The Inelastic Scattering of Neutrons*

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The total cross section and the differential cross section for the inelastic scattering of neutrons are considered. It is assumed that the compound nucleus is sufficiently excited so that the statistical model may be applied. If the statistical model may be applied as well to the residual nucleus, it is shown that the angular distribution of the inelastically scattered neutrons is isotropic. If only a few levels of the target nucleus can be excited, the angular distribution is anisotropic. Tables are provided which permit the calculation of the angular distribution if the incident and emergent neutron angular momenta are less than or equal to $3\hbar$. Examples of the evaluation of total cross sections are given, providing examples of the sensitivity of the results to the quantum numbers of the excited state.

I. INTRODUCTION

THE inelastic scattering of neutrons is of interest not only for the theory of the passage of neutrons through matter but also because of the light it casts upon the properties of the target nucleus. For moderate neutron energies in which only a few levels are excited, the total cross section as well as the energy and angle distribution of the emergent neutrons is sensitive to the energy, angular momentum, and parity differences which exist between the ground state of the target nucleus and the states excited. When the neutron energy is sufficiently high, the measurements are no longer as sensitive to detailed properties of the excited states. It will be shown that the angular distribution of the emergent neutrons is then isotropic. However, the energy distribution is dependent strongly upon the density of energy levels of the target nucleus as a function of the energy of excitation.

In this paper, which may be regarded as a sequel to one of Wolfenstein's[1] on the same subject, the analysis required for the prediction of the relevant experimental quantities is performed. The discussion divides the inelastic processes into two categories according to the validity of the statistical model for the compound and residual nucleus. In category I we place those situations for which the statistical model may only be applied to the compound nucleus but not to the residual nucleus. This will occur whenever the neutron energy is so small that only a few levels can be excited and whenever the target nucleus is a middleweight or heavy nucleus, for then the density of levels in the compound nucleus is sufficiently great. In category II, we assume that the neutron energy is so large that many levels of the target nucleus are excited so that the statistical theory applies to both the compound and residual nuclei. The angular distribution of neutrons in category I is generally anisotropic. Tables are provided which permit a prediction of this distribution when the orbital angular momenta of the incident or emergent neutrons is less than or equal to $3\hbar$. For neutrons in category II, we give a proof of the isotropy of the inelastically scattered neutrons. The familiar dependence of the energy distribution on level density may then be obtained.

Our emphasis is on neutrons. However, the discussion is applicable with only minor changes for protons or alpha-particles either incident or emerging.

II. GENERAL CONSIDERATIONS

The general formulas upon which our discussion will be based have been given by Wolfenstein.[1] We shall restate them here for the purposes of fixing our notation and reviewing the simplifications which follow from the application of the statistical model to the compound nucleus.

Notation.—Let the target nucleus have a spin of i (in units of $\hbar$), the residual nucleus a spin of i', the initial and final orbital neutron angular momentum by l and l', respectively, the corresponding initial and final energies E and E'. The spin of a level in the compound nucleus will be denoted by J. It is convenient to combine the neutron spin and the spin of the nucleus to form the channel spin j_i and j_i' for the initial and final states:

$$j_{1,2}=i\pm\tfrac{1}{2},\quad j'_{1,2}=i'\pm\tfrac{1}{2}.$$

The spin of the compound nucleus J is formed by combining l and j or l' and j'. We shall drop the subscript on j and j' employing the subscript only when it is necessary to sum over initial and final states. For a given J the values of l and l' which may contribute to a reaction are given by

$$|J-j|\leq l\leq(J+j),\quad |J-j'|\leq l'\leq(J+j').$$

The z axis will be taken along the direction of incident neutron so that the z component of its orbital angular momentum will be zero. The z component associated with the initial channel spin is m and is equal to the z component of J. The z component of the angular momentum of the emergent neutron is m' so that the corresponding quantity for the final channel spin j' is $m-m'$.

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[1] L. Wolfenstein, Phys. Rev. 82, 690 (1951).

Parity must, of course, be preserved. Changes in parity will be carried by the neutron orbital angular momenta, even l and l' corresponding to no change in parity whereas odd l and l' introducing a parity change.

Simplifications following from the statistical model.—We assume that, at the excitation energy of the compound nucleus, there are many energy levels of all types. More precisely, the energy resolution of the incident beam is broad enough so that many levels of the compound nucleus are excited. The corresponding wave functions are assumed to have a random phase so that when phase averages are performed all interference terms will vanish. From the concept of the compound nucleus, it follows that it is then possible to divide the inelastic scattering process into two parts comprising (1) the formation of the compound nucleus and (2) the decay of the compound nucleus by particle emission. For the energy region under consideration we shall neglect the competition arising from gamma-ray emission and, except for the lightest nuclei, from charged particle emission. In a paper submitted for publication Margolis will consider the effects of gamma-ray competition.

Application of the statistical model permits us, therefore, to neglect the interference among the various l's which contribute to the formation of the compound nucleus as well as the interference among the l''s which are involved in the decay of the compound nucleus. Hence we may write

$$\sigma(i|i')=\sum\nolimits_{l,\,l'}\sigma(l,i|l',i'),\tag{1}$$

where

$$\sigma(l,i|l',i')=(1/2(2i+1))\sum\nolimits_{\alpha,\beta}\sigma(l,j_\alpha|l',j_\beta').\tag{2}$$

Here $\sigma(l,i|l',i')$ is the cross section for the process involving the indicated initial and final values of the neutron orbital angular momenta and spins of the target nucleus. On the other hand, $\sigma(l,j_\alpha|l',j_\beta')$ is the cross section for the process involving an initial and final channel spin j_α and j'_β, respectively.

A certain fraction of the neutrons with orbital angular momentum l which strike the nucleus, "stick," and form a compound nucleus. The cross section for this process is denoted $\sigma_c^{(l)}$. It is often expressed in terms of the penetrabilities $T_l(E)$ as follows:

$$\sigma_c^{(l)}=(2l+1)\pi\lambda\!\!\!^{-2}T_l(E).\tag{3}$$

The cross section for the formation of the compound nucleus of spin J by "l" neutrons will be given by $\sigma_c^{(l)}$ multiplied by the probability that the incident neutron and target nucleus form a system of spin J. This probability is equal to the square of the Clebsch-Gordan coefficient $|(lj;0m|lj;Jm)|^2$ so that the cross section in question is

$$\sigma_c^{(l)}|(lj;0m|lj;Jm)|^2 \\ =(2l+1)\pi\lambda\!\!\!^{-2}T_l(E)|(lj;0m|lj;Jm)|^2.\tag{4}$$

To obtain the cross section for a particular inelastic process we must multiply (4) by the relative probability of that process. This may be obtained from (4) through the use of the reciprocity theorem.[2] The cross section $\sigma(l,j|l',j'|\vartheta)$ for the production of neutrons of energy E' of angular momentum l', channel spin j' and moving in a direction ϑ turns out to be

$$\sigma(l,j|l',j'|\vartheta) \\ =\pi\lambda\!\!\!^{-2}(2l+1)T_l(E)\sum_J\frac{A_J(l,j|l',j'|\vartheta)}{1+\sum'_{p,q,r}T_p(E_q')/T_{l'}(E')}.\tag{5}$$

The r index refers to possible channel spins, p to possible final neutron angular momenta, E_q' to possible final neutron energies. The prime on the summation indicates that the term for which $p=l'$, $E_q'=E'$, and $j_r'=j'$ is omitted. The factor A_J is defined as

$$A_J(l,j|l',j'|\vartheta)=\sum\nolimits_{m,m'}|(lj;0m|lj;Jm)|^2 \\ \times|(l',j';m',m-m'|l'j';Jm)|^2|Y_{l'm'}(\vartheta,\varphi)|^2.\tag{6}$$

There is no dependence on φ in (7) because of the absolute value sign on $Y_{l',m'}$. Note that Eq. (6) refers to the excitation of a single level of energy E'.

The cross section $\sigma(l,j|l',j')$ may be obtained by integration and the sum rule

$$\sum\nolimits_m|(lj;0m|lj;Jm)|^2=(2J+1)/(2l+1).\tag{7}$$

Hence

$$\sigma(l,j|l',j') \\ =\pi\lambda\!\!\!^{-2}T_l(E)\sum_J\left[(2J+1)\Big/\left(1+\sum\nolimits'_{p,q,r}\frac{T_p(E_q')}{T_{l'}(E')}\right)\right],\tag{8}$$

a familiar expression.[2] At this point the sum of $\sigma(l,j|l',j')$ over j, j', l, and l' may be readily indicated obtaining $\sigma(i|i')$:

$$\sigma(i|i')=\frac{\pi\lambda\!\!\!^{-2}}{2(2i+1)}\sum_l T_l(E) \\ \times\sum_J\frac{\epsilon_{j,l}{}^J(2J+1)}{1+\sum'_{j'',l'',q}\epsilon_{j'',l''}{}^JT_{l''}(E_q')/\sum_{l',j'}\epsilon_{j',l'}{}^JT_{l'}(E')},\tag{9}$$

where

$$\epsilon_{j,l}{}^J=\begin{Bmatrix}2 \text{ if both } j_1 \text{ and } j_2\\ 1 \text{ if } j_1 \text{ or } j_2\text{, not both}\\ 0 \text{ if neither } j_1 \text{ nor } j_2\end{Bmatrix}$$

$$\text{satisfy } |J-l|\le j_i\le(J+l).\tag{10}$$

The prime in the sum (9) requires the omission of those terms in the sum for which $E_q'=E'$, $l'=l''$, and j'' equal to either value of j'. The sum over l' is meant to include all values of the angular momentum of the

[2] J. Blatt and V. F. Weisskopf, forthcoming book on nuclear theory.

TABLE I. Values of initial and final angular momenta and of the spin of the compound nucleus, entering into calculation of $\sigma(i|i')$ for the transition $0^+\to 1^+$.

J	l	$\epsilon_{j,l}{}^J$	l'	$\epsilon_{j',l'}{}^J$
$(1/2)^+$	0	1	0	1
			2	1
$(1/2)^-$	1	1	1	2
$(3/2)^-$	1	1	1	2
			3	1
$(3/2)^+$	2	1	0	1
			2	2
$(5/2)^+$	2	1	2	2
$(5/2)^-$	3	1	1	1
			3	2
$(7/2)^-$	3	1	3	2

emergent neutrons which can lead to the particular excited state being considered.

In these expressions the properties of the compound nucleus enters only in J and T_l. For J we make the statistical assumption that all J's consistent with the conservation of angular momentum enters into the sum. The penetrabilities T_l have been computed on the bases of the schematic theory for nuclear cross section and will be given in a paper in preparation.[3] The properties of the residual nucleus occurs through the quantities E_q' and j', and of course the parity. It should be borne in mind that the penetrabilities as given by the schematic theory are averages over resonances. For a given level in the residual nucleus there can be strong fluctuations from the average.

Although (7) and (8) have been primarily developed for application to inelastic neutron scattering, it is clear that they may be employed to deal with inelastic scattering of other particles and for nuclear reactions provided, of course, that the assumptions underlying their derivation holds. The main change occurs in the factor common to both (7) and (8),

$$T_l(E)T_{l'}(E')/\sum_{p,q,r} T_p(E_q').$$

For $T_l(E)$ and $T_{l'}(E')$ we must insert the penetrabilities of the incident and emergent particles. In the denominator we must insert the penetrabilities for those particles which are readily emitted by the compound nucleus. These particles are usually neutrons. Application to reactions involving charged particles will be presented in another paper.

III. INELASTIC SCATTERING WITH EXCITATION OF FEW LEVELS

We first consider the integrated cross section, $\sigma(i|i')$ as given by Eq. (9). We shall be interested here in showing the sensitivity of $\sigma(i|i')$ to the quantum numbers of the excited levels.

Sufficiently close to threshold the energy dependence of $\sigma(i|i')$ is given by $T_0(E)$, since it is always possible for l' to equal zero and since $T_0(E)$ approaches zero less rapidly than $T_{l'}(E')$, $l'\neq 0$, as may be seen from the limiting form

$$T_{l'}(E')\xrightarrow[E'\to 0]{}[(l'-1)!/(2l'-1)!]^2 \times[x'X_0/(X_0^2+l'^2)](2x')^{2l'}, \quad (11)$$

where $x'=k'R$, k' = wave number of emergent neutron, R, the nuclear radius, and X_0 is a constant defined by Feshbach and Weisskopf.[4] Hence $T_{l'}(E')$ goes to zero like $E'^{(l'+\frac{1}{2})}$. The additional angular momentum l required to make the transition from i to i' possible is furnished by the incident neutron. In general, the term $T_0(E)$ will not dominate the series for $\sigma(i|i')$ for long. This is true even when $|i-i'|$ is small. As $|i-i'|$ increases and the energy of the excited level decreases, the energy region in which the $l'=0$ term is dominant becomes smaller and smaller. As a consequence, the shape of the total cross section curve as a function of energy may be quite sensitive to $|i-i'|$.

The results obtained for a typical example will now be exhibited. The cross section $\sigma(i|i')$ was first calculated for the case of a single excited level 0.5 Mev above the ground state, the latter having a spin $i=0$. Four assignments for the angular momentum and parity of the excited state were investigated: 0^+, 0^-, 1^+, 1^-, where the numeral indicates the value of i, $+$ means same parity as that of the ground state, and $-$ means opposite parity. Angular momenta up to 3 were considered. In Table I we list the values of l, l', J, $\epsilon_{j,l}{}^J$, $\epsilon_{j',l'}{}^J$ for one case, 1^+. The final expression for the cross section is

$$\sigma(i|i')=(\pi\lambda\!\!\!^{-2}/2)\Bigg\{\frac{2T_0(E)}{1+[T_0(E')/\{T_0(E')+T_2(E')\}]} +T_1(E)\Bigg[\frac{2}{1+[T_1(E)/2T_1(E')]} +\frac{4}{1+[T_1(E)/\{2T_1(E')+T_3(E')\}]}\Bigg] +T_2(E)\Bigg[\frac{4}{1+[T_2(E)/\{T_0(E')+2T_2(E')\}]} +\frac{6}{1+[T_2(E)/2T_2(E')]}\Bigg] +T_3(E)\Bigg[\frac{6}{1+[T_3(E)/\{T_1(E')+2T_3(E')\}]} +\frac{8}{1+[T_3(E)/2T_3(E')]}\Bigg]\Bigg\}.$$

The resulting values of $\sigma(i|i')/(\pi\lambda\!\!\!^{-2}/2)$ for a nucleus of radius 8×10^{-13} cm are plotted in Fig. 1. We see that $\sigma(i|i')$ is quite sensitive in magnitude though not in

[3] Feshbach, Shapiro, and Weisskopf, to be published.

[4] H. Feshbach and V. F. Weisskopf, Phys. Rev. **76**, 1550 (1949).

shape (for the $|i-i'|$ is small) to the value of i' but not nearly as sensitive to the parity. On the same figures we have given the predicted $\sigma(i|i')$ for $i'=3$, with a change in parity. We note the characteristic change in shape. In this case the $l'=0$ term dominates for $E'\lesssim 20$ kev while for the cases where $|i-i'|$ is small the $l'=0$ term is dominant up to E' equal to 200 kev. The sensitivity of the shape of $\sigma(i|i')$ to $|i-i'|$ was first noted by Ebel.[5] We again remark that the above calculations are based on averages as given by the schematic theory. Note also that gamma-ray competition may be important for some of the terms in $\sigma(i/i')$ close to threshold.

Angular distribution.—Returning to Eq. (6) we see that the dependence of the cross section on angle enters through the factor

$$A_J(l,j|l',j'|\vartheta)=\sum_{m,m'}|(lj;0m|lj;Jm)|^2 \times|(l'j';m',m-m'|l'j';Jm)|^2|Y_{l'm'}(\vartheta,\varphi)|^2. \quad (12)$$

Most of this section will be concerned with the evaluation of this sum. Our results will maintain a separation of the quantum numbers of the initial and final state permitting the rapid evaluation for various assumptions for these numbers. We therefore expand the sum in (5) over m in a power series in m

$$\sum|(l'j';m',m-m'|l'j';Jm)|^2|Y_{l'm'}(\vartheta,\varphi)|^2 =\sum_{n=0}^{l'} m^{2n}F_n(J|l',j'|\vartheta), \quad (13)$$

where the function F_n is the coefficient of m^{2n}. Letting

$$s_n(l,j|J)=\sum_{m=0}^{j} m^{2n}|(lj;0m|lj;Jm)|^2, \quad (14)$$

A_J is written as follows:

$$A_J(l,j|l',j'|\vartheta)=\sum_{n=0}^{l'} s_n(l,j|J)F_n(J|l',j'|\vartheta). \quad (15)$$

The s_n terms contain the dependence on the initial

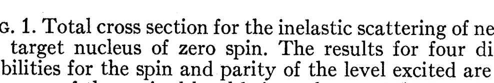

FIG. 1. Total cross section for the inelastic scattering of neutrons by a target nucleus of zero spin. The results for four different possibilities for the spin and parity of the level excited are given, the energy of the excited level being taken as 0.5 Mev above the ground state.

[5] A. Ebel, Massachusetts Institute of Technology thesis (unpublished).

TABLE II. The factors s_n in Eq. (15). The symbol C.F. stands for common factor. For example, $s_1(2,j|j+2)$ is $[(2j+5)/5]\times[j(j+3)/7]$.

$s_n(1,j|J)$

$J\diagdown n$	C.F.	0	1	2
$j+1$	$\frac{2j+3}{3}$	1	$\frac{j(j+2)}{5}$	$\frac{(j)(j+2)(3j^2+6j-2)}{35}$
j	$\frac{2j+1}{3}$	1	$\frac{3j^2+3j-1}{5}$	$\frac{3j^4+6j^3-3j+1}{7}$
$j-1$	$\frac{2j-1}{3}$	1	$\frac{(j+1)(j-1)}{5}$	$\frac{(j+1)(j-1)(3j^2-5)}{35}$

$s_n(2,j|J)$

$J\diagdown n$	C.F.	0	1	2
$j+2$	$\frac{2j+5}{5}$	1	$\frac{j(j+3)}{7}$	$\frac{j(j+3)(j^2+3j-1)}{21}$
$j+1$	$\frac{2j+3}{5}$	1	$\frac{3j^2+6j-2}{7}$	$\frac{5j^4+20j^3+10j^2-20j+6}{21}$
j	$\frac{2j+1}{5}$	1	$\frac{11j^2+11j-15}{21}$	$\frac{9j^4+18j^3-16j^2-25j+21}{21}$
$j-1$	$\frac{2j-1}{5}$	1	$\frac{3j^2-5}{7}$	$\frac{5j^4-20j^2+21}{21}$
$j-2$	$\frac{2j-3}{5}$	1	$\frac{(j-2)(j+1)}{7}$	$\frac{(j+1)(j-2)(j^2-j-3)}{21}$

$s_n(3,j|J)$

$J\diagdown n$	C.F.	0	1	2
$j+3$	$\frac{2j+7}{7}$	1	$\frac{j(j+4)}{9}$	$\frac{j(j+4)(3j^2+12j-4)}{99}$
$j+2$	$\frac{2j+5}{7}$	1	$\frac{j^2+3j-1}{3}$	$\frac{5j^4+30j^3+30j^2-45j+13}{33}$
$j+1$	$\frac{2j+3}{7}$	1	$\frac{7j^2+14j-15}{15}$	$\frac{53j^4+212j^3-30j^2-484j+315}{165}$
j	$\frac{2j+1}{7}$	1	$\frac{23j^2+23j-66}{45}$	$\frac{39j^4+78j^3-190j^2-229j+366}{99}$
$j-1$	$\frac{2j-1}{7}$	1	$\frac{7j^2-22}{15}$	$\frac{53j^4-348j^2+610}{165}$
$j-2$	$\frac{2j-3}{7}$	1	$\frac{j^2-j-3}{3}$	$\frac{5j^4-10j^3-30j^2+35j+63}{33}$
$j-3$	$\frac{2j-5}{7}$	1	$\frac{(j+1)(j-3)}{9}$	$\frac{(j+1)(j-3)(3j^2-6j-13)}{99}$

quantum numbers l and j only, while F_n contains the dependence on the quantum numbers of the final state. The functions s_n are given in Table II for arbitrary j and J for $1\leq l\leq 3$. The functions F_n are given in Table III for arbitrary J with $1\leq l'\leq 2$. The values of F_n at

TABLE III. The factors F_n in Eq. (15). The symbol C.F. stands for common factor. For example, $F_0(j'+1|1,j'|\vartheta)$ is $[1/(2j'+1)]\times[j'|Y_{11}|^2+(j'+1)|Y_{10}|^2]$ while $F_1(j'+1|1,j'|\vartheta)$ is $-(3/4\pi)[(1/(2j'+1)(j'+1)]P_2$. The functions $|Y_{lm}|^2$ and P_l are the square of the absolute value of the normalized spherical harmonics and the Legendre polynomials of order l, respectively. $|Y_{11}|^2$ equals $(3/8\pi)\sin^2\vartheta$; $|Y_{10}|^2$ is $(3/4\pi)\cos^2\vartheta$.

$J\diagdown n$ Angular dependence	C.F.	$F_n(J\|1,j'\|\vartheta)$ 0 $\|Y_{11}\|^2$	$\|Y_{10}\|^2$	1 P_2
$j'+1$	$\frac{1}{(2j'+1)}$	j'	$j'+1$	$-\frac{3}{4\pi}\frac{1}{(j'+1)}$
j'	1	1	0	$\frac{3}{4\pi}\frac{1}{j'(j'+1)}$
$j'-1$	$\frac{1}{(2j'+1)}$	$j'+1$	j'	$-\frac{3}{4\pi}\frac{1}{j'}$

$J\diagdown n$ Angular dependence	C.F.	$F_n(J\|2,j'\|\theta)$ 0 $\|Y_{22}\|^2$	$\|Y_{21}\|^2$	$\|Y_{20}\|^2$	1 $(15/8\pi)P_2$	$(15/8\pi)P_4$	2 $(15/8\pi)P_4$
$j'+2$	$\frac{1}{(2j'+1)(2j'+3)}$	$\frac{j'(j'-1)}{2}$	$2j'(j'+2)$	$\frac{3(j'+2)(j'+1)}{2}$	$-\frac{4(j'+1)(2j'+1)}{7(j'+1)(j'+2)}$	$-\frac{6j'^2+30j'+31}{7(j'+1)(j'+2)}$	$\frac{1}{(j'+1)(j'+2)}$
$j'+1$	$\frac{1}{2j'+1}$	$j'-1$	$j'+2$	0	$\frac{2(j')(j'-4)}{7j'(j'+1)(j'+2)}$	$\frac{2(6j'^2+18j'+7)}{7j'(j'+1)(j'+2)}$	$-\frac{2}{j'(j'+1)(j'+2)}$
j'	$\frac{1}{(2j'-1)(2j'+3)}$	$3(j'-1)(j'+2)$	3	$(j')(j'+1)$	$\frac{2(4j'^2+4j'-15)}{7j'(j'+1)}$	$-\frac{6(6j'^2+6j'-5)}{7(j')(j'+1)}$	$\frac{6}{j'(j'+1)}$
$j'-1$	$\frac{1}{2j'+1}$	$j'+2$	$j'-1$	0	$\frac{2(j'+1)(j'+5)}{7j'(j'+1)(j'-1)}$	$\frac{2(6j'^2-6j'-5)}{7j'(j'+1)(j'-1)}$	$-\frac{2}{j'(j'+1)(j'-1)}$
$j'-2$	$\frac{1}{(2j'-1)(2j'+1)}$	$\frac{(j'+1)(j'+2)}{2}$	$2(j'^2-1)$	$\frac{3}{2}j'(j'-1)$	$-\frac{4j'(2j'+1)}{7j'(j'-1)}$	$-\frac{6j'^2-18j'+7}{7j'(j'-1)}$	$\frac{1}{j'(j'-1)}$

0° and 90° are contained in Tables IV and V. For l' equal to 0, A_J equals $(2J+1)/[4\pi(2l+1)]$, independent of ϑ. For $l=0$, $A_J=(2J+1)/[4\pi(2l'+1)]$. The case $l'=3$ may be included by employing the symmetry of our results between initial and final states, which we shall now discuss.

Following Blatt and Biedenharn,[6] it is possible to show that

$$A_J(l,j|l',j'|\vartheta)=\frac{(2l'+1)(2J+1)^2}{4\pi}\Big|\sum_L(2L+1)V(llL;000)W(JJll;Lj)\cdot V(l'l'L;000)W(JJl'l';Lj')P_L(\cos\vartheta)\Big|.$$

The coefficients V and W have been defined by Racah.[7] Blatt and Biedenharn introduce the coefficients $Z(a\,b\,c\,d;e\,f)$, which they have tabulated and in terms of which A_J takes on a particularly simple appearance as follows:

$$A_J(l,j|l',j'|\vartheta)=\frac{1}{4\pi(2l+1)}\Big|\sum_L Z(lJlJ;jL)Z(l'Jl'J;j'L)P_L(\cos\vartheta)\Big|. \tag{16}$$

From the general properties of V and W or of Z we may conclude that the summation index L takes on only even values and that

$$L\leq\min(2l,2l',2J),$$

an observation which has been made by several authors.[8,9] Hence if l or l' are zero or if J is either zero or $\frac{1}{2}$, A_J will be independent of ϑ. The fact that L is even is of course obvious from the definition of A_J. The symmetry between initial and final states may now be

[6] J. Blatt and L. C. Biedenharn, Phys. Rev. **86**, 399 (1952).
[7] G. Racah, Phys. Rev. **61**, 186 (1942).
[8] C. N. Yang, Phys. Rev. **74**, 764 (1948).
[9] E. Eisner and R. G. Sachs, Phys. Rev. **72**, 680 (1947); L. Wolfenstein and R. G. Sachs, Phys. Rev. **73**, 528 (1948).

TABLE IV. $F_n(J|l'j'|0)$.

$J\diagdown n$	$F_n(J\|1,j'\|0)$ C.F.	0	1
$j'+1$	$\frac{3}{4\pi}\frac{1}{2j'+1}$	$j'+1$	$-\frac{1}{(j'+1)}$
j'	$\frac{3}{4\pi}$	0	$\frac{1}{j'(j'+1)}$
$j'-1$	$\frac{3}{4\pi}\frac{1}{2j'+1}$	j'	$-\frac{1}{j'}$

$J\diagdown n$	C.F.	$F_n(J\|2,j'\|0)$ 0	1	2
$j'+2$	$\frac{15}{8\pi}\frac{1}{(2j'+1)(2j'+3)}$	$(j'+2)(j'+1)$	$-\frac{(2j'^2+6j'+5)}{(j'+1)(j'+2)}$	$\frac{1}{(j'+1)(j'+2)}$
$j'+1$	$\frac{15}{4\pi}\frac{1}{(j')(j'+1)(j'+2)(2j'+1)}$	0	$(j'+1)^2$	-1
j	$\frac{5}{4\pi}\frac{1}{(2j'-1)(2j'+3)}$	$(j')(j'+1)$	-6	$\frac{9}{j'(j'+1)}$
$j'-1$	$\frac{15}{4\pi}\frac{1}{j'(j'+1)(j'-1)(2j'+1)}$	0	j'^2	-1
$j'-2$	$\frac{15}{8\pi}\frac{1}{(2j'+1)(2j'-1)}$	$(j')(j'-1)$	$-\frac{(2j'^2-2j'+1)}{j'(j'-1)}$	$\frac{1}{j'(j'-1)}$

stated:

$$A_J(l,j|l'j'|\vartheta)=[(2l'+1)/(2l+1)]A_J(l'j'|lj|\vartheta). \quad (17)$$

Because of the length of these calculations it was important to make a number of checks which were obtained from the following relations:

$$\int F_n(J|l',j'|\vartheta)d\Omega=\delta_{n0}. \quad (18)$$

$$\textstyle\sum_{j'}(2j'+1)F_n(J|l',j'|\vartheta) \text{ is independent of } \vartheta. \quad (19)$$

$$\textstyle\sum_n m^{2n}F_n(J|l',j'|0)$$
$$=[(2l'+1)/4\pi]|(l'j';0m|l'j';Jm)|^2. \quad (20)$$

$$\textstyle\sum_J s_n(l,j|J)=\sum_m m^{2n}. \quad (21)$$

An example of the calculation of A_J will now be given in order to make the use of the tables clear. We take $A_{\frac{3}{2}}(1\frac{1}{2}|1\frac{1}{2}|\vartheta)$ and therefore require $F_0(\frac{3}{2}|1\frac{1}{2}|\vartheta)$ and $F_1(\frac{3}{2}|1\frac{1}{2}|\vartheta)$, where $J=j'+1$, $(2j'+1)=2$. Then

$$F_0(\tfrac{3}{2}|1\tfrac{1}{2}|\vartheta)=\tfrac{1}{2}[\tfrac{1}{2}|Y_{11}|^2+\tfrac{3}{2}|Y_{10}|^2],$$
$$F_1(\tfrac{3}{2}|1\tfrac{1}{2}|\vartheta)=\tfrac{1}{2}[(-3/4\pi)\cdot\tfrac{2}{3}]P_2.$$

The values of s_n, $J=j+1$, are

$$s_0(1\tfrac{1}{2}|\tfrac{3}{2})=4/3, \quad s_1(1\tfrac{1}{2}|\tfrac{3}{2})=\tfrac{1}{3}.$$

Hence

$$A_{\frac{3}{2}}(1\tfrac{1}{2}|1\tfrac{1}{2}|\vartheta)=(1/4\pi)[\tfrac{1}{2}\sin^2\vartheta+3\cos^2\vartheta-\tfrac{1}{3}P_2(\cos\vartheta)].$$

Once A_J is available, sufficient data is available for the calculation of the angular distribution. We note that as a consequence of the statistical assumption for the compound nucleus that the predicted distributions are symmetrical about 90°. If either $l=0$ neutrons incoming or $l'=0$ neutrons outgoing or if the compound nuclear states with $J=\frac{1}{2}$ form the most important components of cross section, then the distribution will be isotropic. This will generally be the case near threshold for then only $l'=0$ neutrons have appreciable transmission factors. We can be certain of large anisotropy in the angular distribution only if the spin change between the initial and final levels is large as would occur in the excitation of isomeric levels. For smaller spin changes the anisotropy will be correspondingly smaller. In Fig. 2 we give an angular distribution in which a target nucleus $i=0$ is struck by neutrons having 1-Mev energy. These excite a level at 0.5 Mev with $i=1$ and opposite in parity to the ground state. These angular distributions are essentially average results since fluctuations away from the schematic theory will affect them considerably. For this reason it is important to employ the parameters R and X_0 which best fit the

TABLE V. $F_n(J|l'j'|\pi/2)$.

$J\diagdown n$	C.F.	$F_n(J\|1,j'\|\pi/2)$ 0	1
$j'+1$	$\frac{3}{8\pi}\frac{1}{2j'+1}$	j'	$\frac{1}{j'+1}$
j'	$\frac{3}{8\pi}$	1	$-\frac{1}{j'(j'+1)}$
$j'-1$	$\frac{3}{8\pi}\frac{1}{2j'+1}$	$j'+1$	$\frac{1}{j'}$

$J\diagdown n$	C.F.	$F_n(J\|2,j'\|\pi/2)$ 0	1	2
$j'+2$	$\frac{15}{64\pi}\frac{1}{(2j'+1)(2j'+3)}$	$3j'^2+5j'+4$	$\frac{2j'^2-6j'-11}{(j'+1)(j'+2)}$	$\frac{3}{(j'+1)(j'+2)}$
$j'+1$	$\frac{15}{32\pi}\frac{1}{2j'+1}$	$j'-1$	$\frac{2j'^2+10j'+3}{j'(j'+1)(j'+2)}$	$\frac{-3}{j'(j'+1)(j'+2)}$
j'	$\frac{15}{32\pi}\frac{1}{(2j'-1)(2j'+3)}$	$\frac{(11j'^2+11j'-18)}{3}$	$\frac{-5(2j'^2+2j'-3)}{j'(j'+1)}$	$\frac{9}{j'(j'+1)}$
$j'-1$	$\frac{15}{32\pi}\frac{1}{2j'+1}$	$j'+2$	$\frac{2j'^2-6j'-5}{j'(j'+1)(j'-1)}$	$\frac{-3}{j'(j'+1)(j'-1)}$
$j'-2$	$\frac{15}{64\pi}\frac{1}{(2j'-1)(2j'+1)}$	$3j'^2+j'+2$	$\frac{2j'^2+10j'-3}{(j'-1)(j')}$	$\frac{3}{(j')(j'-1)}$

total cross-section data, though this will not take care of fluctuations from level to level.

IV. INELASTIC SCATTERING WHEN THE DENSITY OF LEVELS IN THE RESIDUAL NUCLEUS IS HIGH

In this section the statistical hypothesis will be assumed for the residual as well as compound nucleus. Nuclear levels of all types are then so dense that they may be considered to form a continuous distribution. It is expected that this description will be valid for the heavier nuclei or for the middle-weight nuclei if the incident neutron energy is sufficiently great.

Let then the number of levels having an energy between $(E-E')$ (this is just the excitation energy) and $(E-E')+d(E-E')$ be

$$n(E-E')d(E-E') = [(2j'+1)/D_R(E-E')]d(E-E'), \quad (22)$$

where $D_R(E-E')$ is the distance between levels in the residual nucleus. The factor $(2j'+1)$ implies that there are $(2j'+1)^2$ levels at the energy $(E-E')$ with channel spin j'. One factor of $(2j'+1)$ is already contained in the sum over m' in Eqs. (6) and (7). We now pause to justify this factor, which was also assumed by Wolfenstein.[1]

The motion of any ensemble of particles may be broken up into three parts: motion of the center of mass, rotation about the center of mass, and "vibration" relative to the center of mass. The rotation about the center of mass is just that of a rigid body[10] and carries the angular momentum of the system. The specification of this angular momentum requires three quantum numbers; the total angular momentum j', the projection of the angular momentum along both the figure axis of the system, and an arbitrary z axis which in the present problem is taken to be the direction of the incident particle. Hence a state of a given angular momentum has a $(2j'+1)^2$ degeneracy, since each of the projections has $(2j'+1)$ possibilities.

Level density formula (22) leads rigorously to the conclusion that the angular distribution of the emergent neutrons is isotropic. This may be seen as follows.[11] The final l' is formed by adding j' to a fixed J. In order for l' to have all possible orientations it is necessary for the possible directions of j' to cover the complete sphere with j' as radius. Since every vector j' corresponds to a state of the final system with a given value of j', the

[10] E. Kemble, *Fundamental Principles of Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1937).

[11] We are indebted to V. F. Weisskopf for this discussion.

number of these states is quadratic in j'. A proof of the angular isotropy has been given by Wolfenstein[1] but his proof is not as complete as the one that follows.

We now insert expression (22) into the general formula (7). We shall obtain the energy distribution of the emergent neutrons as well as the isotropic angular distribution. The cross section $\sigma(E|E',\vartheta)$ for inelastic scattering with the emergent neutron having an energy between E' and $E'+dE'$ and a direction ϑ in solid angle $d\Omega$ is

$$\sigma(E|E',\vartheta)dE'd\Omega$$
$$=\frac{\pi\lambda\!\!\!^{-2}}{2(2i+1)}\sum_{l,l',i',j}(2l+1)T_l(E)T_{l'}(E')$$
$$\times\sum_{j'J}\frac{(2j'+1)A_J(l,j|l',j'|\vartheta)}{\sum_{pqr}T_p(E_{q'})}\frac{dE'd\Omega}{D_r(E-E')}.$$

The sum[12] over j' may be performed through the use of a relation due to Racah:[7]

$$\frac{|(lj;m_l,mj|lj;J,m_l+m_j)|^2}{2J+1}$$
$$=\frac{|(lj;m_l,-(m_l+m_j)|lJ;j,-m_j)|^2}{2j+1}. \quad (23)$$

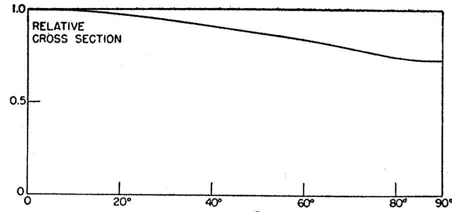

FIG. 2. Angular distribution of inelastically scattered neutrons by a target nucleus of zero spin. The excited level is assumed to have a spin of one opposite parity and an excitation energy of 0.5 Mev, while the incident neutrons have an energy of 1.0 Mev.

Therefore

$$\sum_{j'}(2j'+1)A_J(l,j|l',j'|\vartheta)$$
$$=\sum_{j',m,m'}(2j'+1)|(lj;0m|lj;Jm)|^2$$
$$\times|(l'j';Jm|l'j';m',m-m')|^2|Y_{l'm'}|^2$$
$$=(2J+1)\sum_{m,m'}|(lj;0m|lj;Jm)|^2|Y_{l'm'}|^2$$
$$\times\sum_{j'}|(l'J;m',-m|l'J;j',m'-m)|^2$$
$$=(2J+1)\sum_{m,m'}|(lj;0m|lj;Jm)|^2|Y_{l'm'}|^2.$$

Hence

$$\sum_{j'}(2j'+1)A_J(l,j|l',j'|\vartheta)=(2J+1)^2(2l'+1)/[4\pi(2l+1)], \quad (24)$$

demonstrating the isotropy of the angular distribution which follows from the statistical assumption and energy level density (22).

The energy distribution of the emergent neutrons may be readily obtained since in virtue of (24) the sums over j and J may be performed. We find that

$$\sigma(E|E')=\int\sigma(E|E',\vartheta)d\Omega$$
$$=\pi\lambda\!\!\!^{-2}\frac{(\sum_l(2l+1)T_l(E))(\sum_{l'}(2l'+1)T_{l'}(E'))D_R^{-1}(E-E')}{\sum_{l''}(2l''+1)\int_0^E T_{l''}(E'')D_R^{-1}(E-E'')dE''}. \quad (25)$$

Since the cross section for the formation of the compound nucleus is

$$\sigma_c(E)=\pi\lambda\!\!\!^{-2}\sum_l(2l+1)T_l(E),$$

we may rewrite (25) as

$$\sigma(E|E')=\sigma_c(E)\frac{E'\sigma_c(E')D_R^{-1}(E-E')}{\int_0^E E''\sigma_c(E'')D_R^{-1}(E-E'')dE''}. \quad (26)$$

The cross section σ_c as calculated by the schematic theory for nuclear cross sections are given in reference (3). Formula (26) has been discussed in many places in the literature.[2]

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[12] We are indebted to M. Gell-Mann for suggesting the use of Eq. (23).