Clearly $|\zeta_3'|>1$ so that hypothesis A implies that (65) does not vanish. The above arguments evidently hold for any general value of m and this completes the proof that as $z_1 \to \infty$, z_2 approaches a limit smaller than unity in absolute value.

(4) Keeping z_3, $z_4 \cdots z_m$ fixed one can increase $|z_1|$ and define z_2 as a continuous function of z_1. Since by (59b), $|z_2|$ starts to be $\geqq 1$ in absolute magnitude and tends to a limit <1 in absolute magnitude as $z_1 \to \infty$, there must be a value of z_1 equal to z_1' so that z_2 assumes a value z_2' equal to 1 in absolute magnitude, i.e.,

$$\mathfrak{P}_m(z_1', z_2', z_3, \cdots z_m) = 0 \qquad (66a)$$

and

$$|z_1'|>1, \quad |z_2'|=1, \quad |z_3|, \quad |z_4|, \quad \cdots |z_m| \geqq 1. \qquad (66b)$$

We can fix z_2', z_4, z_5, $\cdots z_m$ and regard z_3 as a function of z_1' and follow the same procedure by increasing $|z_1'|$ till z_3 assumes a value equal to 1 in absolute magnitude.

Continuing this way we finally get a set of values z_1'', z_2'', $\cdots z_n''$ such that

$$\mathfrak{P}_m(z_1'', z_2'', \cdots z_m'') = 0, \qquad (67a)$$

and

$$|z_1''|>1, \quad |z_2''|=|z_3''|=\cdots=|z_m''|=1. \qquad (67b)$$

But $\mathfrak{P}_m$ is linear in z_1''. Writing $\mathfrak{P}_m = Bz_1''+C$ where B and C are independent of z_1'' one verifies easily that

$$B = z_2'' z_3'' \cdots z_m'' \mathfrak{C},$$

where $\mathfrak{C}$ is the complex conjugate of C under the condition (67b). Hence

$$|z_1''| = |C|/|B| = 1, \qquad (68)$$

which contradicts (67b). (It is easy to show that B does not vanish by making a transformation similar to (64) and reduce B to products of some x's with $\mathfrak{P}_{m-1}$.)

This completes the proof by induction.

High Frequency Electrical Breakdown of Gases*

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Previous theoretical treatments of high frequency electrical breakdown in gases based on the Boltzmann transport equation were applied to specific gases and met with mathematical difficulties when applied to higher pressures. We now present a simpler solution applicable to any gas and to a wide pressure range. Agreement between theory and experiment with hydrogen gives confidence that the energy distribution function is correct and the distribution function is therefore used to compute other quantities of physical interest.

I. INTRODUCTION

IN a recent series of papers, the electrical breakdown of helium and hydrogen in a microwave cavity has been predicted from the measured probability of collision P_c and the corresponding probabilities of excitation P_x and ionization P_i. The only processes assumed to take place are acceleration of electrons by the field, elastic and inelastic collisions with the gas, and diffusion to the walls, which in turn have no effect other than to absorb the electrons and ions. Mathematical difficulties arose in the method used in these papers when applied to higher pressures. The simpler solution given in this paper is applicable to any gas and to a wide pressure range.

II. BOLTZMANN EQUATION

When a high frequency electric field $E = E_p \exp(j\omega t)$ is applied to a gas, the velocity distribution $F(\mathbf{v})$ of the

free electrons is determined by the Boltzmann equation:

$$(\partial F/\partial t) = C - \nabla_r \cdot \mathbf{v}F + \nabla_v \cdot e\mathbf{E}F/m. \qquad (1)$$

C represents the effect of collisions and ∇_r and ∇_v are the gradient operators in configuration and velocity space. This equation is solved by expanding the distribution function in spherical harmonics in velocity space and in Fourier series in time,

$$F = \sum_l \sum_k F_k{}^l P_l(\cos\theta) \exp(jk\omega t)$$
$$= F_0{}^0 + \mathbf{v} \cdot [\mathbf{F}_0{}^1 + \mathbf{F}_1{}^1 \exp(j\omega t)]/v. \qquad (2)$$

All terms except the three indicated may be dropped when the geometry, pressure, and frequency fall within certain limits which have been discussed by Brown and MacDonald.[1] These limits require that the mean free path be less than any dimension of the cavity, that the frequency be sufficiently high so that the electrons do not lose appreciable energy between cycles, and that the average motion of the electrons resulting from the action of the field and of collisions be sufficiently small

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[1] S. C. Brown and A. D. MacDonald, Phys. Rev. **76**, 1629 (1949).

so that the field does not clear the electrons out of part of the tube in each half-cycle.

In evaluating the collision term it is convenient to replace the customary mean free paths by the collision frequencies for momentum transfer, for excitation, and for ionization defined by

$$\nu_c = \int\int v p P_c(\theta)(1-\cos\theta)\,\sin\theta\,d\theta\,d\phi, \qquad (3)$$

$$\nu_x = v p P_x, \quad \nu_i = v p P_i, \qquad (4)$$

where P_x and P_i are the experimentally defined probabilities of excitation of all levels and of ionization and $P_c(\theta)$ is the probability of scattering into a unit solid angle inclined at θ to the original direction.

If the temperature of the gas is negligible compared to that of the electrons, the latter will lose a fraction $2m/M$ of their energy, per momentum transfer collision, to recoil of the molecule with which they collide. The mass of the molecule is M and that of the electron m. This fraction will be increased if there is appreciable transfer of energy to rotation or vibration but it is believed that this is negligible.

A collision producing electronic excitation differs from that producing recoil, rotation, and vibration in that the colliding electron loses practically all its energy instead of a very small fraction of it. These processes are treated mathematically as though fast electrons disappeared at the rate $(\nu_x+\nu_i)F_0^0$ and slow electrons appeared at the rate qF_0^0. Making use of these ideas in the evaluation of the collision term, introducing Eq. (2) into Eq. (1) and equating coefficients of similar time and angle functions yields one scalar and two vector equations:

$$(\nu_x+\nu_i-q)F_0^0 = -(v/3)\boldsymbol{\nabla}_r\cdot\mathbf{F}_0^1 + (1/v^2)\partial$$
$$\times[(ev^2/6m)(\mathbf{E}_p\cdot\mathbf{F}_1^1)_{\mathrm{real}}+(m/M)\nu_c v^3 F_0^0]/\partial v, \quad (5)$$

$$\nu_c\mathbf{F}_0^1 = -v\boldsymbol{\nabla}_r F_0^0, \qquad (6)$$

$$(\nu_c+j\omega)\mathbf{F}_1^1 = (e\mathbf{E}_p/m)\partial F_0^0/\partial v. \qquad (7)$$

III. DISTRIBUTION IN SPACE

The direct and alternating current densities,

$$\mathbf{J}_0 = \int_0^\infty -e\mathbf{F}_0^1(4\pi v^3/3)dv = e\boldsymbol{\nabla}_r Dn, \qquad (8)$$

$$\mathbf{J}_1 = \int_0^\infty -e\mathbf{F}_1^1(4\pi v^3/3)dv = e\mu\mathbf{E}n, \qquad (9)$$

are fully determined by the components $\mathbf{F}_0^1$ and $\mathbf{F}_1^1$ of the distribution function. These are, in turn, found to be derivatives of F_0^0. Substitution of Eqs. (6) and (7) into (8) and (9) serves to determine the diffusion and

ac mobility coefficients in terms of F_0^0:

$$nD = \int_0^\infty \frac{v^2}{3\nu_c}F_0^0 4\pi v^2 dv, \qquad (10)$$

$$n\mu = \int_0^\infty \frac{4\pi}{3}\frac{e}{m}F_0^0\frac{d}{dv}\left(\frac{v^3}{\nu_c+j\omega}\right)dv. \qquad (11)$$

The components $\mathbf{F}_0^1$ and $\mathbf{F}_1^1$ can be eliminated from Eq. (5) by substitution from Eqs. (6) and (7) and this yields the differential equation for F_0^0:

$$(\nu_x+\nu_i-q)F_0^0 = (v^2/3\nu_c)\boldsymbol{\nabla}_r^2 F_0^0$$
$$+(1/v^2)\partial[(eu_c/3m)\nu_c v^2(\partial F_0^0/\partial v)+m\nu_c v^3 F_0^0/M]/\partial v. \quad (12)$$

The energy, in electron volts, $u_c = eE_p^2/2m(\nu_c^2+\omega^2)$ which is introduced here turns out to be the average energy transferred from the field to an electron between collisions, and $\nu_c u_c$ is the power transfer. It is in general a function of the electron's energy through the collision frequency ν_c, and it is also a function of the external parameters E, p, and $\lambda = 2\pi c/\omega$. The power transfer has a maximum for $\nu_c = \omega$ and this corresponds to the pressure for easiest breakdown. At pressures above this value, u_c varies as $(E_p/p)^2$.

The total excitation and ionization rates may be defined by

$$n\bar\nu_x = \int_0^\infty \nu_x F_0^0 4\pi v^2 dv, \qquad (13)$$

$$n\bar\nu_i = \int_0^\infty \nu_i F_0^0 4\pi v^2 dv, \qquad (14)$$

and, as every exciting collision yields one, and every ionizing collision two, slow electrons

$$\int_0^\infty qF_0^0 4\pi v^2 dv = n(\bar\nu_x+2\bar\nu_i). \qquad (15)$$

Use is made of these relations in multiplying Eq. (12) by $4\pi v^2 dv$ and integrating over all velocities. The term in brackets vanishes at both limits and one obtains

$$n\bar\nu_i + \nabla^2 Dn = 0. \qquad (16)$$

This is a diffusion equation and expresses the fact that at breakdown the ionization rate equals the diffusion rate. It has a solution which is everywhere positive only if $\bar\nu_i = D/\Lambda^2$, where Λ is the diffusion length for the cavity. This may be called the breakdown condition.

IV. DISTRIBUTION IN ENERGY

If the function F_0^0 is assumed to be the product of a function $n(x)$ of space and a function $f(u)$ of the energy u, defined by $u = mv^2/2e$, we can make use of Eq. (16) to replace $\nabla_r^2 F_0^0$ by $-nf/\Lambda^2$ and obtain the following equation for $f(u)$:

$$\left(\nu_x+\nu_i-q+\frac{2eu}{3m\nu_c\Lambda^2}\right)f = \frac{2}{3\sqrt{u}}\frac{d}{du}\nu_c u^{\frac{3}{2}}\left(u_c\frac{df}{du}+\frac{3m}{M}f\right). \qquad (17)$$

The terms on the left side represent the electrons leaving unit volume of phase space through excitation, ionization, and diffusion, and their reappearance at low energy at the rate qf. On the right are the terms due to energy gained from the field and lost to recoil.

The excitation frequency v_x sets in discontinuously at a potential u_x so that it is always necessary to divide the energy range into two parts and solve Eq. (17) for two functions, f_e and f_{xi}, appropriate to the elastic and inelastic ranges and join them at u_x. On the other hand, (v_x+v_i) may generally be approximated by a continuous function so it is not necessary to join functions at u_i. The method of solution appropriate to the two ranges is quite different so that they must be discussed separately.

V. INELASTIC RANGE

When inelastic collisions are possible, they dominate all other collision processes because of the large energy losses involved. Accordingly the recoil and diffusion terms may be left out of Eq. (17). We may also neglect q in this range. The equation to be solved is then

$$2(d/du)(v_c u_c u^{\frac{3}{2}} df_{xi}/du) = 3(v_x+v_i)u^{\frac{1}{2}} f_{xi}. \qquad (18)$$

The conditions imposed on the solution are somewhat contradictory; we must choose that solution which vanishes at infinity, but we want greatest accuracy just above u_x where most of the excitations take place. The conventional asymptotic expansion does not satisfy the second requirement without an unreasonable number of terms, and the WKB method diverges at u_x, but we can use a somewhat similar approximation. Setting

$$u^{\frac{1}{2}} f_{xi} = e^{-s}, \qquad (19)$$

the equation for s is

$$s'^2 - s'' - s'/2u - (s'+1/2u)(d/du) \ln(v_c u_c)$$
$$= 3(v_x+v_i)/2v_c u_c u. \qquad (20)$$

An analytic approximation to the experimental data must now be substituted for $v_c u_c$ and v_x+v_i and a power series in $1/u$ is substituted for s', the last term in the series being reserved to obtain exact agreement at u_x. This procedure will be illustrated later in the case of hydrogen.

Knowing s, the average ionization frequency per electron is given by

$$\bar{v}_i = 2\pi(2e/m)^{\frac{3}{2}} \int_{u_i}^{\infty} v_i u^{\frac{1}{2}} f_{xi} du, \qquad (21)$$

and the total inelastic frequency is given [see Eq. (25)] by

$$(\bar{v}_x+\bar{v}_i) = -(4\pi/3)(2e/m)^{\frac{3}{2}}(v_c u_c u^{\frac{3}{2}} df_{xi}/du)_x. \qquad (22)$$

The subscript indicates that the quantity in the parenthesis is to be taken at u_x. Both these expressions contain an unknown normalization constant and so cannot

be evaluated as they stand, but their ratio

$$N_{xi} = 1 + \frac{\bar{v}_x}{\bar{v}_i} = -\frac{2}{3}\left(v_c u_c u^{\frac{3}{2}} \frac{df_{xi}}{du}\right)_x \bigg/ \int_{u_i}^{\infty} v_i u^{\frac{1}{2}} f_{xi} du, \qquad (23)$$

can be evaluated and has a physical meaning. Because one electron must leave the tube for every one produced, $1/\bar{v}_i$ is the average lifetime of a free electron from its liberation at an ionization to its absorption at the walls. $\bar{v}_x/\bar{v}_i$ is the average number of excitations produced by an electron during its lifetime, and the number N_{xi} represents the total inelastic collisions during an electron's free lifetime.

Because of the exponential nature of f_{xi}, the number N_{xi} depends primarily on $\exp(s_x-s_i)$. From Eq. (20) it is seen that the main part of s' is given by $[3(v_x+v_i)/2v_c u_c u]^{\frac{1}{2}}$ which, when $v_c>\omega$, is proportional to p/E_p. Accordingly the variation of N_{xi} with p/E_p is given very nearly by

$$N_{xi} = \alpha \exp(\beta p/E_p), \qquad (24)$$

where α and β are constants obtainable from Eq. (20).

VI. ELASTIC RANGE

Below u_x the excitation and ionization frequencies are zero but we must discuss the appearance rate qF_0^0. In order to calculate it as a function of energy it is necessary to take the product of the distribution function times the excitation function of each level and shift the product down the energy scale by the energy of the particular level. The excitation functions of allowed transitions have a sharp maximum just above the excitation potential, so the scattered electrons have very little energy. The excitation functions of forbidden transitions have a maximum far above the excitation potential, but there are a negligible number of electrons with sufficient energy to excite these. Accordingly most inelastically scattered electrons have very little energy, and no appreciable error is made in assuming that q is a delta-function at zero energy.

Multiplying Eq. (17) by $4\pi v^2 dv$ gives the net rate of loss of electrons from the spherical shell dv. Integrating and making use of Eq. (15) gives

$$\bar{v}_x + 2\bar{v}_i - \frac{4\pi}{3\Lambda^2} \int_0^v \frac{f_e}{v_c} v^4 dv = -\frac{4\pi}{3} v_c v^3 \left(u_c \frac{df_e}{du} + \frac{3m}{M} f_e\right). \qquad (25)$$

$\bar{v}_x + 2\bar{v}_i$ is the rate of appearance of electrons at small velocities. The integral represents losses by diffusion of electrons of speeds between zero and v, and its value at infinity would, by Eq. (16), equal $\bar{v}_i$. The difference represents the rate at which electrons pass the energy u in the upward direction in order to supply the inelastic processes occurring at higher energies. Equation (25) was derived by Smit[2] directly from this principle of balance between electrons going up in energy and the

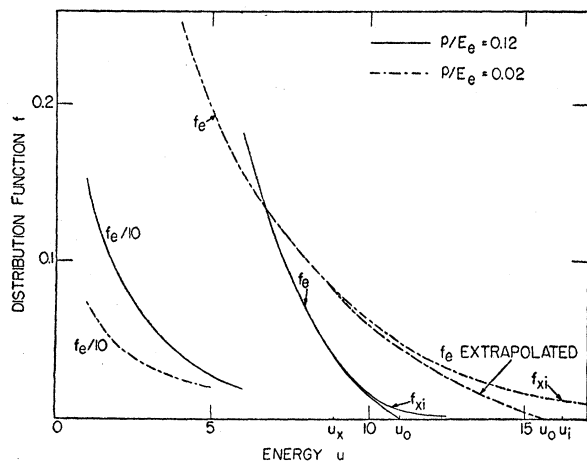[2] J. A. Smit, Physica 3, 543 (1936).

FIG. 1. Distribution functions for hydrogen for two values of p/E_e.

rate of inelastic collisions; however, Smit includes the thermal energy of the gas but does not include diffusion. In glow discharges the diffusion term is much larger than the thermal one. At very low pressures the diffusion term is quite large and one must solve the second-order equation (17) for f_e, but in most cases the diffusion term is small and there is then a great advantage in replacing the integral by an approximation such as

$$\frac{4\pi}{3\Lambda^2}\int_0^v \frac{f_e v^4}{\nu_c}dv = \frac{\bar{\nu}_i\nu_c v^3}{(\nu_c v^3)_0}. \quad (26)$$

This expression gives the full diffusion loss $\bar{\nu}_i$ at the velocity v_0, which corresponds to the energy u_0 to be defined shortly, and the third power of the velocity was actually found to be the best in the case for which extensive numerical calculations were made. With this substitution, Eq. (25) becomes a first-order inhomogeneous equation. If we define an energy variable w by $dw=3mdu/Mu_c$, the solution of the homogeneous part is

$$f_M = Ae^{-w}, \quad (27)$$

which is the same function as that given by Margenau.[3] The solution of the complete equation for f_e is

$$f_e = \frac{3}{4\pi}\left\{(\bar{\nu}_x+2\bar{\nu}_i)e^{-w}\int_u^{u_0} e^w\frac{du}{\nu_c u_c v^3}\right.$$

$$\left. -\bar{\nu}_i\left(\frac{M}{3m}\right)\left(\frac{e^{w_0-w}-1}{\nu_c v_0{}^3}\right)\right\}, \quad (28)$$

which, at higher pressures, can be replaced by the simpler function

$$f_e = \frac{3}{4\pi}(\bar{\nu}_x+\bar{\nu}_i)e^{-w}\int_u^{u_0} e^w du/\nu_c u_c v^3. \quad (29)$$

[3] H. Margenau, Phys. Rev. **69**, 508 (1946).

Both these functions are infinite at the origin, but this is a consequence of our assumption about the reappearance function q. The constant of integration in both of these expressions appears as an energy u_0 at which the function f_e crosses the axis when extended beyond u_x. The meaning of this energy is seen by noting that f_e would be unchanged if the actual excitation and ionization functions ν_x and ν_i were replaced at u_0 by delta-functions with the proper relative magnitudes so that all inelastic collisions would take place at exactly that energy. Thus u_0 is the equivalent single excitation potential. By this equivalence the diffusion should also vanish above u_0 and hence the integral (26) must equal $\bar{\nu}_i$ at $v=v_0$.

The potential u_0 is determined by equating the logarithmic derivatives of f_e and f_{xi} at u_x. In general the extrapolation u_0-u_x is small and when this is so a linear extrapolation formula may be used. The first-order derivative may be eliminated from Eq. (17) by the standard transformation

$$g = (\nu_c u_c)^{\frac{1}{2}}u^{\frac{1}{4}}e^{w/2}f. \quad (30)$$

Then $g''=0$ when $g=0$. The function $g(u)$ has a point of inflection at u_0 and may be extrapolated linearly back to u_x, giving

$$\frac{1}{u_0-u_x}\left(\frac{g'}{g}\right)_x = \frac{1}{2}\frac{d\ln(\nu_c u_c)}{du}+\frac{1}{4u_x}+\frac{3m}{2Mu_c}-s_x', \quad (31)$$

the whole right-hand side being taken at u_x. When this extrapolation is valid, the effective excitation potential u_0 may be calculated from the inelastic function f_{xi} alone.
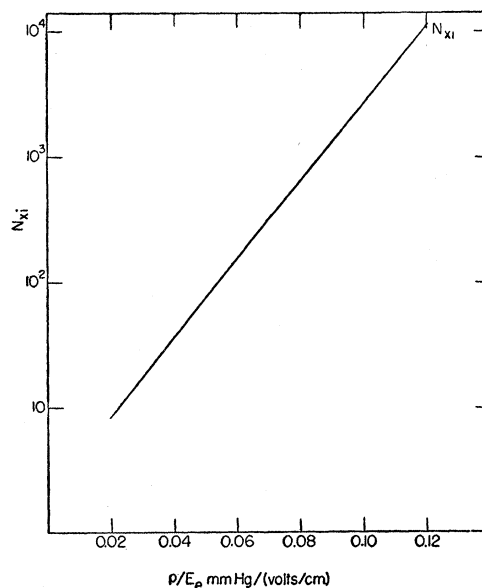

FIG. 2. Number of excitations and ionizations per electron lifetime as a function of p/E_e.

VII. BREAKDOWN EQUATION

The diffusion coefficient D may now be calculated by substituting f_e and f_{xi} into Eq. (10). A negligible error is made by integrating f_e from 0 to u_0 and not using f_{xi} (see Fig. 1), the difference being readily computed and shown to be small. Using Eq. (28) we get

$$D = \frac{2\pi}{3}\left(\frac{2e}{m}\right)^{5/2} \int_0^{u_0} f_e u^{\frac{3}{2}} \frac{du}{\nu_c} = \frac{2u_0^2}{E_p^2}[(\bar\nu_x + \bar\nu_i)\mathfrak{D} + \bar\nu_i\delta], \quad (32)$$

where $\mathfrak{D}$ and δ are two dimensionless functions of v_0:

$$\mathfrak{D} = \frac{4}{v_0^4} \int_0^{v_0} \frac{v^4 e^{-w}}{\nu_c} dv \int_v^{v_0} \frac{\nu_c^2 + \omega^2}{\nu_c v^2} e^w dv, \quad (33)$$

$$\delta = \mathfrak{D} - \frac{M E_p^2}{6m u_0^2 v_0^3} \int_0^{v_0} \frac{v^4}{\nu_c^2}[e^{w_0 - w} - 1]dv. \quad (34)$$

The breakdown condition is then

$$\Lambda^2 E_p^2 / 2u_0^2 = N_{xi}\mathfrak{D} + \delta, \quad (35)$$

where Eq. (23) or (24) can be substituted for the inelastic collision number.

VIII. CONSTANT COLLISION FREQUENCY

The above theory has been verified by comparison with measurements of breakdown in hydrogen. For this gas the collision frequency ν_c is nearly independent of energy and given by $\nu_c = 5.9 \times 10^9 p$ at all energies above 4 volts. The effective field E_e, defined by

$$E_e^2 = \nu_c^2 E_p^2 / 2(\nu_c^2 + \omega^2),$$

is then a constant and in terms of this the average energy gain per collision is $u_c = 5 \times 10^{-5}(E_e/p)^2$ electron-volts.

When ν_c is constant the variable $2w/3$ is the ratio of recoil loss to energy gain per collision, the loss exceeding the gain if $w > \frac{3}{2}$. At the higher pressures breakdown is observed for w_i approximately 4, so that in these cases the electrons are losing more energy to recoil, in the average, than they gain from the field, over most of the energy range. There are a sufficient number of statistically lucky electrons, however, to overcome this handicap and reach the ionization potential, producing breakdown.

The integrals in Eqs. (33) and (34) can now be evaluated in terms of incomplete gamma-functions or, more conveniently, by the series

$$\mathfrak{D} = \frac{\nu_c^2 + \omega^2}{\nu_c^2} w_0^{-2} \int_0^{w_0} w^{\frac{3}{2}} e^{-w} \int_w^{w_0} t^{-\frac{3}{2}} e^t dt dw$$

$$= \frac{\nu_c^2 + \omega^2}{\nu_c^2} 4! \sum_0^\infty \frac{(k+1)!}{(2k+5)!}(4w_0)^k, \quad (36)$$

$$\delta = 3\frac{\nu_c^2 + \omega^2}{\nu_c^2} 4! \sum_0^\infty \frac{(k+1)!}{(2k+5)!} \frac{(4w_0)^k}{2k+7}.$$
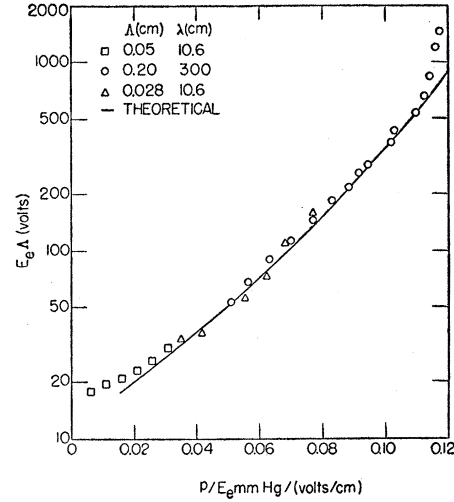
FIG. 3. Experimental breakdown voltage $E_e\Lambda$ for hydrogen as a function of p/E_e and the calculated curve.

Ramien[4] has measured the excitation and ionization probabilities in hydrogen and his data can be represented by the functions

$$(\nu_x + \nu_i)/\nu_c = h_0 u - h_1 - h_2/u, \quad (37)$$

$$\nu_i/\nu_c = h_i(u - u_i), \quad (38)$$

with the constants $u_i = 16.2$ volts, $h_i = 9.2 \times 10^{-3}$ volt^{-1}, $u_x = 8.9$ volts. The values $h_0 = 8.7 \times 10^{-3}$ volt^{-1} and $h_1 = 76 \times 10^{-3}$ are in agreement with his data but were actually obtained by fitting the present theory to the breakdown data as this is a more critical test of their value. Agreement with the breakdown data could not, however, be obtained if the losses observed by Ramien below 8.9 volts and ascribed to the excitation of vibrations were included in the theory.

With the above inelastic collision functions we set

$$s = au - b \ln u + c/u. \quad (39)$$

The coefficients a and b are determined in the usual way for series near infinity,

$$a^2 = 3h_0/2u_c, \quad b = 3h_1/4au_c - \frac{1}{4}. \quad (40)$$

The coefficient c is used to obtain exact agreement at u_x. This gives

$$c/u_x = au_x - b + \frac{3}{4} - (2au_x - b + \frac{9}{16})^{\frac{1}{2}}. \quad (41)$$

The approximation is then tested by substituting Eq. (39) with these constants back into Eq. (18) and solving for $(\nu_i + \nu_x)$. This gives the theoretical excitation frequency for which Eqs. (19) and (39) are the exact solution, and it must agree closely with the experimental data for P_x and P_i particularly between u_x and u_i. Substitution of Eqs. (19) and (39) in Eq. (23) gives
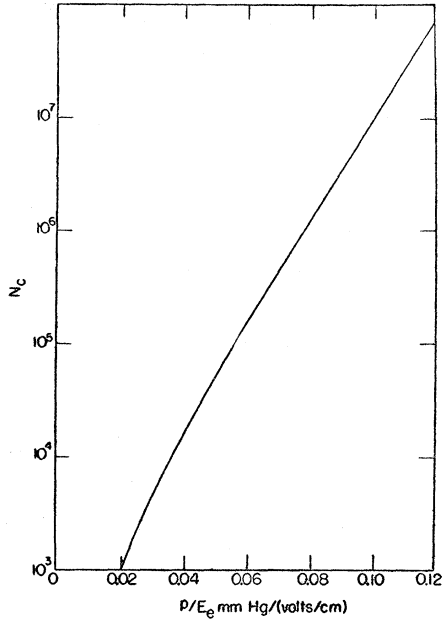
[4] H. Ramien, Z. Physik **70**, 353 (1931).

FIG. 4. Number of collisions per electron lifetime as a function of p/E_e.

the number of inelastic collisions per electron

$$N_{xi}=\frac{h_0}{h_i}\frac{au_x-b+\frac{1}{2}-c/u_x}{\mathcal{G}(u_i)}\left(\frac{u_x}{u_i}\right)^b$$

$$\times\exp[(a-c/u_iu_x)(u_i-u_x)], \quad (42)$$

where

$$\mathcal{G}(u_i)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(k+l+1)!}{k!(au_i)^k}\frac{(b-l)!}{(b-k-l)!}\frac{(c/au_i^2)^l}{l!}.$$

This function is shown in Fig. 2 and agrees very well with the approximation (24) with $\alpha=2$, $\beta=71.7$ volts/cm-mm of Hg, for almost the whole range of the measurements. The limit $N_{xi}=2$ as $p/E\rightarrow0$ comes from the near equality of h_i and h_0, so that at high energies there are about equal numbers of excitations and ionizations. Introducing Eq. (39) into Eq. (31) we find the effective excitation potential from

$$1/(u_0-u_x)=a-3m/2Mu_c-(b+\tfrac{1}{4})/u_x-c/u_x{}^2. \quad (43)$$

Expressions (36), (42), and (43) may then be introduced in (35) to obtain a direct comparison with the quantities measured at breakdown. The results are shown in Fig. 3. The agreement is good over a wide range of pressure for several different values of Λ and at two different frequencies. The disagreement at low pressures is due to the neglect of the diffusion term in f_{xi} and to the ap-

proximations made in the formula for the effective excitation potential u_0 and at these pressures the more exact confluent hypergeometric functions should be used.[5]

The agreement with breakdown data gives considerable confidence that the distribution function is correct, and therefore it may be used to compute other quantities of physical interest. Normalizing the distribution f_e yields the total number of collisions by an electron

$$N_c=\nu_c/\bar{\nu}_i=(u_0/u_c)[N_{xi}(1+w_0\mathfrak{D})+\tfrac{3}{5}+w_0\delta], \quad (44)$$

and this is shown in Fig. 4. The tremendous number of collisions experienced by an electron before being absorbed by the walls is a measure of the probability of its finding an impurity, however small. At $p\Lambda=40$ cm-mm of Hg, which corresponds in hydrogen to $p/E_e=0.1$, there are 10^7 collisions and therefore an impurity occurring as one part in 10^6 will be struck, in the average, ten times by each electron. If the impurity is such as to attach the electron this will make breakdown more difficult, and this is probably the explanation of
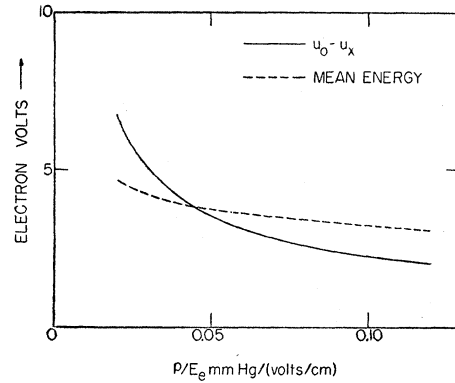

FIG. 5. Mean energy and equivalent single excitation potential as functions of p/E_e.

the experimental points lying above the theoretical curve at the higher pressures.

The mean energy $\bar{u}$ is obtained from the same integral as the diffusion constant and is given by

$$\bar{u}/\bar{u}_0=\tfrac{3}{2}(N_{xi}\mathfrak{D}+\delta)/[N_{xi}(1+w_0\mathfrak{D})+\tfrac{3}{5}+w_0\delta]. \quad (45)$$

At high pressures the mean energy tends towards $3u_0/10$. The mean energy and u_0-u_x are plotted against p/E_e in Fig. 5.

[5] A. D. MacDonald and S. C. Brown, Phys. Rev. **76**, 1634 (1949).