Differential Cross Section for Bremsstrahlung and Pair Production

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WE have calculated the cross section for bremsstrahlung and pair production in the limit of high relativistic energies without the use of the Born approximation. The initial and final states of the electron are described by the wave function of Furry,[1] and Sommerfeld and Maue,[2] which is also identical with the first two terms of that of Bess.[3,4] The exact expansion in spherical harmonics of Mott and Massey[5] goes over into our wave function by neglecting $a^2=(Ze^2/\hbar c)^2$ compared with l^2, where l is the angular momentum.

A major part of our calculation consists in the demonstration that this wave function is accurate enough for the calculation of the differential cross section. This was found to be true for energies $\epsilon\gg mc^2$ and scattering angles θ of the order of mc^2/ϵ, which are the only ones contributing significantly to the total cross section. The final electron energy must also be large.

The integration of the matrix element was performed by a method due to Nordsieck.[6] We denote the initial electron energy by ϵ_1, the final one by ϵ_2, their momenta by $\mathbf{p}_1$ and $\mathbf{p}_2$, the quantum energy by k, the rest energy of the electron by μ, the angles between the quantum momentum $\mathbf{k}$ and the two electron momenta by θ_1 and θ_2, the azimuth between them by ϕ, the momentum change by $\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2-\mathbf{k}$, and the quantity $Ze^2/\hbar c$ by a. We further introduce the abbreviation

$$X=1-\frac{q^2(p_1-p_2-k)(p_1-p_2+k)}{4k^2(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\approx 1-\frac{q^2\mu^2}{(\epsilon_1^2\theta_1^2+\mu^2)(\epsilon_2^2\theta_2^2+\mu^2)},\quad (1)$$

the last expression being valid for large ϵ and for θ of order μ/ϵ or less. X varies from 0 to 1. We use the hypergeometric functions

$$V(X)=F(ia,\,-ia;\,1,\,X),\qquad W(X)=a^{-2}dV/dX=F(1+ia,\,1-ia;\,2,\,X),\quad (2)$$

which are clearly real.

Then the differential cross section for bremsstrahlung is

$$\begin{aligned}d\sigma=&\left\{\frac{\pi a}{\sinh\pi a}\right\}^2\frac{a^2}{2\pi}\left(\frac{\hbar}{mc}\right)^2\mu^2\frac{e^2}{\hbar c}\frac{p_2}{p_1}\frac{dk}{k}\sin\theta_1\sin\theta_2 d\theta_1 d\theta_2 d\phi\\ &\times\left\{\frac{V^2(X)}{q^4}\left[\frac{p_1^2\sin^2\theta_1(4\epsilon_2^2-q^2)}{(\epsilon_1-p_1\cos\theta_1)^2}+\frac{p_2^2\sin^2\theta_2(4\epsilon_1^2-q^2)}{(\epsilon_2-p_2\cos\theta_2)^2}\right.\right.\\ &-\frac{(4\epsilon_1\epsilon_2-q^2+2k^2)2p_1p_2\sin\theta_1\sin\theta_2\cos\phi}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\\ &\left.+2k^2\frac{(p_1^2\sin^2\theta_1+p_2^2\sin^2\theta_2)}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\right]\\ &+\frac{a^2[k^2-(p_1-p_2)^2]^2W^2(X)}{(4k^2)^2(\epsilon_1-p_1\cos\theta_1)^2(\epsilon_2-p_2\cos\theta_2)^2}\\ &\times\left[\frac{p_1^2\sin^2\theta_1(4\epsilon_2^2-q^2)}{(\epsilon_1-p_1\cos\theta_1)^2}+\frac{p_2^2\sin^2\theta_2(4\epsilon_1^2-q^2)}{(\epsilon_2-p_2\cos\theta_2)^2}\right.\\ &+\frac{(4\epsilon_1\epsilon_2-q^2+2k^2)2p_1p_2\sin\theta_1\sin\theta_2\cos\phi}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\\ &-2k^2\frac{(p_1^2\sin^2\theta_1+p_2^2\sin^2\theta_2)}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\\ &\left.\left.+\frac{4k^2}{\mu^2}(\epsilon_1\epsilon_2+p_1p_2\cos\theta_1\cos\theta_2)\right]\right\}.\quad (3)\end{aligned}$$

The first square bracket here is identical with the corresponding expression in the theory of Bethe and Heitler which is based on the Born approximation. It is multiplied by $(\pi a/\sinh\pi a)^2V^2(X)$. It can be shown that for $X=1$, which corresponds to small q, this factor is unity; for larger q it is less. The second square bracket, which is multiplied by $W^2(X)$, is new; the slight difference in structure between the two terms is real. For small q, the term with W^2 is negligible. Therefore, in this limit, in which screening may be important, there is no correction to the Born approximation formula.

For pair production, we must substitute $p_1p_2d\epsilon_1/k^3$ for p_2dk/p_1k, and change the sign of ϵ_2 and p_2 everywhere. Then ϵ_2 is the positron energy and θ_2 the angle between positron and quantum. In contrast to the customary Bethe-Heitler theory, θ_2 should not be changed into $\pi-\theta_2$; this substitution is replaced by the change of sign of p_2.

A more complete account of this calculation will be published later.

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[1] W. H. Furry, Phys. Rev. **46**, 391 (1934).
[2] A. Sommerfeld and A. W. Maue, Ann. Physik **22**, 629 (1935).
[3] L. Bess, Phys. Rev. **77**, 550 (1950), Eq. (4).
[4] The last term in Bess's wave function is wrong, as are therefore all the further calculations in his paper. We are indebted to Professor A. T. Nordsieck for pointing out this fact to us.
[5] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), second edition, Chap. IV, Eq. (39).
[6] See reference 3, and private communication for which we are indebted to Professor Nordsieck.

Integral Cross Section for Bremsstrahlung and Pair Production

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WE have integrated the differential cross section of the preceding paper over angles. Since the differential cross section contains complicated (hypergeometric) functions of a certain variable X, Eq. (1) of the preceding paper, the integration over this variable must be left to the end. This can be done by a method similar to that used earlier by one of the authors.[1]

The independent variables used over most of the region of integration are

$$\xi=\mu^2/(\epsilon_1^2\theta_1^2+\mu^2),\quad \eta=\mu^2/(\epsilon_2^2\theta_2^2+\mu^2),\quad (1)$$

and X. Analytic integration over ξ and η proves possible. For $\theta\gg\mu/\epsilon$, the contribution to the total cross section is negligible and is left out. For very small q, i.e., X near 1, the differential cross section reduces to the Born approximation and therefore the integration can be done as in reference 1, including the case of screening.

After integration over ξ and η, the total cross section reduces to the surprisingly simple form (in the case of no screening)

$$\begin{aligned}\sigma=\frac{2}{3}\frac{e^2}{\hbar c}\left(\frac{\hbar c}{\mu}\right)^2\frac{a^2}{\epsilon_1^2}\frac{dk}{k}(3\epsilon_1^2+3\epsilon_2^2-2\epsilon_1\epsilon_2)\left\{2\log\frac{2\epsilon_1\epsilon_2}{\mu k}-1\right.\\ +\left(\frac{\sinh\pi a}{\pi a}\right)^2a^2\left[2\int_0^1V(X)W(X)\log(1-X)dX\right.\\ \left.\left.+\int_0^1XW^2(X)dX\right]\right\}.\quad (2)\end{aligned}$$

The first two terms in the curly bracket are the result of Bethe and Heitler; the last is the correction to the Born approximation. Both of the integrals over X converge at both limits. For pair production, the sign of the term $2\epsilon_1\epsilon_2$ in the first set of parentheses must be changed, and dk/ϵ_1^2 must be replaced by $d\epsilon_1/k^2$.

For small a (small nuclear charge), the integrals over X can be carried out analytically and the curly bracket in (2) becomes

$$\{\ \}=2\log(2\epsilon_1\epsilon_2/\mu k)-1-2.414a^2,\quad (3)$$

where 2.414 stands for $2(1+2^{-3}+3^{-3}+\cdots)$. The correction is therefore proportional to $a^2\sim Z^2$ as would be expected, and as was anticipated and verified in the many experimental papers on the absorption of gamma-rays by pair production.[2–4] In the case