Differential Cross Section for Bremsstrahlung and Pair Production

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WE have calculated the cross section for bremsstrahlung and pair production in the limit of high relativistic energies without the use of the Born approximation. The initial and final states of the electron are described by the wave function of Furry,[1] and Sommerfeld and Maue,[2] which is also identical with the first two terms of that of Bess.[3,4] The exact expansion in spherical harmonics of Mott and Massey[5] goes over into our wave function by neglecting $a^2=(Ze^2/\hbar c)^2$ compared with l^2, where l is the angular momentum.

A major part of our calculation consists in the demonstration that this wave function is accurate enough for the calculation of the differential cross section. This was found to be true for energies $\epsilon \gg mc^2$ and scattering angles θ of the order of mc^2/ϵ, which are the only ones contributing significantly to the total cross section. The final electron energy must also be large.

The integration of the matrix element was performed by a method due to Nordsieck.[6] We denote the initial electron energy by ϵ_1, the final one by ϵ_2, their momenta by $\mathbf{p}_1$ and $\mathbf{p}_2$, the quantum energy by k, the rest energy of the electron by μ, the angles between the quantum momentum $\mathbf{k}$ and the two electron momenta by θ_1 and θ_2, the azimuth between them by ϕ, the momentum change by $\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2-\mathbf{k}$, and the quantity $Ze^2/\hbar c$ by a. We further introduce the abbreviation

$$X=1-\frac{q^2(p_1-p_2-k)(p_1-p_2+k)}{4k^2(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\approx 1-\frac{q^2\mu^2}{(\epsilon_1^2\theta_1^2+\mu^2)(\epsilon_2^2\theta_2^2+\mu^2)}, \quad (1)$$

the last expression being valid for large ϵ and for θ of order μ/ϵ or less. X varies from 0 to 1. We use the hypergeometric functions

$$V(X)=F(ia,-ia;1,X),$$
$$W(X)=a^{-2}dV/dX=F(1+ia,1-ia;2,X), \quad (2)$$

which are clearly real.

Then the differential cross section for bremsstrahlung is

$$\begin{aligned}d\sigma=&\left\{\frac{\pi a}{\sinh\pi a}\right\}^2\frac{a^2}{2\pi}\left(\frac{\hbar}{mc}\right)^2\mu^2\frac{e^2}{\hbar c}\frac{p_2}{p_1}\frac{dk}{k}\sin\theta_1\sin\theta_2 d\theta_1 d\theta_2 d\phi\\
&\times\left\{\frac{V^2(X)}{q^4}\left[\frac{p_1^2\sin^2\theta_1(4\epsilon_2^2-q^2)}{(\epsilon_1-p_1\cos\theta_1)^2}+\frac{p_2^2\sin^2\theta_2(4\epsilon_1^2-q^2)}{(\epsilon_2-p_2\cos\theta_2)^2}\right.\right.\\
&-\frac{(4\epsilon_1\epsilon_2-q^2+2k^2)2p_1p_2\sin\theta_1\sin\theta_2\cos\phi}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\\
&\left.+2k^2\frac{(p_1^2\sin^2\theta_1+p_2^2\sin^2\theta_2)}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\right]\\
&+\frac{a^2[k^2-(p_1-p_2)^2]^2W^2(X)}{(4k^2)^2(\epsilon_1-p_1\cos\theta_1)^2(\epsilon_2-p_2\cos\theta_2)^2}\\
&\times\left[\frac{p_1^2\sin^2\theta_1(4\epsilon_2^2-q^2)}{(\epsilon_1-p_1\cos\theta_1)^2}+\frac{p_2^2\sin^2\theta_2(4\epsilon_1^2-q^2)}{(\epsilon_2-p_2\cos\theta_2)^2}\right.\\
&+\frac{(4\epsilon_1\epsilon_2-q^2+2k^2)2p_1p_2\sin\theta_1\sin\theta_2\cos\phi}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\\
&-2k^2\frac{(p_1^2\sin^2\theta_1+p_2^2\sin^2\theta_2)}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\\
&\left.\left.+\frac{4k^2}{\mu^2}(\epsilon_1\epsilon_2+p_1p_2\cos\theta_1\cos\theta_2)\right]\right\}. \quad (3)\end{aligned}$$

The first square bracket here is identical with the corresponding expression in the theory of Bethe and Heitler which is based on the Born approximation. It is multiplied by $(\pi a/\sinh\pi a)^2V^2(X)$. It can be shown that for $X=1$, which corresponds to small q, this factor is unity; for larger q it is less. The second square bracket, which is multiplied by $W^2(X)$, is new; the slight difference in structure between the two terms is real. For small q, the term with W^2 is negligible. Therefore, in this limit, in which screening may be important, there is no correction to the Born approximation formula.

For pair production, we must substitute $p_1p_2d\epsilon_1/k^3$ for p_2dk/p_1k, and change the sign of ϵ_2 and p_2 everywhere. Then ϵ_2 is the positron energy and θ_2 the angle between positron and quantum. In contrast to the customary Bethe-Heitler theory, θ_2 should not be changed into $\pi-\theta_2$; this substitution is replaced by the change of sign of p_2.

A more complete account of this calculation will be published later.

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[2] A. Sommerfeld and A. W. Maue, Ann. Physik **22**, 629 (1935).
[3] L. Bess, Phys. Rev. **77**, 550 (1950), Eq. (4).
[4] The last term in Bess's wave function is wrong, as are therefore all the further calculations in his paper. We are indebted to Professor A. T. Nordsieck for pointing out this fact to us.
[5] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), second edition, Chap. IV, Eq. (39).
[6] See reference 3, and private communication for which we are indebted to Professor Nordsieck.

Integral Cross Section for Bremsstrahlung and Pair Production

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WE have integrated the differential cross section of the preceding paper over angles. Since the differential cross section contains complicated (hypergeometric) functions of a certain variable X, Eq. (1) of the preceding paper, the integration over this variable must be left to the end. This can be done by a method similar to that used earlier by one of the authors.[1]

The independent variables used over most of the region of integration are

$$\xi=\mu^2/(\epsilon_1^2\theta_1^2+\mu^2), \quad \eta=\mu^2/(\epsilon_2^2\theta_2^2+\mu^2), \quad (1)$$

and X. Analytic integration over ξ and η proves possible. For $\theta \gg \mu/\epsilon$, the contribution to the total cross section is negligible and is left out. For very small q, i.e., X near 1, the differential cross section reduces to the Born approximation and therefore the integration can be done as in reference 1, including the case of screening.

After integration over ξ and η, the total cross section reduces to the surprisingly simple form (in the case of no screening)

$$\begin{aligned}\sigma=&\frac{2}{3}\frac{e^2}{\hbar c}\left(\frac{\hbar c}{\mu}\right)^2\frac{a^2}{\epsilon_1^2}\frac{dk}{k}(3\epsilon_1^2+3\epsilon_2^2-2\epsilon_1\epsilon_2)\left\{2\log\frac{2\epsilon_1\epsilon_2}{\mu k}-1\right.\\
&+\left(\frac{\sinh\pi a}{\pi a}\right)^2a^2\left[2\int_0^1V(X)W(X)\log(1-X)dX\right.\\
&\left.\left.+\int_0^1XW^2(X)dX\right]\right\}. \quad (2)\end{aligned}$$

The first two terms in the curly bracket are the result of Bethe and Heitler; the last is the correction to the Born approximation. Both of the integrals over X converge at both limits. For pair production, the sign of the term $2\epsilon_1\epsilon_2$ in the first set of parentheses must be changed, and dk/ϵ_1^2 must be replaced by $d\epsilon_1/k^2$.

For small a (small nuclear charge), the integrals over X can be carried out analytically and the curly bracket in (2) becomes

$$\{\ \}=2\log(2\epsilon_1\epsilon_2/\mu k)-1-2.414a^2, \quad (3)$$

where 2.414 stands for $2(1+2^{-3}+3^{-3}+\cdots)$. The correction is therefore proportional to $a^2\sim Z^2$ as would be expected, and as was anticipated and verified in the many experimental papers on the absorption of gamma-rays by pair production.[2–4] In the case of screening, partial or complete, the *correction* should be the same as in (3), because for small q the differential cross section is unchanged. Therefore we may simply use the Bethe-Heitler formula, subtracting from the logarithm the amount $1.207a^2$.

The correction amounts to a decrease of the cross section, in accord with experiment. The decrease occurs in bremsstrahlung (which has not been well investigated experimentally) to the same extent as in pair production. The main factor in the energy distribution of the resulting electrons is unchanged; only a constant is added to the logarithmic term. That the correction to the Born approximation is small (experimentally about 10 percent for Pb) arises from the fact that the correction must be compared with the logarithm [first term in (3)] which is of the order of 5.

We have also calculated the integrals in (2) numerically for the case of lead for which $a=0.6$. In this case, the last term in (3) is placed by 0.67 whereas $2.4a^2=0.87$. Thus the correction for heavy elements is somewhat less than the Z^2 law would indicate.

Integration over the energy ϵ_1 gives for the total cross section for pair production for lead

$$\sigma=\frac{28}{9}\frac{Z^2r_0^2}{137}\left(\log 2k-\frac{109}{42}-0.33_5\right), \tag{4}$$

in which the last term is the correction calculated in this paper. In the case of complete screening we get (for smaller Z)

$$\sigma=\frac{28}{9}\frac{Z^2r_0^2}{137}\left[\log 183Z^{-\frac{1}{3}}-1.207\left(\frac{Z}{137}\right)^2\right]. \tag{5}$$

At 88 Mev, the calculated reduction of cross section for Pb is 11.8 percent, the observed[3] 11 percent; at 280 Mev, the numbers are 10.0 and[4] 10 percent. The agreement is excellent. Around 20 Mev where several experiments are available, our approximations based on $\epsilon\gg\mu$ are probably no longer good enough; our theory would give a reduction of 20 percent for Pb, whereas Walker's experimental value[2] is only 15.5 percent. This is not surprising; at still lower energies Hulme *et al.*[5] found a theoretical cross section larger than the Born approximation.

A fuller account of the calculations will be given later.

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Production of 40-Mev π^+ and π^- Mesons in Seven Elements by 240-Mev Protons*

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A MEASUREMENT has been made of the relative differential cross section for production of 40-Mev π^+ mesons in the angular range 130° to 150° by 240-Mev protons in Be, C, Al, Cu, Ag, W, and Pb. A similar measurement has been made for 40-Mev π^- mesons in the angular range 30° to 50°.

The targets in which the mesons were produced were exposed to the internal circulating proton beam of the Rochester synchrocyclotron. The π^+ mesons were detected with a scintillation counter telescope in the arrangement shown in Fig. 1, taking advantage of the focusing of the mesons in the fringing field of the cyclotron magnet. The detection scheme allowed the mesons to be distinguished from the background radiation on the basis of their rate of energy loss in the first counter, their range in matter, and by the requirement of coincident pulses from the three counters. The arrangement used for detecting π^- mesons can be visualized by reflecting the shield and telescope through the radius drawn through the target.

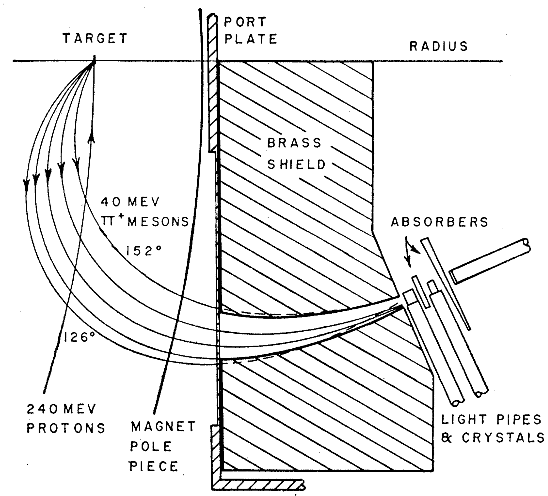

FIG. 1. Arrangement used for observing mesons.

The background correction was made by subtracting the counting rate observed with sufficient absorbing material, included in the telescope to stop the mesons, from that observed with the mesons traversing all three crystals. This correction amounted to about 10 percent.

The recirculation of the proton beam, i.e., the average number of traversals of protons through targets, the multiple scattering of which varied over a wide range, was measured by a method similar to that described by Knox.[1] The targets in which the mesons were produced were designed so that the effect of multiple traversals would not be large, and a correction was applied to the proton current from the multiple traversal measurement. This correction was largest for the Be target, for which it amounted to about a factor of 2, and was known within 10 percent.

The measured relative cross sections are given in Table I; the π^+ and π^- cross sections are separately normalized to unity for Be. The actual π^+/π^- ratio for Be is 3 ± 1, but as pointed out above, the π^+ and π^- cross sections were not measured in the same angular range. The errors quoted in the table are probable errors and include contributions from counting statistics, proton current fluctuations, and time variation of the detector efficiency.

TABLE I. Relative cross sections for meson production.

Element	σ^+	σ^-	$\sigma^+/A^{2/3}$	$\sigma^-/A^{2/3}$
Be	1.00±0.03	1.00±0.07	1.00±0.03	1.00±0.07
C	3.52±0.09	1.74±0.06	2.91±0.07	1.44±0.05
Al	7.95±0.11	6.0 ±0.2	3.83±0.05	2.9 ±0.1
Cu	13.7 ±0.3	14.6 ±0.4	3.74±0.08	4.0 ±0.1
Ag	16.6 ±0.3	19.1 ±0.7	3.18±0.06	3.66±0.13
W	19.4 ±0.4	23.8 ±0.9	2.60±0.05	3.20±0.12
Pb	19.0 ±0.5	23.5 ±1.0	2.35±0.06	2.91±0.12

It is evident from Table I that the π^+ relative cross sections vary in about the same manner as the π^- relative cross sections, except for C and Al. The difference in these two elements may well be due to their considerably different energy thresholds for production of π^+ and π^- mesons.[2]

In contrast to a previous experiment on the production of mesons by γ-rays,[3] in which it was found that the sums of the relative cross sections varied as $A^{\frac{2}{3}}$ over a surprisingly wide range, there appears to be no such simple correlation of the cross sections of the present experiment with the mass number A. This is illustrated in Table I, where the relative cross sections have been divided by $A^{\frac{2}{3}}$. If, instead of $A^{\frac{2}{3}}$, one divides by the proton absorption cross section (which varies as A for small A and $A^{\frac{2}{3}}$ for large A) the variation is slightly less.