;\ \dot{g}_{\mu\nu}$	$\longrightarrow$	$\delta_4{}^\sigma;\ \dot{g}_{\mu\nu}\equiv g_{\mu\nu,4}$	
$\lambda_\sigma;\ \pi^{\mu\nu}$	$\longrightarrow$	$0;\ \pi^{\mu\nu}$	(1)
$J;\ l_\mu;\ l^2$	$\longrightarrow$	$1;\ \delta_\mu{}^4;\ g^{44}$	
ϕ^σ	$\longrightarrow$	ϕ^σ	
φ_4	$\longrightarrow$	h	

The Lagrangian [PS (37)] of the gravitational field is given by

$$L\equiv\tfrac{1}{4}(-g)^{\frac{1}{2}}\{2g^{\alpha\sigma}g^{\beta\rho}g^{\mu\nu}-g^{\alpha\beta}g^{\rho\sigma}g^{\mu\nu}-2g^{\alpha\mu}g^{\nu\sigma}g^{\beta\rho}+g^{\alpha\mu}g^{\beta\nu}g^{\rho\sigma}\}g_{\alpha\beta,\sigma}g_{\mu\nu,\rho}. \quad (2)$$

The canonical momenta [PS (38)] are

$$\pi^{\mu\nu}=\tfrac{1}{2}(\partial L/\partial\dot{g}_{\mu\nu}+\partial L/\partial\dot{g}_{\nu\mu})\equiv\tfrac{1}{4}(-g)^{\frac{1}{2}}\{2g^{\alpha\sigma}g^{\beta4}g^{\mu\nu}+g^{\mu4}g^{\nu\sigma}g^{\alpha\beta}+g^{\nu4}g^{\mu\sigma}g^{\alpha\beta}-2g^{\alpha\beta}g^{\mu\nu}g^{\sigma4}-2g^{\alpha\mu}g^{\nu\sigma}g^{\beta4}-2g^{\alpha\nu}g^{\mu\sigma}g^{\beta4}+2g^{\alpha\mu}g^{\beta\nu}g^{\sigma4}\}g_{\alpha\beta,\sigma}. \quad (3)$$

The Hamiltonian density [PS (43)] is defined by

$$H\equiv\pi^{\alpha\beta}\dot{g}_{\alpha\beta}-L. \quad (4)$$

Applying the recipe (1) to the results of [PS III (B)] we find[4]

$$H\equiv h+\beta_\sigma\phi^\sigma, \quad (5)$$

where ϕ^σ vanishes in the weak sense, and h and ϕ^σ do not involve the velocities $\dot{g}_{\mu\nu}$. The Hamiltonian density H is now no longer weakly zero, but it is still weakly independent of the velocities.[5] The expressions in (5) are given by

$$\phi^\sigma\equiv\pi^{\sigma4}-\tfrac{1}{4}(-g)^{\frac{1}{2}}\{2g^{\alpha s}g^{\sigma4}g^{\beta4}-2g^{\alpha4}g^{\beta4}g^{\sigma s}-g^{s4}g^{\alpha\beta}g^{\sigma4}+g^{\alpha\beta}g^{44}g^{\sigma s}\}g_{\alpha\beta,s}=0, \quad (6)$$

$$\beta_\sigma\equiv(g^{44})^{-1}g^{\alpha\beta}(2g_{\alpha\sigma,\beta}-g_{\alpha\beta,\sigma}), \quad (7)$$

$$\begin{aligned}h\equiv{}&\tfrac{1}{2}(g^{44})^{-1}(-g)^{-\frac{1}{2}}(2g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\beta}g_{\mu\nu})\pi^{\alpha\beta}\pi^{\mu\nu}\\&+(g^{44})^{-1}(2g^{\beta4}\pi^{\alpha s}-2g^{\beta s}\pi^{\alpha4}-g^{s4}\pi^{\alpha\beta})g_{\alpha\beta,s}\\&+(g^{44})^{-1}(-g)^{\frac{1}{2}}\{\tfrac{1}{8}(g^{\alpha\beta}g^{\epsilon\iota}-2g^{\alpha\epsilon}g^{\beta\iota})\\&\times(g^{44}g^{mn}-g^{m4}g^{n4})-g^{\alpha4}g^{\beta4}g^{\epsilon m}g^{\iota n}-g^{\alpha4}g^{\beta\epsilon}g^{\iota m}g^{n4}\\&+\tfrac{1}{2}g^{\alpha4}g^{\beta\epsilon}g^{\iota4}g^{mn}+\tfrac{1}{2}g^{\alpha m}g^{\beta4}g^{\epsilon4}g^{\iota n}+\tfrac{1}{2}g^{\alpha4}g^{\beta n}g^{\epsilon m}g^{\iota4}\\&+\tfrac{1}{2}g^{\alpha n}g^{\beta\epsilon}g^{\iota m}g^{44}\}g_{\alpha\beta,m}g_{\epsilon\iota,n}.\end{aligned} \quad (8)$$

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[1] F. A. E. Pirani and A. Schild, Phys. Rev. **79**, 986 (1950). (This paper will be referred to as [PS]); also Bergmann, Penfield, Schiller, and Zatzkis, Phys. Rev. **80**, 81 (1950).

[2] Anderson, Bergmann, and Penfield, Phys. Rev. **82**, 321 (1951); **83**, 1018 (1951). (The latter will be referred to as [AB]); also F. A. E. Pirani, D. Sc. thesis, Carnegie Institute of Technology, May, 1951. Correction: In [PS], Sec. II (D), the last two sentences are wrong.

[3] See reference 2, Pirani, Sec. 2.4; also R. H. Penfield, Phys. Rev. **84**, 737 (1951).

[4] "Strong" equations ($\equiv$) remain valid after performing a variation, where the "coordinates" $g_{\mu\nu}$, $g_{\mu\nu,s}$, the "velocities" $\dot{g}_{\mu\nu}$, and the "momenta" $\pi^{\mu\nu}$ are varied independently—in particular, independently of (3). "Weak" equations ($=$) do not remain valid after such a variation. It is important for the present theory that the transition from (4) to (5) be in the strong sense.

[5] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).

The nonzero Poisson brackets between pairs of the canonical variables $g_{\alpha\beta}$, $g_{\alpha\beta,s}$, $\pi^{\mu\nu}$ are

$$[g_{\alpha\beta}, \pi'^{\mu\nu}] \equiv \tfrac{1}{2}(\delta_\alpha{}^\mu\delta_\beta{}^\nu + \delta_\alpha{}^\nu\delta_\beta{}^\mu)\delta, \tag{9}$$

$$[g_{\alpha\beta,s}, \pi'^{\mu\nu}] \equiv \tfrac{1}{2}(\delta_\alpha{}^\mu\delta_\beta{}^\nu + \delta_\alpha{}^\nu\delta_\beta{}^\mu)\delta_s, \tag{10}$$

where $g_{\alpha\beta} \equiv g_{\alpha\beta}(\mathbf{x})$, $\pi'^{\mu\nu} \equiv \pi^{\mu\nu}(\mathbf{x}')$,

$$\delta \equiv \delta(\mathbf{x}-\mathbf{x}') \equiv \delta(x^1-x'^1)\delta(x^2-x'^2)\delta(x^3-x'^3),$$

and $\delta_s \equiv \partial\delta/\partial x^s$. The equation of motion of a function or functional F of the canonical field variables is

$$\dot{F} = [F, \mathcal{H}], \tag{11}$$

where $\mathcal{H}$ is the Hamiltonian

$$\mathcal{H} \equiv \int H d\mathbf{x}, \tag{12}$$

and where $d\mathbf{x} \equiv dx^1dx^2dx^3$, the integral being taken over the 3-surface $x^4 \equiv$ constant. Equation (11) is the canonical form of the classical field equations.

Since the weak equation $\phi^\sigma = 0$ must remain valid on all 3-surfaces $x^4 \equiv$ constant, we must have

$$\dot{\phi}^\sigma = [\phi^\sigma, \mathcal{H}] = \int \{[\phi^\sigma, h'] + [\phi^\sigma, \beta_\rho'\phi'^\rho]\} d\mathbf{x}' = 0. \tag{13}$$

These equations are not satisfied identically, and so lead to new constraints, the χ-equations of Dirac[5] or the secondary constraints of Bergmann.[2]

A direct calculation shows that

$$[\phi^\sigma, \phi'^\rho] \equiv 0, \tag{14}$$

and therefore

$$[\phi^\sigma, \beta_\rho'\phi'^\rho] \equiv \beta_\rho'[\phi^\sigma, \phi'^\rho] + [\phi^\sigma, \beta_\rho']\phi'^\rho = 0,$$

by (6) and (14). Thus (13) reduces to

$$\int [\phi^\sigma, h'] d\mathbf{x}' = 0. \tag{15}$$

Again, direct calculation shows that

$$\begin{aligned}[\phi^\sigma, h'] = \chi^\sigma\delta - \{\pi^{r\sigma} + \tfrac{1}{4}(-g)^{\frac{1}{2}}[&g^{\sigma 4}(g^{\alpha\beta}g^{rs} - 2g^{\alpha r}g^{\beta s}) \\ &+ g^{\sigma r}(g^{\alpha\beta}g^{s4} - 2g^{\alpha 4}g^{\beta s}) \\ &+ g^{\sigma s}(4g^{\alpha 4}g^{\beta r} - 2g^{\alpha\beta}g^{r4})]g_{\alpha\beta,s}\}\delta_r',\end{aligned} \tag{16}$$

where use has been made of the weak equations $\phi^\sigma = 0$. In (16), $\delta_r' \equiv \partial\delta/\partial x'^r$, and

$$\begin{aligned}\chi^\sigma \equiv &-\tfrac{1}{4}(g^{44})^{-1}(-g)^{-\frac{1}{2}}g^{\sigma 4}(2g_{\epsilon\mu}g_{\iota\nu} - g_{\epsilon\iota}g_{\mu\nu})\pi^{\epsilon\iota}\pi^{\mu\nu} + \pi^{\sigma s}{}_{,s} \\ &+ (g^{44})^{-1}\{\pi^{\mu s}(g^{\sigma\nu}g^{44} - g^{\sigma 4}g^{\nu 4}) \\ &- \tfrac{1}{2}\pi^{\mu\nu}(g^{\sigma s}g^{44} - g^{\sigma 4}g^{s4})\}g_{\mu\nu,s} \\ &+ \tfrac{1}{4}(-g)^{\frac{1}{2}}\{(g^{44})^{-1}g^{\sigma 4}[-g^{\alpha\beta}g^{\mu r}g^{\nu s}g^{44} \\ &+ \tfrac{1}{4}(g^{\alpha\beta}g^{\mu\nu} - 2g^{\alpha\mu}g^{\beta\nu})(g^{rs}g^{44} + g^{r4}g^{s4}) \\ &+ g^{\alpha 4}g^{s4}(2g^{\beta\mu}g^{\nu r} - g^{\beta r}g^{\mu\nu}) \\ &+ g^{\alpha 4}g^{\nu 4}(g^{\beta r}g^{\mu s} - g^{\beta\mu}g^{rs} - g^{\beta s}g^{\mu r}) \\ &+ g^{\alpha\mu}g^{44}(g^{\beta s}g^{\nu r} + 2g^{\beta r}g^{\nu s})] \\ &+ g^{\sigma r}[g^{\alpha\beta}(g^{\mu s}g^{\nu 4} - \tfrac{1}{2}g^{\mu\nu}g^{s4}) + g^{\alpha 4}(g^{\mu\nu}g^{\beta s} - 2g^{\mu\beta}g^{\nu s}) \\ &+ g^{\alpha\mu}(g^{\beta\nu}g^{s4} - 2g^{\beta s}g^{\nu 4})]\}g_{\alpha\beta,r}g_{\mu\nu,s} \\ &+ \tfrac{1}{4}(-g)^{\frac{1}{2}}\{g^{\sigma 4}(g^{\alpha\beta}g^{rs} - 2g^{\alpha r}g^{\beta s}) \\ &\qquad + g^{\sigma r}(2g^{\alpha 4}g^{\beta s} - g^{\alpha\beta}g^{s4})\}g_{\alpha\beta,rs}.\end{aligned} \tag{17}$$

Integrating (16) with respect to $\mathbf{x}'$, the second term drops out and (15) becomes

$$\chi^\sigma = 0. \tag{18}$$

According to the general theory of [AB], further consistency equations $\dot{\chi}^\sigma = 0$ should be satisfied automatically by virtue of $\phi^\sigma = 0$, $\chi^\sigma = 0$, and should therefore not give additional χ-equations.

B. COMBINED GRAVITATIONAL AND ELECTROMAGNETIC FIELDS

The electromagnetic field variables are the potentials A_μ and the Lagrangian is now

$$L \equiv L_1 - \kappa(-g)^{\frac{1}{2}}F_{\mu\nu}F^{\mu\nu}, \tag{19}$$

where

$$F_{\mu\nu} \equiv A_{\mu,\nu} - A_{\nu,\mu}, \quad F^{\mu\nu} \equiv g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}, \tag{20}$$

where κ is the gravitational constant, and where L_1 is the Lagrangian (2) of the gravitational field. The momentum variables $\pi^{\mu\nu}$ conjugate to the $g_{\mu\nu}$ remain unchanged and are given by (3). The new momentum variables conjugate to the A_μ are

$$\pi^\mu = -4\kappa(-g)^{\frac{1}{2}}F^{\mu 4}. \tag{21}$$

The Hamiltonian density is obtained immediately by applying the recipe (1) to the results of [PS III (C)]:

$$H \equiv h + \beta_\sigma\phi^\sigma + \beta\phi, \tag{22}$$

where ϕ^σ and β_σ are still given by (6) and (7), and

$$\phi \equiv \pi^4 = 0, \tag{23}$$

$$\beta \equiv g^{\alpha 4}(g^{44})^{-1}\dot{A}_\alpha, \tag{24}$$

$$\begin{aligned}h \equiv h_1 &- (8\kappa)^{-1}(g^{44})^{-1}(-g)^{-\frac{1}{2}}g_{\alpha\beta}\pi^\alpha\pi^\beta \\ &+ (g^{44})^{-1}(g^{\alpha 4}\pi^s - g^{s4}\pi^\alpha)A_{\alpha,s} \\ &- 2\kappa(g^{44})^{-1}(-g)^{\frac{1}{2}}\{g^{44}g^{\alpha n}g^{\beta m} - g^{44}g^{\alpha\beta}g^{mn} \\ &\quad - 2g^{\alpha n}g^{\beta 4}g^{m4} + g^{\alpha\beta}g^{m4}g^{n4} + g^{mn}g^{\alpha 4}g^{\beta 4}\}A_{\alpha,m}A_{\beta,n},\end{aligned} \tag{25}$$

where h_1 is the expression (8) for the gravitational field.

We demand that $\dot{\phi}^\sigma = 0$ and $\dot{\phi} = 0$. Since (14) remains valid and also

$$[\phi^\sigma, \phi'] = 0, \quad [\phi, \phi'] = 0, \tag{26}$$

these conditions reduce to (15) and

$$\int [\phi, h']dx' = 0. \tag{27}$$

This last equation immediately gives the χ-equation:

$$\chi \equiv \pi^{s}{}_{,s} = 0. \tag{28}$$

We must remember that the h in (15) is now given by (25). Therefore the expressions that are given by (17), and which we shall now denote by $\chi_1{}^{\sigma}$, do not vanish weakly. Instead we obtain the χ-equations:

$$\begin{aligned}\chi^{\sigma} \equiv \chi_1{}^{\sigma} &+ (16\kappa)^{-1}(g^{44})^{-1}(-g)^{-\frac{1}{2}}g^{\sigma 4}g_{mn}\pi^{m}\pi^{n} \\ &+ \tfrac{1}{2}(g^{44})^{-1}(g^{\sigma 4}g^{n4} - g^{\sigma n}g^{44})F_{mn}\pi^{m} \\ &- \kappa(g^{44})^{-1}(-g)^{\frac{1}{2}}g^{\sigma 4}\{g^{\mu\nu}g^{rs}g^{44} - g^{\mu r}g^{\nu s}g^{44} \\ &+ 2g^{\mu 4}g^{\nu s}g^{r4} - g^{\mu\nu}g^{r4}g^{s4} - g^{\mu 4}g^{\nu 4}g^{rs}\}A_{\mu,s}A_{\nu,r} = 0. \end{aligned} \tag{29}$$

According to the general theory of [AB] there should be no other χ-equations.

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The Disintegration of Cs^{130} †

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The disintegration of Cs^{130} (30 min) has been studied with the help of a magnetic lens spectrometer and a NaI(Tl) scintillation spectrometer. Cs^{130} decays to both Xe^{130} by positron emission and K electron capture and to Ba^{130} by negatron emission. The end-point energy of the positrons is 1.97 Mev and of the negatrons 0.442 Mev. Since no gamma-rays other than annihilation radiation were observed, it is assumed that the disintegration leads to the ground state of both Xe^{130} and Ba^{130}.

I. INTRODUCTION

THE disintegration of Cs^{130} was studied with a view of getting some additional information on the energy levels of Xe^{130}. Information exists[1] on the levels of Xe^{126}, Xe^{128}, and Xe^{130} which has been obtained from the study of the decay by beta-ray emission of the appropriate iodine isotopes. The spectrum accompanying the disintegration of I^{130} was investigated by Roberts, Elliott, Downing, Peacock, and Deutsch[2] from which the energy levels of Xe^{130} were obtained. The disintegration of I^{130} gives rise to gamma-rays of energy 0.417, 0.537, 0.667, and 0.744 Mev. The gamma-ray of energy 0.417 Mev was shown to arise from the two most highly excited levels of Xe^{130} but the remaining three gamma-rays are in cascade and, to date, it has not been determined which of these gamma-rays arises as a result of a transition from the first excited state to the ground state. The present work was an attempt to determine the energy of the first excited state of Xe^{130} from a study of the disintegration of Cs^{130} which decays, in part, by positron emission and orbital electron capture to Xe^{130}. This hope was not realized since, as we shall show, the disintegration of Cs^{130} does not lead to an excited state of Xe^{130}. However, the present work has shown that Cs^{130} decays to Xe^{130} by positron emission and orbital electron capture on the one hand and to Ba^{130} by electron emission on the other. In addition, certain conclusions concerning the configuration of Cs^{130} have been obtained.

Fink, Reynolds, and Templeton[3] produced a caesium activity, whose half-life is 30 min by the bombardment of I^{127} with alpha-particles. They reported that this activity emits x-rays and gamma-rays of about 0.5-Mev energy.

II. PREPARATION OF SOURCES

Resublimed elemental iodine, free of bromine and chlorine impurities, was used as the target for bombardments. The iodine was pressed into a copper target, which was water-cooled, and covered by a palladium foil 0.0005 in. thick and was bombarded by the 23-Mev alpha-particle beam of the Indiana University cyclotron. This thickness of palladium was calculated to be such as to slow down the high energy alpha-particles in the cyclotron beam to an energy for which the α-$2n$ reaction should be very improbable. The targets were bombarded at low beam intensity to keep the iodine from overheating.

The active caesium was separated in the following manner. The iodine was washed off the target with CCl_4, and the caesium and any other soluble salts were separated by extracting with a small amount of 1 normal HCl to which a small amount of caesium and heavy metal carrier had been added. Since some cadmium and silver activities can be produced in the palladium cover and can be driven into the iodine target, cadmium was used as the heavy metal carrier. The

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[1] A. C. G. Mitchell, Revs. Modern Phys. **22**, 36 (1950).
[2] Roberts, Elliott, Downing, Peacock, and Deutsch, Phys. Rev. **64**, 268 (1943).
[3] Fink, Reynolds, and Templeton, Phys. Rev. **77**, 614 (1950).