Neutron-Proton Scattering in the Region 0–5 Mev

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The theoretical cross section for scattering of neutrons by protons has been calculated on the basis of a pure central Yukawa potential with the same range in ^1S and ^3S states, corresponding to meson mass 200 and 300, respectively. The results indicate that the best over-all fit with most recent scattering experiments would be obtained with a meson mass of roughly 350 m_e. Allowing different ranges in singlet and triplet states and making use of the deuteron binding energy, we found a "triplet mass" of 290 and a "singlet mass" of about 380.

A further result of this work is that, at least in the case of Yukawa potential, the "effective range" approximation is prone to give the total cross section much more accurately than the singlet and triplet cross sections separately. Accordingly, this approximation ought to be handled with some care in other problems than n–p scattering.

THE present work aims at an accurate theoretical calculation of the neutron-proton scattering in the energy region up to 5 Mev where influence from P and higher states can still be neglected.[1]

We assume a pure central Yukawa potential with the same range in the ^3S as well as in the ^1S state, thus

$$\left.\begin{aligned} V_{^1S}(r) &= -Ae^{-\kappa r}/r, \\ V_{^3S}(r) &= -Be^{-\kappa r}/r, \end{aligned}\quad \kappa=\mu mc/\hbar.\right\} \tag{1}$$

To begin with we take singlet and triplet ranges equal, simply because it is a natural assumption if we want to retain the connection between meson theory and nuclear force theory.

The calculations were performed with two different values of the meson mass μm: 200 and 300 m (m=electron mass). In each case the constants A and B were determined from the singlet and triplet scattering amplitudes, a_s and a_t, respectively. For these quantities we have the well-known equations

$$\tfrac{3}{2}a_t+\tfrac{1}{2}a_s=f,\quad 3\pi a_t^2+\pi a_s^2=\sigma_0, \tag{2}$$

where σ_0 is the epithermal neutron-proton scattering cross section and f the coherent scattering amplitude. Starting from

$$\begin{aligned} f&=-(3.78\pm0.03)\cdot10^{-13}\text{ cm},^{2} \\ \sigma_0&=(20.32\pm0.10)\cdot10^{-24}\text{ cm}^2,^{3} \end{aligned} \tag{3}$$

and remembering that a_t must be positive (triplet interaction corresponding to a bound state), we obtain

$$\begin{aligned} a_t&=(5.37\pm0.03)\cdot10^{-13}\text{ cm}, \\ a_s&=(-23.67\pm0.06)\cdot10^{-13}\text{ cm}. \end{aligned} \tag{4}$$

According to Blatt and Jackson[4] the effective range can be defined as a coefficient in the expansion of $k\cot\eta$ in powers of k, the reduced wave number of the neutron-proton system, η being the asymptotic phase of the singlet or triplet S wave function, thus

$$k\cot\eta=-1/a+\tfrac{1}{2}rk^2+O(k^4), \tag{5}$$

where a is the scattering amplitude. The quantity k is related to the energy E_i of the incident neutrons in the laboratory system through the following formula[5]

$$k=\left(\frac{M_pE_i}{2\hbar^2}\right)^{\frac{1}{2}}=\frac{M_pc}{\hbar}\left(\frac{E_i}{2M_pc^2}\right)^{\frac{1}{2}}, \tag{6}$$

(M_p=proton mass).

The method of calculating the effective range r_s (singlet) and r_t (triplet) when a_s and a_t are known will be outlined below. The results in the present case are given in Table I. Accepting now the effective range approximation, we get the following expression for the total cross section:

$$\sigma=\tfrac{1}{4}\sigma_s+\tfrac{3}{4}\sigma_t, \tag{7}$$

with

$$\sigma_i=\frac{4\pi a_i^2}{1+(a_i^2-a_ir_i)k^2+\frac{1}{4}a_i^2r_i^2k^4}. \tag{8}$$

TABLE I. Scattering amplitude a from (4). Calculated effective range r (Yukawa potential).

State	Meson mass assumed	$a\cdot10^{13}$ cm	$r\cdot10^{13}$ cm
Singlet	200	−23.67	4.717
Singlet	300	−23.67	3.006
Triplet	200	5.37	1.759
Triplet	300	5.37	1.599

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[1] See, however, L. Hulthén and A. Pais, Physical Society of London, Cambridge Conference Report (1947), p. 177, Table II. For meson mass 200 the P states may contribute to the total cross section by 3–5 percent at 5 Mev, depending on the P interaction assumed. On the other hand, the contribution does not exceed 5 parts in a thousand for meson mass 300.

[2] Ringo, Burgy, and Hughes, Phys. Rev. **82**, 344 (1951).

[3] This figure differs slightly from the value given by E. Melkonian [Phys. Rev. **76**, 1744 (1949)] (20.36±0.10) which, according to private information kindly given by Professor Rainwater, is still the best value.

[4] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).

[5] Present best values give $k^2=1.2051\cdot10^{24}\,E_i$(Mev) cm^{-2}.

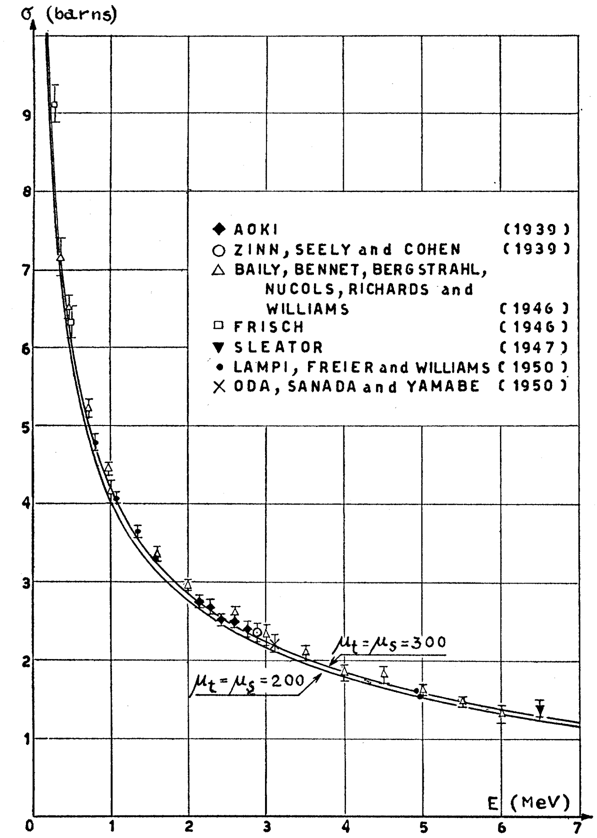

FIG. 1. Neutron-proton scattering cross section based on Yukawa potential with same range in ^1S and ^3S corresponding to meson masses 200 and 300, plotted against incident neutron energy E and compared with various measurements.

The results obtained in this way are compared with experimental results[6–12] in Fig. 1.

THE MATHEMATICAL METHOD

We start by recalling the main features of the variational method as formulated in a previous paper.[13] Take the equation

$$\{d^2/dx^2+q^2+v(x)\}\phi(x)=0, \tag{9}$$

where $v(x)$ is supposed to be a short-range potential, for instance, be^{-x}/x. The boundary conditions are

$$\phi(0)=0, \quad \phi(x)\rightarrow\cot\eta \sin qx+\cos qx \text{ as } x\rightarrow\infty, \tag{10}$$

η being the unknown asymptotic phase. Writing $\phi(x)$ in the form

$$\phi(x)=\cot\eta \sin qx+\cos qx-y(x), \tag{11}$$

we have for $y(x)$ the following boundary conditions:

$$y(0)=1, \quad y(\infty)=0. \tag{12}$$

Inserting (11) into (9) we obtain

$$y''+q^2y+v(x)y=v(x)(\cos qx+\cot\eta \sin qx). \tag{13}$$

This is the Euler equation of the following variational problem. Require

$$J=\int_0^\infty[-y'^2+q^2y^2+v(x)\,(\cos qx-y)^2]dx \tag{14}$$

to be stationary under the accessory condition

$$N=\int_0^\infty v(x)\sin qx(\cos qx-y)dx=\text{const.} \tag{15}$$

If we introduce a multiplier λ, we get a variational equation

$$\delta J+2\lambda\delta N=0, \tag{16}$$

which leads to Eq. (13), with λ for $\cot\eta$. Furthermore, it is easy to show[13] that the integral in (15) can be expressed in terms of η if y is an exact solution of (13); we have

$$N=q(1-\eta_B\cot\eta), \tag{17}$$

where

$$\eta_B=\frac{1}{q}\int_0^\infty v(x)\sin^2 qx dx, \tag{18}$$

is the eigenphase in Born's approximation. Regarding $y(x)$ as a trial function which depends linearly on a number of indeterminate parameters, we obtain from (16) and (17) a unique approximate solution, if λ is identified with $\cot\eta$. In this way we get a value for $\lambda=\cot\eta$ which is not stationary; this is however easily remedied, as will be shown presently.[14]

Define

$$\mathfrak{L}=\int_0^\infty\phi\left(\frac{d^2}{dx^2}+q^2+v(x)\right)\phi dx, \tag{19}$$

and insert here Eq. (11) for $\phi(x)$, and we obtain after partial integrations with due regard to the boundary conditions (12),

$$\mathfrak{L}=-q\lambda+q\eta_B\lambda^2+J+2\lambda N, \quad (\lambda=\cot\eta). \tag{20}$$

On an infinitesimal variation of ϕ we get from (19) after partial integrations

$$\delta\mathfrak{L}=2\int_0^\infty\delta\phi\left(\frac{d^2}{dx^2}+q^2+v(x)\right)\phi dx+q\delta\lambda. \tag{21}$$

For the exact solution of (9) the integral in (21) obviously vanishes. We now fix an approximate solution by requiring this integral to vanish for all possible $\delta\phi$, thus (compare Zeilon[15])

$$\int_0^\infty\delta\phi\left(\frac{d^2}{dx^2}+q^2+v(x)\right)\phi dx=0. \tag{22}$$

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6 H. Aoki, Proc. Phys. Math. Soc. Japan **21**, 232 (1939).
7 Zinn, Seely, and Cohen, Phys. Rev. **56**, 260 (1939).
8 Bailey, Bennet, Bergstrahl, Nuckols, Richards, and Williams, Phys. Rev. **70**, 583 (1946).
9 D. H. Frisch, Phys. Rev. **70**, 589 (1946).
10 W. Sleator, Jr., Phys. Rev. **72**, 207 (1947).
11 Lampi, Freier, and Williams, Phys. Rev. **80**, 853 (1950).
12 Oda, Sanada, and Yamabe, Phys. Rev. **80**, 469 (1950).
13 L. Hulthén, Arkiv. Mat. Astron. Fysik. **A35**, No. 25, 1 (1948). See also the original paper, K. Fysiogr. Sällsk. Lund, Förh. **14**, 1 (1944).
14 See reference 13, pp. 7–9.
15 N. Zeilon, Lunds Univ. Årsskr. Avd. 2 (C. W. K. Gleerup, Lund, 1947), New Series, Sec. 2, Vol. 43, Nr. 10.

Then we obtain the equation

$$\delta\mathfrak{L}=q\delta\lambda, \tag{23}$$

where $\delta\mathfrak{L}$ and $\delta\lambda$ denote variations out from the approximate solution. But J and N in (20) do not depend on λ. Hence (23) can be split up in two equations; the first one is (16), the second one,

$$\partial\mathfrak{L}/\partial\lambda=q,^{16} \tag{24}$$

is easily found to be identical with (17) (with $\cot\eta=\lambda$). Thus the condition (22) is equivalent with the formalism quoted in the beginning of this section.

The λ-value obtained in this way is not stationary. However, the first-order deviation from the stationary value can be obtained from Eq. (23), letting now $\delta\mathfrak{L}$ and $\delta\lambda$ denote deviations from the exact solution of (9) and (10). Neglecting second-order terms, we have $\delta\mathfrak{L}=\mathfrak{L}$, since $\mathfrak{L}=0$ for the exact solution, and thus obtain, by using (17) and (20),

$$\delta\lambda=\mathfrak{L}/q=(J+\lambda N)/q=\Delta/q, \tag{25}$$

J, λ, and N being fixed by the approximate solution determined from (16) and (17). The improved λ-value is then

$$\lambda_0=\lambda-\Delta/q, \tag{26}$$

which is stationary because the first-order deviation has been removed, the remaining error in λ_0 being of the second order (or higher) in δy.[17]

Taking for $y(x)$ in (11) a trial function containing a number of indeterminate parameters $c_1\cdots c_n$, we get from (16) and (17)

$$\partial J/\partial c_\nu+2\lambda\partial N/\partial c_\nu=0,\quad \nu=\ ,2,3,\cdots,n, \tag{27a}$$

$$N=q(1-\eta_B) \tag{27b}$$

which determine the approximate solution. If $y(x)$ depends linearly on the parameters c_ν, Eqs. (27) are linear in the c_ν's and λ, and the solution is unique.

In the numerical calculation of Δ from (25) it is convenient to make direct use of (27). With a trial function depending linearly on the parameters c_ν, we can write

$$\left.\begin{aligned} J&=J^{(0)}+\sum_{\nu=1}^{n}\beta_\nu c_\nu+\tfrac{1}{2}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}c_\mu c_\nu,\\ N&=N^{(0)}+\sum_{\nu=1}^{n}\gamma_\nu c_\nu,\quad \alpha_{\mu\nu}=\alpha_{\nu\mu},\end{aligned}\right\} \tag{28}$$

[16] See S. S. Huang, Phys. Rev. **76**, 1878 (1949) and L. Hulthén and P. O. Olsson, Phys. Rev. **79**, 532 (1950). The identity of (24) with (17) was first pointed out to us by Dr. P. O. Olsson and Dr. B. C. H. Nagel (unpublished manuscript). See also T. Kato, Phys. Rev. **80**, 475 (1950).

[17] See reference 13, original paper, pp. 4–6. An explicit stationary expression for λ can be obtained by putting (20) = 0 and solving with respect to λ. We then have

$$\lambda=-\frac{1}{q\eta_B}\left(N-\frac{q}{2}-[(N-\tfrac{1}{2}q)^2-q\eta_B J]^{\frac{1}{2}}\right), \tag{20a}$$

the sign of the square root being fixed for instance by the condition that λ must approach $1/\eta_B$ (Born approximation) when the potential $v(x)$ decreases sufficiently ($N\ll q$). Requiring λ to be stationary, we obtain an equation (system) equivalent to (16), which together with (20a) determines an approximate solution. This is virtually the same method as that described in the original paper (1944).

and Eqs. (27) become

$$\left.\begin{aligned}&\sum_{\nu=1}^{n}\alpha_{\mu\nu}c_\nu+2\lambda\gamma_\mu=-\beta_\mu,\quad \mu=1,2,\cdots,n\\ &\sum_{\nu=1}^{n}\gamma_\nu c_\nu+q\eta_B\lambda=q-N^{(0)}.\end{aligned}\right\} \tag{29}$$

Thus the quadratic terms can be eliminated from $J+\lambda N$, the result being

$$\Delta=J+\lambda N=J^{(0)}+\lambda N^{(0)}+\tfrac{1}{2}\sum_{\nu=1}^{n}\beta_\nu c_\nu. \tag{30}$$

Turning now to the Yukawa potential $v(x)=be^{-x}/x$, we have the equation

$$(d^2/dx^2+q^2+be^{-x}/x)\phi(x)=0, \tag{31}$$

where

$$q^2=ME/\hbar^2\kappa^2=k^2/\kappa^2,\quad \kappa=\mu mc/\hbar.^{18} \tag{32}$$

$M/2$ is the reduced mass of the neutron-proton system and E the total energy in the center-of-mass system [see Eq. (6)]. μm and m are the masses of meson and electron, respectively.

Comparison with (1) gives

$$b=\begin{cases}{}^1b=MA/\kappa\hbar^2,\\ {}^3b=MB/\kappa\hbar^2,\end{cases}\quad M=\tfrac{1}{2}(M_n+M_p). \tag{33}$$

As a trial function for $y(x)$ in (11) we choose the following expression, which satisfies the boundary conditions (12)

$$y(x)=e^{-x}+(1-e^{-x})\sum_{1}^{n}c_\nu e^{-\nu x}. \tag{34}$$

Inserting this into (14), (15), and (18), putting $v(x)=be^{-x}/x$, we obtain with the notations (28)

$$\left.\begin{aligned}J^{(0)}&=-\tfrac{1}{2}(1-q^2)+b[\ln\{(4+q^2)/3\}-\tfrac{1}{4}\ln(1+4q^2)],\\ \beta_\nu&=\frac{2(1+q^2)}{(\nu+1)(\nu+2)}-b\ln\frac{(\nu+2)^2((\nu+2)^2+q^2)}{(\nu+3)^2((\nu+1)^2+q^2)},\\ \tfrac{1}{2}\alpha_{\mu\nu}&=\frac{2q^2-2\mu\nu-\mu-\nu}{(\mu+\nu)(\mu+\nu+1)(\mu+\nu+2)}+b\ln\frac{(\mu+\nu+2)^2}{(\mu+\nu+1)(\mu+\nu+3)},\\ N^{(0)}&=b(\tfrac{1}{2}\operatorname{arctg}2q-\operatorname{arctg}\tfrac{1}{2}q),\\ \gamma_\nu&=-b\left(\operatorname{arctg}\frac{q}{\nu+1}-\operatorname{arctg}\frac{q}{\nu+2}\right),\\ \eta_B&=(b/4q)\ln(1+4q^2).\end{aligned}\right\} \tag{35}$$

[18] Present best values give $1/\kappa=(386.12/\mu)\cdot10^{-13}$ cm.

TABLE II. Approximate solutions of (31) from Eqs. (27a, b) and (26), with trial functions according to (11) and (34). $b=1.5$.

q	$-c_1$	$+c_2$	$-c_3$	λ	Δ/q	λ_0	$\eta=\text{arc cot}\lambda_0$
0.1	0.4752	···	···	1.02192	0.03742	0.98450	0.79321
	0.5246	0.0720	···	0.99936	0.01569	0.98367	0.79363
	0.5643	0.2076	0.1104	0.99084	0.00738	0.98346	0.79374
0.2	0.5090	···	···	0.69803	0.02616	0.67187	0.97920
	0.5731	0.0931	···	0.68249	0.01130	0.67119	0.97967
	0.6248	0.2679	0.1417	0.67655	0.00553	0.67102	0.97978
0.3	0.5596	···	···	0.66521	0.02564	0.63957	1.00179
	0.6451	0.1229	···	0.65014	0.01133	0.63881	1.00233
0.4	0.6211	···	···	0.69873	0.02786	0.67087	0.97989
	0.7313	0.1561	···	0.68242	0.01243	0.66999	0.98050
0.5	0.6880	···	···	0.75417	0.03143	0.72274	0.94497
	0.8231	0.1869	···	0.73596	0.01416	0.72180	0.94559
0.6	0.7561	···	···	0.81752	0.03600	0.78152	0.90743
	0.9147	0.2126	···	0.79689	0.01629	0.78060	0.90800
	1.0213	0.5312	0.2424	0.78919	0.00872	0.78047	0.90808
0.7	0.8220	···	···	0.88373	0.04169	0.84204	0.87094
	1.0012	0.2300	···	0.86021	0.01896	0.84125	0.87140
	1.1184	0.5529	0.2347	0.85089	0.00972	0.84117	0.87145
0.8	0.8829	···	···	0.95099	0.04878	0.90221	0.83676
	1.0783	0.2360	···	0.92419	0.02254	0.90165	0.83707
	1.1849	0.4926	0.1706	0.91418	0.01255	0.90163	0.83708
0.9	0.9357	···	···	1.01914	0.05794	0.96120	0.80518
	1.1316	0.2174	···	0.99012	0.02916	0.96096	0.80531
1.0	0.9758	···	···	1.08928	0.07057	1.01871	0.77613
	1.5606	0.5744	···	0.99261	−0.02636	1.01897	0.77600
	1.3199	0.3733	0.0447	1.03386$_5$	0.01494$_6$	1.01891$_9$	0.77602$_8$

Equations (29) are then solved for instance by Gauss' method, and the improved value of $\cot\eta$, λ_0, is obtained from (26) and (30).

Numerical results have already been given in reference 13 for $b=1.5$ and 1.6. These results are quoted in our Tables II and IV. In Tables III, V–XI, the calculations have been extended to some other b-values of immediate interest. A comparison between the results obtained with one and two parameters, respectively, gives an idea of the accuracy of the corrected phase value η.[19] Moreover, the eigenfunctions were checked by means of certain integral identities;[13,20] with two parameters the deviations were about five parts in a thousand or less. This seems satisfactory since the uncertainty in the corrected phase η is of the second order, errors of first order being removed by the procedure comprised in Eq. (26).[21]

TABLE III. Approximate solutions of (31) from Eqs. (27a, b) and (26), with trial functions according to (11) and (34). $b=1.55$.

q	$-c_1$	$+c_2$	λ	Δ/q	λ_0	$\eta=\text{arc cot}\lambda_0$
0.1	0.4988	···	0.76260	0.03754	0.72506	0.94345
	0.5478	0.0715	0.74029	0.01605	0.72424	0.94399
0.2	0.5329	···	0.56083	0.02601	0.53482	1.07968
	0.5963	0.0922	0.54553	0.01137	0.53416	1.08020
0.3	0.5839	···	0.56575	0.02530	0.54045	1.07531
	0.6680	0.1210	0.55096	0.01125	0.53971	1.07589
0.4	0.6459	···	0.61619	0.02732	0.58887	1.03860
	0.7538	0.1530	0.60028	0.01225	0.58803	1.03922
0.5	0.7134	···	0.68042	0.03064	0.64978	0.99458
	0.8454	0.1829	0.66268	0.01380	0.64888	0.99521
0.6	0.7822	···	0.74865	0.03492	0.71373	0.95091
	0.9371	0.2079	0.72854	0.01569	0.71285	0.95150

TABLE IV. Approximate solutions of (31) from Eqs. (27a, b) and (26), with trial functions according to (11) and (34). $b=1.6$.

q	$-c_1$	$+c_2$	$-c_3$	λ	Δ/q	λ_0	$\eta=\text{arc cot}\lambda_0$
0.1	0.5232	···	···	0.51566	0.03755	0.47811	1.12482
	0.5716	0.0708	···	0.49368	0.01638	0.47730	1.12547
0.2	0.5575	···	···	0.43018	0.02580	0.40438	1.18652
	0.6199	0.0908	···	0.41517	0.01143	0.40374	1.18707
0.3	0.6089	···	···	0.47103	0.02491	0.44612	1.15117
	0.6913	0.1188	···	0.45660	0.01119	0.44541	1.15176
0.4	0.6714	···	···	0.53760	0.02672	0.51088	1.09848
	0.7769	0.1498	···	0.52209	0.01202	0.51007	1.09912
0.5	0.7395	···	···	0.61023	0.02981	0.58042	1.04490
	0.8683	0.1787	···	0.59297	0.01341	0.57956	1.04554
0.6	0.8090	···	···	0.68309	0.03377	0.64932	0.99490
	0.9602	0.2032	···	0.66352	0.01504	0.64848	0.99549
	1.0671	0.5229	0.2434	0.65582	0.00746	0.64836	0.99558
0.7	0.8768	···	···	0.75451	0.03865	0.71586	0.94950
	1.0483	0.2205	···	0.73211	0.01697	0.71514	0.94998
	1.1645	0.5406	0.2326	0.72287	0.00781	0.71506	0.95004
0.8	0.9404	···	···	0.82425	0.04461	0.77964	0.90860
	1.1318	0.2316	···	0.79806	0.01893	0.77913	0.90891
0.9	0.9974	···	···	0.89286	0.05213	0.84073	0.87171
	1.2147	0.2412	···	0.86069	0.02023	0.84046	0.87187
1.0	1.0445	···	···	0.96149	0.06215	0.89934	0.83834$_6$
	1.1526	0.1060	···	0.94359	0.04423	0.89936	0.83833$_5$
	1.3101	0.2356	−0.0314$_5$	0.91648$_6$	0.01715$_6$	0.89933$_0$	0.83835$_2$

TABLE V. Approximate solutions of (31) from Eqs. (27a, b) and (26), with trial function (34) [see (11)]. $b=1.65$.

q	$-c_1$	$+c_2$	λ	Δ/q	λ_0	$\eta=\text{arc cot}\lambda_0$
0.1	···	···	···	···	···	···
	0.5637	0.0224	0.27282	0.03081	0.24200	1.33336
0.2	···	···	···	···	···	···
	0.6441	0.0891	0.29075	0.01151	0.27923	1.29850
0.3	···	···	···	···	···	···
	0.7153	0.1162	0.36653	0.01109	0.35544	1.22816
0.4	···	···	···	···	···	···
	0.8003	0.1459	0.44755	0.01182	0.43573	1.15987
0.5	···	···	···	···	···	···
	0.8915	0.1739	0.52651	0.01301	0.51350	1.09641

TABLE VI. Approximate solutions of (31), with $b=2.1$. For trial function see (11) and (34).

q	$-c_1$	$+c_2$	λ	Δ/q	λ_0	$\eta=\text{arc cot}\lambda_0$
0.1	0.8306	···	−1.49943	+0.04729	−1.54672	2.56766
	0.8666	0.0529	−1.51453	+0.03286	−1.54739	2.56786
0.2	0.8567	···	−0.63878	+0.02020	−0.65897	2.15345
	0.8941	0.0553	−0.64738	+0.01182	−0.65920	2.15361
0.3	0.9098	···	−0.30240	+0.01797	−0.32036	1.88083
	0.9592	0.0723	−0.31066	+0.00995	−0.32061	1.88105
0.4	0.9746	···	−0.10267	+0.01755	−0.12052	1.69074
	1.0372	0.0903	−0.11147	+0.00933	−0.12080	1.69101
0.5	1.0460	···	+0.03968	+0.01833	+0.02135	1.54945
	1.1218	0.1070	+0.02992	+0.00886	+0.02106	1.54974
0.6	1.1204	···	+0.15137	+0.01887	+0.13250	1.43906
	1.2102	0.1225	+0.14014	+0.00793	+0.13221	1.43935

19 In Tables II and IV we also find some three parameter results.

20 L. Hulthén, K. Fysiogr. Sällsk. Lund, Förh. 14, Nr. 8 (1944).

21 In one particular case ($\mu_t=290$) we compared the triplet effective range calculated by the variational method ($r_t=1.618$) with the value obtained from the integral formula (see reference 4)

$$\kappa_t r_t = 2\int_0^\infty \left\{\left(1-\frac{x}{\kappa a_t}\right)^2 - [\phi_t^{(0)}(x)]^2\right\} dx,$$

TABLE VII. Approximate solutions of (31), $b=2.3$. For trial functions see (11), (34).

q	$-c_1$	$+c_2$	λ	Δ/q	λ_0	$\eta=\text{arc cot}\lambda_0$
0.1	0.9771	...	−2.19459	0.02503	−2.21962	2.71830
	0.9882	0.0166	−2.19932	0.02036	−2.21968	2.71831
0.2	1.0117	...	−1.00031	0.01549	−1.01580	2.36403
	1.0285	0.0253	−1.00413	0.01165	−1.01578	2.36402
0.3	1.0636	...	−0.56248	0.01317	−0.57565	2.09312
	1.0887	0.0370	−0.56657	0.00916	−0.57573	2.09318
0.4	1.1271	...	−0.31655	0.01237	−0.32892	1.88857
	1.1604	0.0484	−0.32112	0.00788	−0.32900	1.88864
0.5	1.1976	...	−0.14965	0.01174	−0.16139	1.73081
	1.2393	0.0593	−0.15491	0.00657	−0.16148	1.73089
0.6	1.2722	...	−0.02408	0.01072	−0.03480	1.60558
	1.3238	0.0708	−0.03042	0.00448	−0.03490	1.60568

TABLE VIII. Approximate solutions of (31), $b=2.4$. For trial functions see (11), (34).

q	$-c_1$	$+c_2$	λ	Δ/q	λ_0	$\eta=\text{arc cot}\lambda_0$
0.1	...	...	...	...	...	...
	1.0651	−0.0011	−2.52963	0.02098	−2.55061	2.76795
0.2	...	...	...	...	...	...
	1.1032	+0.0055	−1.17711	0.01163	−1.18874	2.44221
0.3	...	...	...	...	...	...
	1.1596	+0.0141	−0.69005	0.00871	−0.69876	2.18069
0.4	...	...	...	...	...	...
	1.2274	+0.0224	−0.42178	0.00703	−0.42881	1.97589
0.5	...	...	...	...	...	...
	1.3024	+0.0300	−0.24314	0.00531	−0.24845	1.81431

TABLE IX. Approximate solutions of (31), $b=2.5$. For trial functions see (11), (34).

q	$-c_1$	$+c_2$	λ	Δ/q	λ_0	$\eta=\text{arc cot}\lambda_0$
0.1	1.1633	...	−2.86521	0.01513	−2.88034	2.80743
	1.1484	−0.0226	−2.85907	0.02138	−2.88045	2.80744
0.2	1.1954	...	−1.35168	0.00885	−1.36053	2.50776
	1.1834	−0.0181	−1.34908	0.01148	−1.36056	2.50777
0.3	1.2440	...	−0.81369	0.00682	−0.82051	2.25792
	1.2353	−0.0129	−0.81231	0.00818	−0.82049	2.25791
0.4	1.3038	...	−0.52169	0.00540	−0.52709	2.05588
	1.2982	−0.0083	−0.52094	0.00616	−0.52710	2.05589
0.5	1.3713	...	−0.33002	0.00364	−0.33366	1.89284
	1.3688	−0.0036	−0.32971	0.00395	−0.33366	1.89284
0.6	1.4444	...	−0.19027	0.00094	−0.19121	1.75973
	1.4471	+0.0039	−0.19060	0.00061	−0.19121	1.75973

EXPANSION IN k^2. COMPARISON BETWEEN "EXACT" AND "EFFECTIVE RANGE" METHOD

Equation (35) shows that J is an even function of q (or k), whereas N and η_B are odd functions. From (27a) and (27b), which are linear in the parameters c_ν and λ, we then obtain c_ν and $k\lambda$ as even functions of k. The correction Δ in (25) is also an even function, and so becomes $k\lambda_0$. These statements are, in fact, more generally valid and agree with the theory of the "effective range" approximation.[4] For small energies we then put, as in Eq. (5),

$$q\lambda_0=-1/\kappa a+\tfrac{1}{2}\kappa r q^2 \qquad (36)$$

neglecting higher powers than q^2.

where $\phi_t^{(0)}$ is the eigenfunction for zero energy. With two parameters in the trial function $\phi_t^{(0)}$ the result was $r_t=1.642$, which differs from the variational value by 1.5 percent. As a test of the accuracy of the eigenfunction, this must be considered satisfactory since the integrand contains the square of the eigenfunction. It should be remembered that the value of r_t obtained from (37) is a stationary quantity and must therefore be expected to be more accurate than a value obtained from an integral formula without stationary properties. Concerning the accuracy of the variational method see also P.-O. Löwdin and A. Sjölander, Arkiv Fysik. **3**, nr. 11, pp. 155–166 (1951).

TABLE X. Approximate solutions of (31), $b=2.7$. For trial functions see (11), (34).

q	$-c_1$	$+c_2$	λ	Δ/q	λ_0	$\eta=\text{arc cot}\lambda_0$
0.1	1.3894	...	−3.55092	+0.00082	−3.55174	2.86715
	1.3374	−0.0796	−3.53039	+0.02217	−3.55256	2.86721
0.2	1.4165	...	−1.70863	−0.00036	−1.70827	2.61199
	1.3641	−0.0800	−1.69765	+0.01110	−1.70875	2.61211
0.3	1.4538	...	−1.06657	−0.00162	−1.06495	2.38764
	1.4043	−0.0815	−1.05832	+0.00690	−1.06522	2.38776
0.4	1.5109	...	−0.72613	−0.00347	−0.72266	2.19657
	1.4542	−0.0844	−0.71881	+0.00409	−0.72290	2.19673
0.5	1.5721	...	−0.50805	−0.00633	−0.50172	2.03582
	1.5127	−0.0862	−0.50097	−0.00093	−0.50190	2.03596
0.6	1.6411	...	−0.35296	−0.01081	−0.34215	1.90046
	1.5823	−0.0823	−0.34606	−0.00379	−0.34227	1.90057

TABLE XI. Approximate solutions of (31), $b=2.9$. For trial functions see (11), (34).

q	$-c_1$	$+c_2$	λ	Δ/q	λ_0	$\eta=\text{arc cot}\lambda_0$
0.1	1.6697	...	−4.28703	−0.01979	−4.26724	2.91140
	1.5657	−0.1617	−4.24796	+0.02285	−4.27080	2.91159
0.2	1.6877	...	−2.08825	−0.01297	−2.07528	2.69256
	1.5796	−0.1674	−2.06673	+0.01043	−2.07716	2.69291
0.3	1.7173	...	−1.33209	−0.01272	−1.31937	2.49303
	1.6026	−0.1759	−1.31541	+0.00532	−1.32072	2.49352
0.4	1.7571	...	−0.93784	−0.01472	−0.92312	2.31624
	1.6336	−0.1863	−0.92259	+0.00161	−0.92420	2.31682
0.5	1.8072	...	−0.69034	−0.01865	−0.67138	2.16206
	1.6751	−0.1920	−0.67489	−0.00262	−0.67227	2.16267
0.6	1.8681	...	−0.51744	−0.02499	−0.49245	2.02839
	1.7316	−0.1931	−0.50189	−0.00878	−0.49311	2.02892

TABLE XII. Coefficients $a_0(b)$ and $a_2(b)$, defined by (36) and (37), calculated by variational method, using trial function (34) [see (11)].

	a_0			a_2	
b	1 param.	2 param.	3 param.	1 param.	2 param.
1.5	0.086200	0.086135	0.086117	1.23231	1.23074
1.55	...	0.060695	...	...	1.18021
1.6	0.036537	0.036475	0.036455	1.13373	1.13227
1.65	...	0.013346	...	...	1.08670
1.7	−0.008749	−0.008807	...	1.04459	1.04328
1.75	...	−0.030086	...	...	1.00182
1.8	−0.050534	−0.050585	...	0.96330	0.96216
1.9	...	−0.089572	...	...	0.88767
2.0	...	−0.126361	...	...	0.81883
2.1	−0.161439	−0.161460	−0.161491	0.75542	0.75490
2.2	−0.195306	−0.195316	...	0.69564	0.69533
2.25	...	−0.211908	...	...	...
2.3	−0.228337	−0.228339	−0.228377	0.63985	0.63974
2.35	...	−0.244660	...	...	0.61335
2.4	−0.260915	−0.260915	...	0.58783	0.58789
2.5	−0.293412	−0.293420	−0.293465	0.53957	0.53973
2.6	−0.326209	−0.326243	...	0.49522	0.49540
2.7	−0.359706	−0.359794	−0.359846	0.45521	0.45529
2.8	−0.394343	−0.394529	...	0.42026	0.42008
2.9	−0.430625	−0.430975	−0.431033	0.39153	0.39089
3.0	−0.469138	−0.469748	...	0.37083	0.36945

TABLE XIII. Singlet, triplet, and total cross sections (σ_s, σ_t, and σ) in effective range approximation, based on Yukawa potential with range corresponding to meson mass 300 ($\mu_s=\mu_t=300$). Δ is the difference between the "exact"-value (i.e., the value obtained from an accurate phase calculation, without expansion in k^2) and the "effective range" value.

E_l (Mev)	σ_s	Δ_s	σ_t	Δ_t	σ	Δ
0.2	$27.838\cdot 10^{-24}$	$0.007\cdot 10^{-24}$	$3.455\cdot 10^{-24}$	$0.000\cdot 10^{-24}$	$9.551\cdot 10^{-24}$	$0.002\cdot 10^{-24}$
0.6	12.502	0.011	3.158	−0.000	5.494	0.002
1.0	8.007	0.014	2.907	−0.001	4.182	0.002
2.0	4.153	0.017	2.418	−0.004	2.852	0.001
3.0	2.763	0.021	2.063	−0.007	2.238	0.000
4.0	2.048	0.023	1.795	−0.008	1.858	−0.000
5.0	1.613	0.025	1.585	−0.009	1.592	−0.001
6.0	1.322	0.027	1.415	−0.010	1.392	−0.001

By expanding (26), (29), and (30) in powers of q, retaining terms up to q^2 in J and q^3 in N, we obtain through the procedure described in the preceding section the coefficients in (36) as functions of the binding parameter b [see (33)]:

$$-1/\kappa a=a_0(b),\quad \tfrac{1}{2}\kappa r=a_2(b). \tag{37}$$

In this way a_0 and a_2 are turned into stationary quantities. It should, however, be noted that this is true for a_2 only if the parameters c_ν are computed to the order of q^2. In Table XII a_0 and a_2 are given for a succession of b values in the most interesting region.

Using the figures given for a_t and a_s in (4), interpolations in Table XII give us the quantities ${}^1b$, ${}^3b$, r_s, and r_t for different meson masses.

We are now ready to compare the results of the two methods: the "effective range" approximation and the ordinary phase method. In both cases the same trial function (34) with two parameters has been used and the resulting technical errors in the cross sections are believed to be much smaller than the differences between "exact" results and "effective range" values, except at low energies, of course. Examples are found in Tables XIII (meson mass 300), XIV (meson mass 200), and XV, which give the singlet and triplet cross sections separately as well as the total cross sections, all based on the "effective range" approximation. The quantities denoted by Δ are the differences between the σ's calculated directly (i.e., without expansion in powers of k) and the σ's in the table.

Some comments may be called for. We see that the differences in the total cross sections are small, less than one part in a thousand for meson mass 300 and 1 percent for 200. On the other hand, the differences between the singlet cross sections obtained with the two methods run up to 2 percent for meson mass 300 and 7 percent for mass 200. The differences in the triplet cross sections are smaller and have the opposite sign. This fact, in connection with the different weight factors of singlet and triplet states, makes the differences in the total cross sections so small. The conclusion is that the "effective range" approximation is very accurate in calculating the total scattering cross section, the accuracy increasing with decreasing range (i.e., increasing meson mass), as is well known. This accuracy, however, is to a certain extent fortuitous, and therefore it is well advised to use the "effective range" method, even the "shape-dependent" version, with caution in other problems than neutron-proton scattering.

TABLE XIV. Same quantities as in Table XIII, with meson mass 200 ($\mu_s=\mu_t=200$).

E_l (Mev)	σ_s	Δ_s	σ_t	Δ_t	σ	Δ
0.2	$26.694\cdot 10^{-24}$	$0.018\cdot 10^{-24}$	$3.461\cdot 10^{-24}$	$-0.000\cdot 10^{-24}$	$9.270\cdot 10^{-24}$	$0.004\cdot 10^{-24}$
0.6	11.693	0.036	3.175	−0.001	5.304	0.008
1.0	7.372	0.044	2.930	−0.003	4.040	0.009
2.0	3.705	0.058	2.448	−0.006	2.762	0.010
3.0	2.398	0.066	2.094	−0.009	2.170	0.010
4.0	1.733	0.071	1.824	−0.011	1.801	0.010
5.0	1.333	0.074	1.611	−0.012	1.542	0.010
6.0	1.069	0.076	1.439	−0.012	1.346	0.010

TABLE XV. Same quantities as in Table XIII, "triplet meson mass" $\mu_t=290$ and "singlet meson mass" $\mu_s=380$.

E_l	σ_s	Δ_s	σ_t	Δ_t	σ	Δ
1	$8.258\cdot 10^{-24}$	$0.007\cdot 10^{-24}$	$2.909\cdot 10^{-24}$	$-0.001\cdot 10^{-24}$	$4.246\cdot 10^{-24}$	$+0.001\cdot 10^{-24}$
2	4.327	0.009	2.421	−0.004	2.898	−0.001
3	2.905	0.010	2.067	−0.007	2.276	−0.003
4	2.171	0.011	1.798	−0.009	1.891	−0.004
5	1.724	0.012	1.587	−0.010	1.622	−0.004
6	1.423	0.014	1.418	−0.010	1.419	−0.004

DISCUSSION

Figure 1 shows that a meson mass 300 gives a better agreement with the experimental results than meson mass 200, but it is not possible to obtain a very good over-all fit with a meson mass between 200 and 300. A rough estimate, based on a comparison with the most recent and probably most accurate measurements, those of Lampi, Freier, and Williams,[11] indicates that a meson mass of about 350 would give the best agreement with present experimental knowledge. This value is rather far from the π-meson mass, 276 m_e, and it would of course be interesting to see what meson mass would come out if noncentral forces are taken into account with the proper distance dependence. Such calculations are in progress at this institute.

So far the calculations were based on low energy scattering data exclusively [see (3)]. Taking now the deuteron binding energy E_0 into account, we have a possibility to determine the triplet range, and thus the corresponding meson mass. With $|E_0|=2.226\pm 0.003$ Mev[22] and $a_t=(5.37\pm 0.03)\cdot 10^{-13}$ cm [see (4)], we obtain, making use of Table I[23] in the paper of Hulthén and Laurikainen,[24] combined with Table XII above,

$$\mu_t=290\pm 15. \qquad (38)$$

Having thus fixed a "triplet mass," we look at Fig. 1 and find again that we should not get a very close fit with recent scattering data by assuming the same "singlet mass." In fact, to obtain a good agreement with the results of Lampi, Freier, and Williams, we must choose

$$\mu_s=380\pm 50,^{25} \qquad (39)$$

the cross sections being rather insensitive to variations of the singlet mass. The results obtained with $\mu_t=290$ and $\mu_s=380$ are indicated in Table XV and compared with experimental results in Fig. 2. The corresponding effective ranges are, in units of 10^{-13} cm,

$$r_t=1.62\pm 0.03, \quad r_s=2.33\pm 0.35. \qquad (40)$$

In this connection it might be of interest to mention

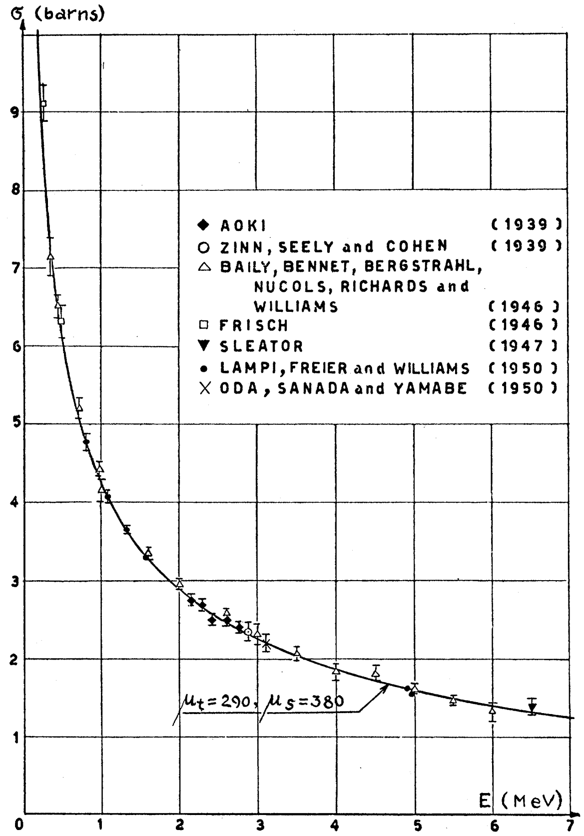

FIG. 2. Neutron-proton scattering based on Yukawa potential with "triplet mass" 290 and "singlet mass" 380 (see Fig. 1).

the results obtained by P. O. Brundell and B. Enander.[26] Using an exponential potential with the same range (real, not "effective") in singlet and triplet states and determining the parameters (range, triplet and singlet binding constants) from the low energy scattering data and the deuteron binding energy, they get a perfect agreement with the results of Lampi, Freier, and Williams. Thus in the case of an exponential potential, all the experimental data in question can be accounted for by three constants only, instead of four with other known potential forms, including Yukawa, as we have seen above.

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[22] R. C. Mobley and R. A. Laubenstein, Phys. Rev. 80, 309 (1950).

[23] With $a=ME_0/\hbar^2\kappa^2$, we get with present best values $(-a)^{\frac{1}{2}}=89.46_7/\mu$.

[24] L. Hulthén and K. V. Laurikainen, Revs. Modern Phys. 23, 1 (1951).

[25] The values (38) and (39) do not quite agree with the results of E. E. Salpeter, Phys. Rev. 82, 60 (1951), Eqs. (12) and (14). Part of the difference is probably due to the fact that the underlying experimental data are not identical (compare Salpeter's Eqs. (5) and (6) with Eq. (3) above). A more detailed discussion of the shape-dependent effective range theory is deferred to a forthcoming paper on the photodisintegration of the deuteron by L. Hulthén and B. C. H. Nagel.

[26] P. O. Brundell and B. Enander, to be published.