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Nonlinear Pseudoscalar Meson Theory

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An attempt is made to explain nuclear saturation by means of a nonlinear pseudoscalar meson field. The same nonlinearity as that introduced by Schiff for the scalar case is employed; the treatment follows similar lines, using classical field theory. The total source strength is calculated for an isolated nucleon at rest, and an expression for the two-nucleon interaction in free space is obtained. It is shown that both of these are infinite, so that a cutoff must be introduced. For the two-nucleon problem the variation method is used with a simple trial function, which is the superposition of the single-nucleon meson field amplitudes; this should be a good approximation for large separation of the nucleons. Then the nonlinear terms of the two-particle interaction are shown to be separable from the linear ones, and are repulsive. The nonlinear terms can also be separated in the many-body problem, but they are not necessarily positive definite here, and hence need not lead to saturation. A nuclear model, based on a lattice structure with stationary nucleons is discussed qualitatively.

I. INTRODUCTION

THERE have been several attempts to explain nuclear saturation consistent with known two-body interactions. Schiff recently proposed a nonlinear meson theory of nuclear forces[1,2] to explain both saturation and the shell structure of nuclei on the basis of many-body forces. Two methods by which a nonlinearity may be introduced into the usual meson theories were discussed. The more promising of these methods, at least from the point of view of classical field theory, appeared to be the addition of a nonlinear term to the free meson wave equation. This case was studied in detail in (S) for a neutral scalar meson. The nonlinearity introduced into the Hamiltonian density was a positive one, proportional to ϕ^4, where ϕ is the meson field amplitude. This term corresponds physically to a point-contact repulsion between mesons. The results seemed sufficiently promising that the investigation of a neutral pseudoscalar meson has now been undertaken along similar lines. The reason for this choice is that all recent evidence indicates that π-mesons are pseudoscalar.[3–5] The neutral meson is studied because of its inherent simplicity. The whole work is only intended to be of an exploratory nature, but the results are significantly different from the scalar case.

We start with a Lagrangian for the meson field and meson-nucleon coupling and derive a field equation by the usual variation procedure. With this equation as a basis, the problem of an isolated nucleon at rest is discussed. The two-nucleon interaction in free space is treated, and a generalization is attempted for the interior of a heavy nucleus. In no instance have detailed calculations been made, since the nature of this study does not justify the lengthy computations that would be required. General inferences, however, are made whenever possible.

[1] L. I. Schiff, Phys. Rev. 84, 1 (1951), referred to here as (S).
[2] L. I. Schiff, Phys. Rev. 84, 10 (1951).
[3] K. M. Watson and K. A. Brueckner, Phys. Rev. 83, 1 (1951).
[4] Panofsky, Aamodt, and Hadley, Phys. Rev. 81, 565 (1951).
[5] R. E. Marshak, Revs. Modern Phys. 23, 137 (1951).

II. FORMULATION

The Lagrangian for a pseudoscalar meson with pseudovector coupling to a nonrelativistic nucleon is (units of $\hbar=c=1$ are used)

$$L=\int\left[\tfrac{1}{2}(\partial\phi/\partial t)^2-\tfrac{1}{2}(\nabla\phi)^2-G(\phi)-f(\mathbf{r},t)\boldsymbol{\sigma}\cdot\nabla\phi/\mu\right]d\tau. \quad (1)$$

Here ϕ is the meson field amplitude, f is the source density, $\boldsymbol{\sigma}$ is twice the nucleon spin, μ is the meson rest mass, and $G(\phi)$ is a nonlinear term which approaches $\frac{1}{2}\mu^2\phi^2$ for weak fields. After a partial integration, dropping the surface term, we get

$$L=\int\left[\tfrac{1}{2}(\partial\phi/\partial t)^2-\tfrac{1}{2}(\nabla\phi)^2-G(\phi)+\phi\nabla\cdot(f\boldsymbol{\sigma})/\mu\right]d\tau, \quad (2)$$

and the wave equation becomes

$$\Delta\phi-\partial^2\phi/\partial t^2-G'(\phi)=-\nabla\cdot(f\boldsymbol{\sigma})/\mu, \quad (3)$$

where the prime indicates differentiation with respect to the argument. If π is the momentum canonically conjugate to ϕ, the Hamiltonian can be written as

$$H=\int\left[\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\nabla\phi)^2+G(\phi)-\phi\nabla\cdot(f\boldsymbol{\sigma})/\mu\right]d\tau. \quad (4)$$

In order to obtain saturation, $G(\phi)$ should be chosen such that ϕ increases less rapidly than lineraly with f.[1] Since ϕ depends only on $\nabla\cdot(f\boldsymbol{\sigma})$, no direct requirement is imposed on $G(\phi)$. However, for $f=0$, that is for a free meson, the wave equation is exactly the same as that of the scalar meson. Thus, it is assumed in analogy with (S) that $G(\phi)$ is

$$G(\phi)=\tfrac{1}{2}\mu^2\phi^2+\tfrac{1}{4}\alpha^2\phi^4, \quad (5)$$

where α is a constant to be determined by comparison of the calculated and experimental nuclear energies.

As for the scalar meson, the ϕ^4 term can be interpreted as a point-contact repulsion between mesons.

The free meson solutions are the same as those in (S) because the wave equations are alike. No further discussion is therefore necessary. The desirability of a variation principle for the solution of the meson field amplitude has also been indicated in (S). It can be shown, by a completely analogous procedure, that the negative of the Lagrangian also gives an upper limit to the energy here.

III. ISOLATED NUCLEON AT REST

The treatment of the pseudoscalar meson field as a classical one is complicated by the introduction of the spin. The usual definition, taken over from the limiting case of quantum mechanics is[6] $\{\sigma_x, \sigma_y\} = -2\sigma_z$, together with cyclic permutations, where $\{\ \ \}$ represents the classical Poisson bracket. In the case of an isolated nucleon at rest, the simplest solution is to treat $\boldsymbol{\sigma}$ as a fixed and constant vector of unit length, oriented along the z axis. Then ϕ will not necessarily be spherically symmetric; for simplicity, however, it is assumed that the angular dependence of ϕ can be separated. The meson field amplitude then becomes

$$\phi(\mathbf{r}) = R(r)\Theta(\theta) = [\chi(r)/r]\Theta(\theta), \tag{6}$$

and the wave equation (3) can be written as

$$\frac{\chi''(r)}{r}\Theta + \frac{\chi}{r^3 \sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\Theta(\theta)] - G'(\phi) = -\frac{\boldsymbol{\sigma}}{\mu}\cdot\nabla f(r,\theta) = -\frac{1}{\mu}\left(\sigma_r\frac{\partial f}{\partial r} + \frac{\sigma_\theta}{r}\frac{\partial f}{\partial\theta}\right). \tag{7}$$

The separation of variables in Eq. (6) is possible if $\Theta = \cos\theta$ and the source density is the physically plausible $f(\mathbf{r}) = P(r) + Q(r)\cos^2\theta$, where $P(r)$ and $Q(r)$ are independent functions. For this case, Eq. (7) separates as follows:

$$\begin{aligned} \frac{\chi''(r)}{r} - \frac{2\chi}{r^3} - \frac{\mu^2\chi}{r} &= -\frac{1}{\mu}\left[P'(r) + \frac{2Q}{r}\right], \\ \frac{\alpha^2\chi^3}{r^3} &= -\frac{1}{\mu}\left[-Q'(r) + \frac{2Q}{r}\right], \end{aligned} \tag{8}$$

and if an integration is performed, this becomes

$$\begin{aligned} Q &= \alpha^2 r^2 \mu \int \frac{\chi^3}{r'^5} dr', \\ P + Q &= -\mu\left[\frac{\chi'(r)}{r} + \frac{\chi}{r^2} - \mu^2 \int \frac{\chi}{r'} dr' - \alpha^2 \int \frac{\chi^3}{r'^3} dr'\right]. \end{aligned} \tag{9}$$

The integration constants are zero, since $f(r)$ must approach zero for large radial distances.

[6] W. Pauli, *Meson Theory of Nuclear Forces* (Interscience Publishers, Inc., New York, 1946), p. 15.

If $G(\phi)$ is taken to be $\frac{1}{2}\mu^2\phi^2 + \alpha^2\phi^n/n$, where n is not equal to 2, the angular separation indicated in Eq. (6) is not possible except for $n=4$, unless a source density is used that is much more complicated and has many more terms than are used here.

It is of interest to study the requirements imposed on the meson field strength and on the spatial extent of the source density necessary to obtain a finite total source strength, g. This is given by

$$g = \int f(\mathbf{r}) d\tau = 4\pi \int_0^\infty [(P+Q) - \tfrac{2}{3}Q] r^2 dr. \tag{10}$$

Substitution of Eq. (9) into Eq. (10) shows that the integral is well behaved everywhere except possibly at the upper and lower limits. Since most sources fall off sufficiently rapidly at large distances, we need only consider the divergences at vanishingly small radii. If $\chi(0)$ is finite, then g in Eq. (10) is well behaved. This requirement is less stringent than that for the scalar case, where $\chi(r)$ must at least be proportional to r for small radii.

The behavior of the meson field amplitude at small r depends on the spatial extent of the source. The broader the source, the less singular is $\chi(0)$. For the narrow source chosen here $\chi(0)$ is not finite, and a cutoff must be introduced in evaluating the total source strength. Because the treatment used is nonrelativistic and neglects nucleon recoil, it is not unreasonable to introduce a cutoff for the source density at the nucleon Compton wavelength. When $r \leqslant 1/M$, where M is the nucleon mass, f is taken to be a constant equal to the value it has at the cut-off radius.

For a distributed source, it is simpler to start with a given meson field amplitude. The narrowest source is obtained if the linear field theory function ϕ is used:

$$\phi_1(\mathbf{r}) = (A/\mu)\boldsymbol{\sigma}\cdot\nabla(e^{-\mu r}/r), \tag{11}$$

$$(\Delta - \mu^2)\phi_1 = -4\pi A\boldsymbol{\sigma}\cdot\nabla\delta(\mathbf{r})/\mu, \tag{12}$$

where A is an arbitrary constant to be determined. A substitution into Eq. (9) and an integration gives

$$\begin{aligned} P &= A\Bigg\{4\pi\delta(\mathbf{r}) + \left(\frac{\alpha A}{\mu}\right)^2 e^{-3\mu r}\Bigg[\frac{2}{35r^5} + \frac{6\mu}{35r^4} + \frac{2\mu^2}{35r^3} - \frac{13\mu^3}{140r^2} + \frac{41\mu^4}{140r} + \frac{3\mu^5}{280} + \frac{9\mu^5 e^{3\mu r}}{10}\int\frac{e^{-3\mu r'}}{r'}dr' \\ &\qquad - \frac{9\mu^6 r}{280} - \frac{27\mu^7 r^2 e^{3\mu r}}{280}\int\frac{e^{-3\mu r'}}{r'}dr'\Bigg]\Bigg\} \\ Q &= A\Bigg\{\left(\frac{\alpha A}{\mu}\right)^2 e^{-3\mu r}\Bigg[\frac{1}{7r^5} + \frac{3\mu}{7r^4} + \frac{12\mu^2}{35r^3} - \frac{\mu^3}{140r^2} + \frac{\mu^4}{140r} - \frac{3\mu^5}{280} + \frac{9\mu^6 r}{280} + \frac{27\mu^7 r^2 e^{3\mu r}}{280}\int\frac{e^{-3\mu r'}}{r'}dr'\Bigg]\Bigg\}. \end{aligned} \tag{13}$$

The total source strength is

$$g=\int f(\mathbf{r})d\tau=4\pi A(1+3.77\alpha^2A^2), \quad (14)$$

where the cutoff mentioned previously has been introduced in evaluating all but the δ-function integral. The divergence in g is $1/r^2$, so that the exact value of the cut-off radius and the cut-off procedure will have a relatively large effect. The nucleon self-energy is infinite and is not evaluated here.

IV. TWO-NUCLEON INTERACTION IN FREE SPACE

For the two-nucleon problem the meson-nucleon coupling is taken as

$$-\frac{\phi_{12}}{\mu}\boldsymbol{\sigma}_{12}\cdot\nabla f_{12}=-\frac{\phi_{12}}{\mu}[\boldsymbol{\sigma}_1\cdot\nabla f_1(\mathbf{r}-\mathbf{r}_1)+\boldsymbol{\sigma}_2\cdot\nabla f_1(\mathbf{r}-\mathbf{r}_2)], \quad (15)$$

with $f_1(\mathbf{r})=P(r)+Q(r)\cos^2\theta$. No assumption is made about the relative direction of the nucleon spins, but the sources are taken to be sufficiently far apart that the variational trial function, $\phi_{12}^v=\phi_1(\mathbf{r}-\mathbf{r}_1)+\phi_1(\mathbf{r}-\mathbf{r}_2)\equiv\phi_1+\phi_2$, with $\phi_1(\mathbf{r})$ given by Eq. (11), represents the meson field to a good approximation, and no minimization with respect to an arbitrarily introduced parameter is carried out. From Eq. (4), the Hamiltonian is given by

$$H^v=\int[\tfrac{1}{2}(\nabla\phi_{12}^v)^2+G(\phi_{12}^v)-(\phi_{12}^v/\mu)\boldsymbol{\sigma}_{12}\cdot\nabla f_{12}]d\tau, \quad (16)$$

and the interaction energy is $H_{\text{inter}}^v=H^v-2H_1$, where H_1 is the single nucleon self-energy. This energy is subtracted before the integration is performed. The result is

$$H_{\text{inter}}^v=\int[\nabla\phi_1\cdot\nabla\phi_2+\mu^2\phi_1\phi_2+\alpha^2(\phi_1^3\phi_2+\phi_1\phi_2^3+\tfrac{3}{2}\phi_1^2\phi_2^2)-(\phi_1/\mu)\boldsymbol{\sigma}_2\cdot\nabla f_2-(\phi_2/\mu)\boldsymbol{\sigma}_1\cdot\nabla f_1]d\tau. \quad (17)$$

We use the wave equation (3), the relation (12), and a partial integration to obtain

$$H_{\text{inter}}^v=(A^2/\mu^2)\boldsymbol{\sigma}_1\cdot\nabla\boldsymbol{\sigma}_2\cdot\nabla(e^{-\mu R}/R)+\tfrac{3}{2}\alpha^2\int\phi_1^2\phi_2^2d\tau$$

$$=\frac{A^2}{\mu^2}\left[S_{12}\left(\frac{3}{R^3}+\frac{3\mu}{R^2}+\frac{\mu^2}{R}\right)e^{-\mu R}+\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\mu^2\frac{e^{-\mu R}}{R}\right]+\tfrac{3}{2}\alpha^2\int\phi_1^2\phi_2^2d\tau, \quad (18)$$

where $\mathbf{R}=\mathbf{r}_2-\mathbf{r}_1$ and $S_{12}=(\boldsymbol{\sigma}_1\cdot\mathbf{R}\boldsymbol{\sigma}_2\cdot\mathbf{R}/R^2-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2/3)$.

The first term of Eq. (18) (in square brackets) is the usual linear interaction energy. The second term, which arises from the introduction of the nonlinearity, is positive definite, and is therefore always repulsive. It might be noted that the integral to be evaluated in Eq. (18) is logarithmically divergent, so that a cutoff must again be introduced. The expression obtained for the interaction energy is only valid for large nucleon separation due to the assumption made for ϕ_{12}^v.

V. NUCLEAR MODEL

A generalization of the two-body problem is that of nuclear matter. In these preliminary considerations a model for the interior of a nucleus, treating all nucleons alike and neglecting edge effects, is discussed qualitatively.

A simple model for nuclear matter is obtained by assuming that the nucleon sources are distributed in such a way that the source density is approximately constant. However, the nucleon spin must also be specified. One of the primary difficulties of any classical nuclear model, using pseudoscalar mesons, is the treatment of this nucleon spin density.

Of the models considered, the most promising one for the interior of the nucleus is a lattice. Stationary, but interacting nucleons are assumed to be at the centers of the basic lattice structures, which are taken to be cubes.

Since the nucleus is treated classically, the nucleons align their spins so as to give a minimum resultant potential energy.[7] If the nonlinearity is neglected momentarily, because maximum binding energy, within saturation, is desired for the linear case, then each lattice cube contains a point source. The effective nucleon source density is taken to be $\sum_i\boldsymbol{\sigma}_i\cdot\nabla f_1(\mathbf{r}-\mathbf{r}_i)$. The total potential energy is then

$$H=\sum_{i<j}\sum_{j}H_{ij}=\frac{A^2}{\mu^2}\sum_{i<j}\sum_{j}\sigma_i\cdot\nabla\sigma_j\cdot\nabla\left(\frac{e^{-\mu R}}{R}\right),$$

with $R=|\mathbf{r}_i-\mathbf{r}_j|$. If only interactions between nearest neighbors are considered, it is found that one of the configurations which gives a minimum potential energy is that shown in Fig. 1. This lattice, consisting of alternating layers of nucleon spins up and down, has intuitive appeal, as it is known that there exists a correlation in the spatial distribution of particles of opposite spin, due to the Pauli exclusion principle. The potential energy per nucleon for this model is purely attractive. It is

$$H_{\text{inter}}/\text{nucleon}=-\frac{A^2}{\mu^2}\left(\frac{5.514}{a^3}+\frac{6\mu}{a^2}+\frac{3.357\mu^2}{a}\right), \quad (19)$$

where a is the lattice spacing. This is equal to $(4\pi/3)^{\frac{1}{3}}r_0$ if the volume of each cube is $4\pi r_0^3/3$, where r_0 is the nucleon spacing in the nucleus.

[7] R. Serber and S. M. Dancoff, Phys. Rev. 63, 143 (1943).

We want the re-introduction of the nonlinearity to cause saturation. The potential energy, with this term included must therefore be less than that due to the linear terms alone. The initial trial function for the meson field is taken as $\phi_N=\sum_i \phi_1(\mathbf{r}-\mathbf{r}_i)\equiv\sum_i \phi_i$, where $\phi_1(\mathbf{r})$ is defined by Eq. (11). This function can only be expected to be a good approximation for at least moderately large a. With this choice of ϕ_N the interaction energy is

$$\begin{aligned} H_{N\,\mathrm{inter}}=H_N-\textstyle\sum_i H_i=\displaystyle\int \{&\tfrac{1}{2}[\nabla(\textstyle\sum_i \phi_i)]^2+\tfrac{1}{2}\mu^2(\textstyle\sum_i \phi_i)^2 \\ &+\tfrac{1}{4}\alpha^2(\textstyle\sum_i \phi_i)^4-\textstyle\sum_i(\phi_i/\mu)\sum_j \boldsymbol{\sigma}_j\cdot\nabla f_j \\ &-\tfrac{1}{2}\textstyle\sum_i(\nabla\phi_i)^2-\tfrac{1}{2}\mu^2\sum_i \phi_i^2-\tfrac{1}{4}\alpha^2\sum_i \phi_i^4 \\ &-\textstyle\sum_i(\phi_i/\mu)\boldsymbol{\sigma}_i\cdot\nabla f_i\}d\tau. \end{aligned}$$

The integration is to be carried out over a single cube, considering only forces between adjacent nucleons. Use of Eq. (3), Eq. (12), and a partial integration ($\phi_i=0$ on the surface of each basic cube) gives

$$\begin{aligned} H_{N\,\mathrm{inter}}=\tfrac{1}{2}\sum_{i\neq j}\sum_{j}\frac{A^2}{\mu^2}\boldsymbol{\sigma}_i\cdot\nabla\boldsymbol{\sigma}_j\cdot\nabla\left(\frac{e^{-\mu R}}{R}\right) \\ +\frac{\alpha^2}{4}\sum_{i\neq j}\sum_{j\neq k}\sum_{k\neq l}\sum_{l}\int(3\phi_i^2\phi_j^2 \\ +6\phi_i^2\phi_j\phi_k+\phi_i\phi_j\phi_k\phi_l)d\tau, \quad (20) \end{aligned}$$

with $R=|\mathbf{r}_i-\mathbf{r}_j|$. The exact evaluation of the last integral is lengthy, but a simple calculation shows that for the particular model chosen the logarithmic divergence cancels in the limit of vanishingly small r, by reasons of symmetry. Physically this implies that any slight disturbance from exact symmetry causes the potential energy to become infinite. The integral in Eq. (20) must be larger than zero, independent of the particular model chosen, in order to satisfy the saturation requirement. The integrand is, however, not necessarily positive definite, since it can be rewritten as

$$\sum_{i<j}\sum_{j<k}\sum_{k<l}\sum_{l}6\{[\phi_i(\phi_j+\phi_k+\phi_l)+\phi_j(\phi_k+\phi_l)+\phi_k\phi_l]^2-2\phi_i\phi_j\phi_k\phi_l\}.$$

If $\phi_j+\phi_k+\phi_l=0$, and ϕ_i is large and positive, this expression is easily seen to be negative. A sufficient condition for the integrand to be positive definite is that all meson field amplitudes have the same sign wherever they overlap,[8] or that four-body interactions can be neglected. For the symmetry shown in Fig. 1, the integral of Eq. (20) is positive because all meson fields have the same sign wherever the overlap is large. Once the potential energy is found, the parameters α and A can be evaluated as in (S). That is, $E=H_{\mathrm{inter}}$ $+$kinetic energy is minimized with respect to r_0. This gives a relation between r_0, $E_{\min}$, α, and A. Since the experimental values of r_0 and $E_{\min}$ are known, there are two relations from which the two parameters α and A can theoretically be obtained.

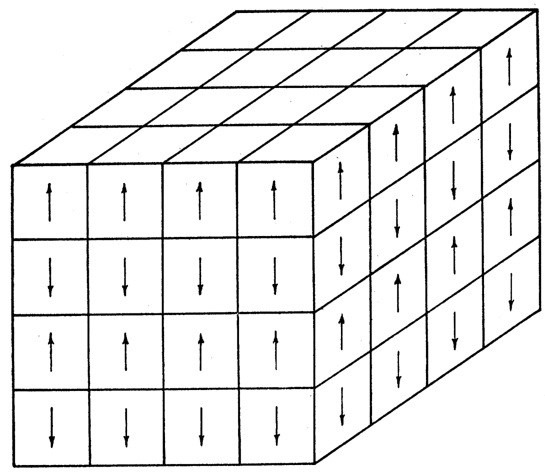

FIG. 1. Lattice model of nuclear matter. The arrows indicate the projections of the individual nucleon spins.

VI. CONCLUSIONS

A nonlinear meson theory analogous to that of Schiff has been investigated for neutral pseudoscalar mesons. The theory was noted to differ from the scalar one[1] in several important respects. Whereas the angular dependence of the pseudoscalar meson field requires the same nonlinearity as that introduced in (S) to allow separation of the wave equation, the theory does not require a cutoff if the radial dependence of the field amplitude is the same as that used for the scalar case, i.e., $e^{-\mu r}/r$. However, the choice of such a radial dependence results in a rather broad source. On the other hand, if the single nucleon meson field is chosen to be the same as for the linear pseudoscalar meson, the source is much narrower, but the strong singularity at the origin necessitates a cutoff to be introduced not only in evaluating the total source strength, but also the potential energy of two or more nucleons. Only one of the integrals with divergent features has been evaluated, because it was felt that these difficulties are real ones and should be eliminated before any further work is carried out with such a theory. Variational expressions for the two-nucleon interaction and for the potential energy of nuclear matter have, however, been found. The latter has been based on a lattice structure for the interior of the nucleus, and a simple variational form for the meson field amplitude. A consistent treatment would require the quantization of the field and a quantum-mechanical treatment of the nucleon spins.

It is a sincere pleasure to acknowledge the continued interest and guidance of Professor L. I. Schiff, who suggested this problem. The author also wishes to thank Dr. S. D. Drell for several stimulating discussions.

[8] Hardy, Littlewood, and Polya, *Inequalities* (Cambridge University Press, London, 1934), p. 51.