Sum Rules in the Dispersion Theory of Nuclear Reactions*

T. TEICHMANN AND E. P. WIGNER
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received March 3, 1952)

In the first five sections the dispersion theory is developed with an internal region V whose boundary S is quite close to the nuclear surface. Two types of quantities then occur: those like the derivative matrix R, which connect the values and derivatives of the wave function at S, and those like the collision matrix U which give the asymptotic behavior of the wave function. These latter are, of course, independent of the position of S. Because of the proximity of S to the nuclear surface, the wave function in the closed channels and its derivative remain appreciable at S. They may, however, be eliminated from the formalism, and this is done in Sec. II, leading to a reduced R which connects the values and derivatives of the wave function in the open channels only. In the next section this reduced R is used to obtain expressions for the S-independent quantities, in particular, for the collision matrix. These expressions are given more explicit, but approximate form in Sec. IV. The development shows that the quantity, which in the usual formulas for the cross sections is interpreted as the nuclear radius, need not be equal to this at all, but is the distance at which two opposite effects compensate. The fifth section gives exact expressions for the poles and residues of some S-independent quantities, which are then compared with the poles and residues of the approximate expressions of the previous chapter. In this way criteria are derived for the accuracy of these approximate expressions.

The last sections contain derivations of two sum rules for the parameters in the original R. The consequences of these sum rules are traced for the reduced R and for the S-independent quantities, in the light of the development of the first five sections. The first sum rule gives a maximum for the partial widths of levels, while the second leads to the well-known proportionality of reduced level width and level spacing. The meaning and validity of this is discussed in some detail, both for the single particle and many particle pictures.

I. INTRODUCTION

MEASUREMENT of the cross section for a nuclear reaction in which a pair of nuclei denoted by s collide to form a compound nucleus, which subsequently disintegrates into a pair of nuclei t, shows that near certain "resonance" energies E_λ of the colliding pair, this cross section can be represented in the rather simple form

$$\sigma_{st}=\frac{\pi}{k_s^2}\cdot\frac{\Gamma_{\lambda s}\Gamma_{\lambda t}}{(E_\lambda-E)^2+\frac{1}{4}\Gamma_\lambda^2}, \tag{0}$$

where k_s is the wave number of the colliding pair (in the center-of-mass system), E their energy, and $\Gamma_{\lambda s}$, $\Gamma_{\lambda t}$, Γ_λ are parameters called "widths" which are only mildly energy dependent.

Both time-dependent perturbation theory treatments,[1,2] and stationary state methods,[3–5] lead one to results of this type: in each case the E_λ are characteristic energies of a suitably restricted compound system, and the $\Gamma_{\lambda s}$ are proportional to the square of the matrix element between the corresponding proper functions and the wave function of the emitted (or colliding) pair s.

In the most general stationary treatment,[5,6] the configuration space of the problem is arbitrarily divided into two regions, the "internal region" V, to which the compound state, and all specifically nuclear effects, are restricted, and the remaining external region in which the various pairs s, t, etc., are presumed to behave like two-particle systems. The E_λ are then the proper values of the quantum-mechanical system in the internal region V satisfying certain self-adjoint boundary conditions on the surface S of V; these boundary conditions are independent of energy but involve an arbitrary parameter b. The $\Gamma_{\lambda s}$ are then given by $\Gamma_{\lambda s}=2k_sP_s\gamma_{\lambda s}^2$, where P_s is the penetration factor for the two-particle system s at S, while $\gamma_{\lambda s}$ is essentially the value of proper function X_λ corresponding to E_λ on the part of S which correspond to the emission of the pair s.

The internal region is generally taken to be of approximately nuclear size, but from the mathematical point of view no such restriction is necessary. It can be shown[7] that the general formula obtained for σ_{st} is quite independent of the size of V and of the parameter b occurring in the boundary condition.

It is the purpose of this paper to indicate more completely the relation between the empirical parameters which occur in (0), and those which arise from the mathematical formalism. This question, together with that of the most suitable choice of V, is discussed in Secs. II to VI. The dependence of the parameters on the boundary conditions (i.e., on b) is dealt with in Sec. VII, and this allows simple derivations of statistical properties (i.e., "sum rules") of the $\gamma_{\lambda s}$ in the remaining sections. A table of experimental values is included.

Only the basic results of the underlying theory[1–7] are stated below, particularly since the various derivations have been published in several forms. For further details the reader is referred to the references mentioned above, where further bibliography may also be found.

* This work was assisted in part by the AEC.
[1] G. Breit and E. P. Wigner, Phys. Rev. **49**, 519 (1936).
[2] S. Flügge, Z. Naturforsch. **3a**, 97 (1948).
[3] P. L. Kapur and R. Peierls, Proc. Roy. Soc. (London) **A166**, 277 (1938).
[4] H. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937).
[5] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).
[6] E. P. Wigner, Phys. Rev. **73**, 1002 (1948).
[7] T. Teichmann, Phys. Rev. **77**, 506 (1950), and Ph.D. thesis, Princeton University, 1949.

II. ELIMINATION OF CLOSED CHANNELS

In defining the basic set of characteristic functions X_λ in the internal region of configuration space,[5] one can use a great variety of boundary conditions. If the characteristic value problem

$$HX_\lambda = E_\lambda X_\lambda \tag{1}$$

for the X_λ is self adjoint, the most general boundary condition can be given the form

$$\int_S ((\text{grad}X_\lambda)_n + b_s X_\lambda)\psi_s dS = 0 \tag{2}$$

in which dS indicates a surface element of the boundary of the internal region; the integration in (2), as in all similar expressions, is to be extended over this boundary S. The ψ_s, ψ_t, $\cdots$ are a set of orthonormal functions

$$\int_S \psi_s^* \psi_t dS = \delta_{st} \tag{2a}$$

defined on S; the b_s are real constants. It is most reasonable to identify the ψ_s as far as possible with the reaction products so that

$$\sum_s \psi_s(i_s, \Omega_s)[c_s F_s(r_s) + c_s' G_s(r_s)] \tag{3}$$

with arbitrary c_s, c_s', represents a solution of the Schrödinger equation in the external region. The ψ_s, for which such an identification is possible, are the products of a spherical harmonic, which depends on the direction Ω_s of the line connecting the centers of mass of the reaction products s, with the product of the real normalized internal wave functions of these products. i_s denotes the internal coordinates of the two particles which constitute the reaction products s; r_s is the distance of their centers of mass. Because of the normalization (2a), the ψ_s also contain a factor $1/r_s$ so that the F_s and G_s, which describe possible motions of these nuclei with respect to each other, are so-called radial wave functions, i.e., solutions of the radial Schrödinger equation multiplied by r_s. They are so normalized that $E=G+iF$ and $I=G-iF$ represent outgoing and incoming waves of flux $\hbar/M$. Hence

$$F'G - G'F = 1. \tag{4}$$

F' and G' denote the components of grad$F$ and grad$G$ normal to the surface S—the same components that occur in integrals over S, such as (2).

The above definition of the ψ_s will be assumed below for all open channels, i.e., for all reaction products which are energetically possible in the energy range considered. The rest of the ψ_s can be arbitrary except that, together with the open channel ψ_s, they must form a complete orthonormal system on S which will be assumed to be real.

The b_s in (2) are still arbitrary. The simplest choice is $b_s=0$ if one does not want to restrict the energy interval under consideration. We shall see, however, that in the case of a restricted energy interval a different choice of the b may provide some simplification.

The energy dependent R matrix which we shall define gives a relation between the values and normal derivatives of an arbitrary φ on the surface S, provided φ satisfies the equation $H\varphi=E\varphi$ in the internal region. We define

$$\alpha_s = (\hbar^2/2M)^{\frac{1}{2}} \int_S \psi_s \varphi dS, \tag{5}$$

$$\beta_s = (\hbar^2/2M)^{\frac{1}{2}} \int_S \psi_s (\text{grad}\varphi)_n dS, \tag{5a}$$

and obtain in the usual way[5,7]

$$\alpha_s = \sum_t R_{st}(\beta_t + b_t \alpha_t), \tag{6}$$

for which we shall also write

$$\alpha = R(\beta + b\alpha). \tag{6'}$$

In this last expression, α and β are vectors with components α_s, β_s, and b is a diagonal matrix with diagonal elements b_s. The R_{st} in (6) are

$$R_{st} = \sum_\lambda \frac{\gamma_{\lambda s}\gamma_{\lambda t}}{E_\lambda - E}, \tag{6a}$$

for which we shall also write symbolically,

$$R = \sum_\lambda \frac{\gamma_\lambda \times \gamma_\lambda}{E_\lambda - E}. \tag{6a'}$$

The R_{st} depend on E only on through the E in the denominationators; both the E_λ, defined in (1), and the $\gamma_{\lambda s}$,

$$\gamma_{\lambda s} = (\hbar^2/2M)^{\frac{1}{2}} \int_S X_\lambda \psi_s dS, \tag{6b}$$

are independent of E. The integration in (6b) is to be extended over the surface S of the internal region.

Given the α and β which satisfy (6), the wave function φ can be expressed in the internal region in terms of the X_λ,

$$\varphi = \sum_\lambda \frac{\sum_s \gamma_{\lambda s}(\beta_s + b_s\alpha_s)}{E_\lambda - E} X_\lambda. \tag{7}$$

One calculates from (7) that the square integral of φ in the internal region is

$$\int_V |\varphi|^2 dV = \sum\sum_{s,t} \dot{R}(E)_{st}(\beta_s + b_s\alpha_s)^*(\beta_t + b_t\alpha_t). \tag{7a}$$

$\dot{R}(E)$ is the derivative of $R(E)$ with respect to E. (Note that F_s' and G_s' were derivatives with respect to r_s.) For (7a) we shall also write

$$\int |\varphi|^2 dV = (\beta + b\alpha, \dot{R}(E)(\beta + b\alpha)). \tag{7a'}$$

Naturally, the R_{st} depend on the b_s because the X_λ and the $\gamma_{\lambda s}$ do so. Hence (6) gives apparently different relations between the α and the β for different values of the b_s. It should be clear, however, that these relations are only apparently different. If we denote by R^0 the R for the set of b which all vanish, the R for another set of b is given by

$$R=R^0(1+bR^0)^{-1}, \tag{7b}$$

and the validity of (6), (7), (7a) is an immediate consequence of (7b) and the validity of the same equations for R^0.

The above formulas are all rigorous if the summations over s, t are extended over the whole orthonormal set; hence R_{st} is an infinite dimensional matrix. It is, of course, possible to place S so far out that the wave function dies out almost completely where a closed channel intersects S. In this case α and β are very small for closed channels and (6) and (7) become valid with the summation over s and t extended only over the open channels. For reasons which will appear at the end of Sec. IV, this will not be assumed at present.

One can eliminate explicit reference to the closed channels in the preceding formulas by introducing the energy dependent logarithmic derivative K_t (a negative quantity) of the exponentially decaying solution. This gives $\beta_t=K_t\alpha_t$ along the channels to be eliminated and permits the elimination from (6a) of the α and β referring to these channels. In order to carry this out, one subdivides R into submatrices

$$R=\begin{pmatrix} R_{rr} & R_{re} \\ R_{er} & R_{ee} \end{pmatrix}. \tag{8}$$

The index r refers to retained, the index e to channels to be eliminated. In general, the closed channels will belong into the latter category. The R_{rr}, R_{re}, etc., are matrices themselves. R_{rr} and R_{ee} are real symmetric square matrices, R_{re} is the transposed matrix to R_{er}. These are real but not square. The elimination leads to a reduced $\mathfrak{R}$ which permits one to express the α corresponding to the retained channels in terms of only those β which also correspond to retained channels (these are, in general, the open channels)

$$\alpha_s=\sum_t{}'\mathfrak{R}_{st}(\beta_t+b_t\alpha_t). \tag{9}$$

The reduced $\mathfrak{R}$ can be expressed in terms of the submatrices of the original R:

$$\mathfrak{R}=R_{rr}+R_{re}[(K+b_e)^{-1}-R_{ee}]^{-1}R_{er}. \tag{9a}$$

K is a diagonal matrix with diagonal elements K_t; its rows and columns correspond, of course, only to eliminated channels.

The reduced $\mathfrak{R}$ given by (9a) does not have as simple properties as the original R. In particular, like the collision matrix to which we shall turn later, it has essential singularities in addition to poles if it is regarded as a function of the complex variable E. These essential singularities include branch points at those (real) values of $E=E_{\text{thr}}$ which constitute thresholds for a type of nuclear reaction which is energetically impossible at a lower E. One can see this immediately from (9a) since a K_t is proportional to $(E_{\text{thr}}-E)^{\frac{1}{2}}$ in the neighborhood of E_{thr}. (Similar statements apply to the collision matrix.) However, if the K_t can be replaced by constants or linear functions of E in which the coefficients of E are positive (in general "R functions"[8] of E), the approximate $\mathfrak{R}$ obtained in this way will be a finite matrix which permits an expansion of the form (6a) with $\gamma_{\lambda s}$ and E_λ which are independent of E though different from the quantities which occur in (6a).

It may be worth while to carry out such an expansion in detail and, at the same time, to estimate the magnitude of the second term of (9a). The omission of this term corresponds to disregarding the eliminated channels altogether. The $\mathfrak{R}$ can easily be calculated if one approximates R_{re}, R_{er}, and R_{ee} by a single term in the expansion (6a). The $\mathfrak{R}$ then becomes equal to R_{rr} except that the E_λ therein has to be decreased by

$$\sum_t{}'(K_t+b_t)\gamma_{\lambda t}^2, \tag{10}$$

where the summation over t is over all eliminated channels. The $\gamma_{\lambda s}$ corresponding to retained channels do not change, neither the E_μ with $\mu\neq\lambda$. The situation is less simple if one does not approximate R_{re}, R_{er}, and R_{rr} in the above way, but $\mathfrak{R}$ will be essentially equal to R_{rr} if the expressions (10) are small compared with the level spacing for every λ. The sum $\sum_t\gamma_{\lambda t}^2$ can be estimated on the basis of a sum rule which will be derived later and is given in (29). It applies if the position of the surface S is quite close to the nuclear surface. If the surface S is at a distance a' from the nuclear surface, the $\gamma_{\lambda t}$ will contain a factor $\exp K_t a'$ as can be seen from (6b): the wave function X_λ contained therein decays outside the nuclear surface exponentially (K_t is negative). Thus, (10) can be crudely estimated to be of the order $(K+b)(3\hbar^2/2Ma_0)\exp 2Ka'$ where K is an average K_t and a_0 is the nuclear radius. In order to reduce (10) for $b=0$ to a small fraction of an electron volt, one must place the surface S almost 10^{-11} cm from the nuclear surface. This distance can be reduced considerably by choosing the b_t in the boundary conditions (2) for the closed channels to be equal to the average K_t for the energy region of interest. Actually, an estimate of the effect of specifically nuclear forces on the wave function of a nucleon outside the nucleus proper indicates that the effect of these forces remains significant for about 10^{-12} cm. This gives at any rate a lower limit on the distance of S from the nuclear surface. As a result, the use of the more general boundary conditions (2), instead of $(\text{grad}X_\lambda)_n=0$, offers only limited advantages.

One is tempted, at this point, to consider energy dependent boundary conditions, as has been done by

[8] E. P. Wigner, Ann. Math. **53**, 36 (1951).

Kapur and Peierls.[3] For the closed channels $K_t=b_t$ would be the natural choice and we shall see that, for the open channels, $b_s=G_s/G_s'$ would seem to offer simplifications (this is not what was used in reference 3). It seems to the present writers, however, that the advantage resulting from the Hermitean orthogonality of the X_λ and the simple properties of the R matrices overbalance the disadvantages inherent in the transition from the original R to the reduced $\mathfrak{R}$ and from this to the Q of the next section.

III. CONNECTION BETWEEN DERIVATIVE MATRIX AND COLLISION MATRIX

Several equivalent formulas have been given in the past to express the collision matrix U in terms of R. We now believe that the most natural way uses an intermediate expression Q, in terms of which

$$U=(1+iQ)/(1-iQ). \tag{11}$$

This Q is $-\pi$ times Heitler's K-matrix.[9] By the definition of the collision matrix, the functions

$$\begin{aligned} &\varphi=\psi_s I_s-\textstyle\sum_t U_{st}\psi_t E_t, \\ &I_s=G_s-iF_s, \qquad E_s=G_s+iF_s \end{aligned} \tag{12}$$

are, for all s, solutions of the wave equations and although they are defined by (12) only in the external region, they can be continued into the internal one in such a way as to satisfy $H\varphi=E\varphi$. It then follows from (11) that the same holds also for

$$\varphi=\psi_s F_s(r_s)+\textstyle\sum_t Q_{st}\psi_t G_t(r_t). \tag{12a}$$

The functions (12a) are linear combinations of the functions (12), but (12a) can suitably be taken as the definition of Q, instead of (11). U as well as Q is symmetric, U is unitary, Q real. They are both independent of the choice of the internal region, but depend, of course, on the energy in a more complicated way than does R. In fact, the number of the dimensions of U and Q depend on the energy, it being equal to the number of open channels. Q can be expressed in terms of the reduced $\mathfrak{R}$ by applying (9) to (12a) to give

$$\begin{aligned} Q=&-(F'+\bar{b}F)/(G'+\bar{b}G) \\ &+[G(G'+\bar{b}G)-(G'+\bar{b}G)\mathfrak{R}(G'+bG)]^{-1}, \end{aligned} \tag{13}$$

where $\bar{b}=b-(1/a)$, a being the radius of the relevant portion of S. This completes the calculation of U. In (11a) $\bar{b}$, F, G, F', G' are diagonal matrices with rows and columns which correspond only to open channels; their diagonal elements are $\bar{b}_s$, F_s, G_s, etc. The only special property of F and G used when going from (12a) to (13) was (4). Hence an expression similar to (13) can be obtained also for

$$\begin{aligned} U=&(I'+\bar{b}I)/(E'+\bar{b}E)+2i[E(E'+\bar{b}E) \\ &-(E'+\bar{b}E)\mathfrak{R}(E'+\bar{b}E)]^{-1}, \end{aligned} \tag{13a}$$

and for

$$\begin{aligned} -Q^{-1}=&(G'+\bar{b}G)/(F'+\bar{b}F)+[F(F'+\bar{b}F) \\ &-(F'+\bar{b}F)\mathfrak{R}(F'+\bar{b}F)]^{-1}. \end{aligned} \tag{13b}$$

The calculation of Q, U, etc., from $\mathfrak{R}$ is a matter of finite dimensional matrix algebra.

The above form of the dispersion theory differs from older formulations by not assuming that the wave function in closed channels has died out where these channels reach S. In this regard, the above are very similar to the formulas which appear necessary for the investigation of the behavior of the collision matrix near a threshold for a "new" reaction.[6] They differ from these by allowing the use of more general boundary conditions on S—a possibility which will be made use of later. The derivation of (13) consists of the application of (9) to the wave function (12a) and need not be given in detail.

It should be remarked that while U and Q are independent of the position of the surface S dividing internal and external regions, the R, as well as $\mathfrak{R}$, are dependent thereon as well as on the values chosen for the b. If one permits a row and column of R to correspond to every member of the orthogonal set ψ_s, the above formulas are, naturally, rigorous. However, the evaluation of $\mathfrak{R}$ by (9a) then becomes the real problem, along with the determination of the K_t.

IV. APPROXIMATE EQUATIONS

If we treat the K_t which occur in (9a) and the F_s, G_s, F_s', G_s' which are implicit in (13) as constants, Q permits an expansion similar to (6a):

$$Q(E)=\sum_\lambda \frac{\omega_\lambda \times \omega_\lambda}{H_\lambda-E}+q, \tag{14}$$

with energy independent ω, H, and q.[10] [Actually, K_t need not be constant; see the remarks after (9a).] It must be realized, however, that the expansion (14) will represent Q only in the energy range in which the actual K, F, G, etc., are still close to the constants which were chosen for these quantities when making the expansion (14). Outside this range, (14) will not represent Q and, in particular, the resonances H_λ which are outside this range have no physical significance. Neither is the expression (14), outside of its range of validity, independent of the location of S. It is true that one can represent Q in almost every sufficiently narrow energy range by an expression like (14) but the H, ω, and q will be different for different energy ranges. A representation like (14) valid for every E is impossible because Q (as well as U), as analytic function of energy, will have not only poles but also branch points and it is not clear that all the poles of Q are real. The possibility of the expansion (14) under the stated assumptions

[9] W. Pauli, Report on International Conference on Fundamental Particles and Low Temperatures I (The Physical Society, London, 1947), p. 5.

[10] The effect on the resonance cross sections of the energy variation of the F's and G's has been investigated by R. G. Thomas, Phys. Rev. 81, 149 (1951).

follows,[8] however, from the fact that the bilinear form of the Q, as defined in (13) and (9a) with constant K, F, G, etc., is an analytic function of E, the imaginary part of which has the same sign as the imaginary part of E. We shall now determine the constants in (14).

In order to determine the H_μ, we may note that the determinant of the square bracket of (13) must vanish where Q becomes infinite. Hence, the linear homogeneous equations for the $\omega_{\mu t}$,

$$\sum_t[G-(G'+\bar{b}G)\Re(H_\mu)]_{st}(G'+\bar{b}G)_t\omega_{\mu t}=0, \quad (15a)$$

will have a nontrivial solution. One can then calculate the reciprocal of the square bracket of (13) in the neighborhood of H_μ and obtain

$$Q=\frac{\boldsymbol{\omega}_\mu\times\boldsymbol{\omega}_\mu}{H_\mu-E}+Q_\mu, \quad (15)$$

in which Q_μ is regular at $E=H_\mu$. The ω_μ of (14) are the solutions of (15a) so normalized that the scalar product

$$\sum_s\sum_t\omega_{\mu s}[(G'+\bar{b}G)\dot{\Re}(H_\mu)(G'+\bar{b}G)]_{st}\omega_{\mu t}=1. \quad (15b)$$

The dot again signifies differentiation with respect to the energy.

The above determines the H_μ and ω. In order to determine the q of (14), one may insert for E a very large imaginary value. The $\Re$ then vanishes in (13) and one obtains

$$q=-F/G. \quad (16)$$

It is possible to simplify the above formulas considerably by choosing $\bar{b}_s=-G_s'/G_s$. One sees from either (12a) or (15a) that the H_μ then become equal to the E_μ and one has also

$$H_\mu=E_\mu;\ \omega_{\mu s}=G_s^{-1}\gamma_{\mu s} \quad \text{for} \quad \bar{b}_s=-G_s'/G_s. \quad (17)$$

In order to calculate the collision matrix U from (14), it is customary to assume that only one term of the infinite sum of (14) is significant, the one closest to E. The others are then neglected in comparison with $q=-F/G$. With the $\bar{b}_s$ as chosen for (17), this gives

$$U=\frac{1-iF/G}{1+iF/G}+2i\frac{(1+iF/G)^{-1}\boldsymbol{\omega}_\mu\times(1+iF/G)^{-1}\boldsymbol{\omega}_\mu}{H_\mu-E-i(\boldsymbol{\omega}_\mu\cdot(1+iF/G)^{-1}\boldsymbol{\omega}_\mu)}. \quad (18)$$

Squaring (18) one obtains the usual potential scattering, resonance, and interference terms. The level H_μ is shifted to $H_\mu+\Delta_\mu$ and has partial and total widths, $\Gamma_{\mu s}$ and Γ_μ

$$\Gamma_{\mu s}=2(G_s^2+F_s^2)^{-1}\gamma_{\mu s}^2;\quad \Gamma_\mu=\sum_s\Gamma_{\mu s};\quad \Delta_\mu=\sum_s(F_s/G_s)\Gamma_{\mu s}. \quad (18a)$$

The first term of the approximate expression (18) for U clearly depends on a_s while the actual U is, of course, independent thereof. The approximation which is least justified, and which is directly responsible for the dependence of this expression on a_s, is the neglect of all but one term of the infinite sum in (14). The most significant contribution of the $\lambda\neq\mu$ terms, in the neighborhood of $E=H_\mu$, comes not from the H_λ which are close to H_μ, but from the cumulative effect of the distant resonances and these are the ones which compensate for the a_s dependence of q. That the close resonances are in general less important can be seen by a comparison of the quantities $q_s=-F_s/G_s$ and $\omega_{\mu s}^2/D\simeq\gamma_{\mu s}^2/G_s^2D$ where D is the level spacing. On the basis of the well-known estimate[11] for $\gamma_{\mu s}^2/D\simeq 1/K$, which will also be derived in the present article, the ratio of these quantities is of the order F_sG_sK, where K is of the order of 10^{13} or 10^{14} cm^{-1}. The quantity F_sG_s can be estimated from (4) to be of the order of the reciprocal of the wave number k_s for disintegration into the pair s in the energy range in question. Hence q is, in general, several times larger than any single term of the sum. Naturally, this holds only in a general way and the effects of adjoining resonances on each other cannot always be neglected.

The cumulative effect of the distant resonances is not very important as far as the nondiagonal elements of Q are concerned: the $\omega_{\lambda s}\omega_{\lambda t}$ are as likely to be positive as negative for $s\neq t$ and they cancel on the average. However, as far as the diagonal elements are concerned, the positive contribution of the large H_λ terms can be expected to be larger than the negative contribution of the $H_\lambda<E$ terms. The cumulative effect of the $\lambda\neq\mu$ terms on the diagonal elements of Q is probably quite appreciable.[12] Hence, the proper significance of the radius a_s which one finds by fitting the experimental data to (18) is the value at which $-F_s/G_s=Q_{\mu,ss}$. It would therefore be more accurate not to replace the Q_μ of (15) by $-F/G$ but to assume simply that it is a diagonal matrix which varies but slowly with E. The form of U is then still given by (18), but the F therein no longer represents the value of the regular solution of Schrödinger's equation at any particular point but simply $G_s\bar{Q}$, where $\bar{Q}$ is the diagonal part of Q_μ. As the preceding paragraph shows, this quantity depends but little on μ. This change permits one to write U in terms of quantities which are all at least approximately independent of a_s:

$$\begin{aligned}U&=\frac{1+i\bar{Q}}{1-i\bar{Q}}+2i\frac{(1-i\bar{Q})^{-1}\boldsymbol{\omega}_\mu\times(1-i\bar{Q})^{-1}\boldsymbol{\omega}_\mu}{E_\mu-E-i(\boldsymbol{\omega}_\mu\cdot(1-i\bar{Q})^{-1}\boldsymbol{\omega}_\mu)}\\&=\frac{1+i\bar{Q}}{1-i\bar{Q}}+2i\frac{(1-i\bar{Q})^{-1}G^{-1}\boldsymbol{\gamma}_\mu\times(1-i\bar{Q})G^{-1}\boldsymbol{\gamma}_\mu}{E_\mu-E-i(G^{-1}\boldsymbol{\gamma}_\mu\cdot(1-i\bar{Q})^{-1}G^{-1}\boldsymbol{\gamma}_\mu)}.\end{aligned} \quad (18b)$$

The fact that the $\omega_{\mu s}=G_s^{-1}\gamma_{\mu s}$ are practically independent of a_s will be verified again in the next section. Naturally, F/G remains a reasonable estimate for $-\bar{Q}$, though, in general, it will be an overestimate.

The difficulty of bringing Q and U into a simple form which would be valid for all E entails the difficulty of interpreting experimental data in a simple but rigorous

[11] Feshbach, Peaslee, and Weisskopf, Phys. Rev. **71**, 1451 (1947).
[12] E. P. Wigner, Proc. Cambridge Phil. Soc. **47**, 790 (1951).

fashion without having recourse to (13). This equation is difficult to handle, particularly if there are several open channels, i.e., if Q and U have more than one dimension. Although most of the formulas of this section are only approximate, they are a good representation as long as K, F, F/G, etc., vary only slowly with energy. This will be the case only if the a_s can be chosen rather small—which explains the endeavor of the previous sections to lay S as close as possible to the nuclear surface.

V. THE SINGULARITIES OF S-INDEPENDENT QUANTITIES

The Q and U considered in the previous section represent the actual Q and U only in a restricted energy region because it was assumed that the K, G, G', F, F' are independent of energy. It is this assumption which assured the possibility of the expansion (14) for Q. In the present section we shall derive rigorous formulas for the poles of Q. The positions and residues of these poles are, naturally, entirely independent of the size and shape of the internal region and we shall obtain relatively simple expressions for them. However Q's poles and their residues do not determine Q in the way in which R was determined by the corresponding quantities. Thus the exact expressions for Q's poles and residues will be used only to compare them with the corresponding quantities of the Q and U of the preceding section. This will provide an orientation of how closely the latter approximate the actual Q and U.

At the energy values H_μ which are poles of Q, there is a stationary state φ_μ which consists only of the irregular solutions G in the external region. This follows from (12a). If Q becomes infinite at a real H_μ, the determinant of the square bracket of (13) must vanish, i.e., the linear equation (15a) for $\omega_{\mu t}$ must have a nontrivial solution. Hence this last equation is also correct[8] for the actual Q if the G, G' therein are given their actual values rather than treated as constants. One can verify on the basis of this equation that in the external region φ_μ is given by

$$\varphi_\mu=\sum_s \omega_{\mu s}\psi_s G_s(r_s), \tag{19}$$

i.e., has the amplitude $\omega_{\mu s}G_s$ in the sth channel. This gives the physical interpretation of the ratios of the $\omega_{\mu s}$.

One can again expand Q in the neighborhood of H_μ and find that its singular part remains the expression given in (15) which is, therefore, also accurate. In this calculation, one must take into account that F and G also depend on energy and this leads, for the normalization of $\boldsymbol{\omega}_\mu$, to

$$\sum\sum_{st}\omega_{\mu s}[\dot{G}'G-\dot{G}G'+(G'+\bar{b}G)\dot{\mathfrak{R}}(G'+\bar{b}G)]_{st}\omega_{\mu t}=1 \tag{20}$$

instead of (15b). This equation which determines the absolute value of the $\omega_{\mu s}$ can be interpreted in the following way. By (7a), the term which contains $\dot{\mathfrak{R}}$ is equal to

$$(2M/\hbar^2)\int_V \varphi_\mu^2 dV, \tag{21a}$$

the integration to be extended over the internal region V. Furthermore, because of the identity

$$-(2M/\hbar^2)\int_{a_{s0}}^{a_s} G_s^2 dr_s=[\dot{G}_s'G_s-\dot{G}_sG_s']_{a_{s0}}^{a_s}, \tag{21b}$$

the other two terms of (20) are equal to the negative integral of $(2M/\hbar^2)\varphi_{\mu c}^2$ from the a_{s0} at which $\dot{G}_s'G_s-\dot{G}_sG_s'=0$ to the boundary of the internal region; $\varphi_{\mu c}$ is the function which has the form (19) everywhere. This shows that (20) amounts to the normalization of $\int\varphi_\mu^2 dV=\hbar^2/2M$, the integration to be extended inside the surface S_0 which intersects the s channel at a_{s0}, if S_0 lies in the external region. If S_0 does not lie entirely in the external region (which is most unlikely), φ_μ and hence the ω are normalized by

$$\int^S \varphi_\mu^2 dV-\int_{S_0}^S \varphi_{\mu c}^2 dV=\hbar^2/2M. \tag{21}$$

The first integral is to be extended over the internal region, bounded by S; the second over the region between S_0 and S. $\varphi_{\mu c}$ is generally not a solution of the quantum-mechanical equations but is defined everywhere by (19). On the other hand, φ_μ is a solution of the quantum-mechanical equations and represents a state of energy H_μ. In the external region, φ_μ has the same form (19) as $\varphi_{\mu c}$ so that (21) is clearly independent of the position of S as long as S is in the external region. This then holds also for (20), which is only another form of (21).

The a_{s0} which correspond to closed channels are infinite because, in this case, G_s and all its derivatives vanish only at infinity. Hence the part of the second integral of (21) which refers to these channels is always positive. Since, for a free particle, $k_s a_{s0}\approx l$ the same will be true in those channels s which represent disintegrations into particles without long-range interaction and high relative angular momentum, or with repulsive interaction. It is possible, however, for the second term of (21) to be negative and larger than the first term. In this case ω_μ will be imaginary and the sign of the singular term of (15) will be opposite the usual one.

The above considerations may be also carried out using instead of F and G a different set of linearly independent solutions, $\bar{F}$ and $\bar{G}$, of the external (two-particle) motion:

$$\bar{F}=\xi F+\eta G,$$
$$\bar{G}=\zeta F+\chi G.$$

Using the same normalization as before, one must then have

$$\xi\chi-\eta\zeta=1.$$

The above arguments then go through in an analogous way, with Q replaced by

$$\bar{Q}=(-\eta+\xi Q)/(\chi-\zeta Q).$$

In particular, if one places $\eta=1$, $\zeta=-1$, $\xi=\chi=0$, the a_{s0} all become zero, and both integrals in (21) have to be extended over the whole internal region. One may even put $\xi=\zeta=i$, $\chi=-\eta=1$, in which case

$$\bar{Q}=U,$$

so that U itself appears on the left side of (15). However, the roots H_μ of (15a) all become complex in this case, and the F and G are highly singular at infinity.

In the energy interval in which boundary conditions used in the previous section are not substantially different from those used here, (15a) will lead to the same H_μ with the constant G, G' used there as with the variable G, F' used here. The question as to how nearly the Q_μ will be independent of energy—and independent of μ—is more difficult to decide. It is clear that (15) cannot be used just between resonances and nobody will want to use it there. It may be useful to remark, however, that at the point $H_\mu+\Delta_\mu$, where Q_μ was set equal to $\bar{Q}$, this will remain justified except at very high energies because Δ_μ is generally well below the level spacing D. This follows from the formula quoted before, *viz.*, $\gamma^2\simeq D/K$, whence $\Delta_\mu\sim(FD/G)/K(F^2+G^2)$. The denominator of this expression is by (4), of the order of K/k_s, i.e., well in excess of 1 while F/G is usually less than 1. Finally, the $\dot{G}'G-\dot{G}G'$ term in (20) will be much smaller than the term with $\mathfrak{R}$, and can be neglected as in (15b), unless the radii a_s are chosen extremely large or unless the γ^2 are close to their maximum possible values, to be discussed below. If neither of these conditions obtains, the quantities ω defined by (15), (15a), (15b) will be very nearly equal to the ω of the present section and, hence, within limits independent of the a_s. Since (17) is a direct consequence of Eqs. (15), this also verifies that, for $b_s=-G_s'/G_s$, the $G_s^{-1}\gamma_{\mu s}$ are independent of a_s, as claimed in the preceding section. One can see this also more directly from (19): calculating $\gamma_{\mu s}$ from (19) by (6b) we find $(\hbar^2/2M)^{\frac{1}{2}}\omega_{\mu s}G_s(r_s)$ divided by the square root of the integral of φ_μ^2 over the internal region (the X in (6b) is normalized). This square integral is given in (21) as $\hbar^2/2M$ plus a quantity which, though dependent on S, is in general much smaller.

VI. SUMMARY OF THE ABOVE SECTIONS

The purpose of the above sections was to establish the connection between observed level widths and the reduced widths which occur in the R matrix and which are given by (6b). The connection is given approximately by (18a); the $\Gamma_{\mu s}$ of that equation is, also approximately, independent of the size of the internal region.

The singularities of Q, which defines U by (11), are rigorously given by (15). They are situated at those energy values at which there is a solution φ_μ of the wave equation in the internal region, the continuation of which in the external region (19) contains only waves with a phase shift $\pm\pi/2$. The ω_μ of (15) is given by

$$G_s(r_s)\omega_{\mu s}=\int\varphi_\mu\psi_s dV, \tag{22}$$

which is the analog of (6b) but gives a normalization which is strictly independent of the size of the internal region. The normalization of φ_μ is given by (21) [which also takes care of the absence of the $(\hbar^2/2M)^{\frac{1}{2}}$ factor in (22)]. The position of the surface S_0 is defined by the condition that $\dot{G}'G-\dot{G}G'$ vanish on it. This then gives a natural size to the internal region. One is led back to the formulas quoted in the first paragraph of this section if the second integral in (21) can be neglected as compared with the first one. This will be the case if the width of the resonances is considerably below the upper limit to be derived in the following section.

The potential type term Q_μ of (15) contains, in addition to $-F/G$, the effect of other resonances. It appears that the cumulative effect of distant resonances is more important than that of the close-by ones and contributes mainly to the diagonal elements of Q_μ. Both parts of Q_μ depend on the size of the internal re-tion, but their sum is independent of it. The radii a_s as determined from a fitting of the experimental data to the formulas derived from (15), such as (18a), have no other significance but that the second part of Q_μ vanishes for this choice of a_s.

The reduced R matrix $\mathfrak{R}$, the Q, and U all have singularities in addition to the poles discussed above. In particular, they all have a branch point at every reaction threshold. Hence the cross sections will have singularities at the thresholds which generally turn out to be cusps.[6]

Since it is impossible to observe the phase of the outgoing wave directly, neither U nor Q can be determined experimentally without some recourse to theory. The theory, however, remains simplest, and the determination of the U, Q, etc., most unique, if one has widely separated sharp resonances. In this regard, the present more general formulation of the dispersion theory does not differ from older and more simple formulations.

VII. EFFECT OF THE VARIATION OF THE b_s

The purpose of the remaining sections is to derive sum rules which permit an order of magnitude estimate of the $\gamma_{\lambda s}$. The expression (6b) for these has, apart from the factor $(\hbar^2/2M)^{\frac{1}{2}}$, the same form as a transition probability. However, the two sets of functions X and ψ are defined in different spaces: the former in the internal region, the latter on the surface thereof. As a result, no mathematically rigorous formulas can be derived for them in general. The sum rules to be described below both involve assumptions which have

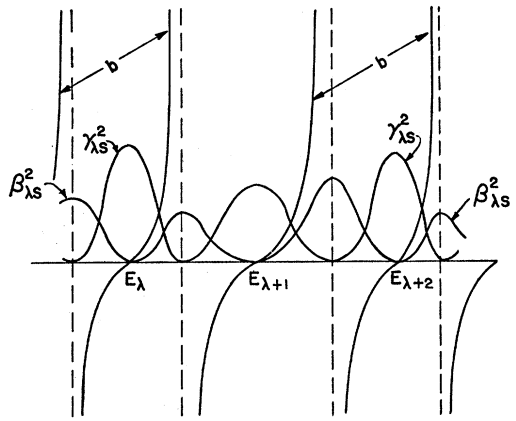

FIG. 1. b is the negative logarithmic derivative of the internal wave function in one of the channels. The abscissa is that value of the energy at which R has a pole when the boundary condition in Eq. (2) is $b_s=b$. Hence $E_\lambda, E_{\lambda+1}, E_{\lambda+2}, \cdots$ are energy values at which R has a pole for the boundary condition $b_s=0$. The value of the normalized wave function at the boundary is proportional to $\gamma_{\lambda s}$, and its derivative to $\beta_{\lambda s}$.

only approximate validity. The only simple interpretation of (6b) which is free of special assumptions is to regard it as an integral over the whole internal region. This is possible after replacing $\psi_s(i_s, \Omega_s)$ by ψ_s times the δ function of the distance of i_s from S in configuration space. Then, (6b) becomes an expansion coefficient of this function in terms of the X_λ. The sum of the squares of all expansion coefficients is equal to the square integral of the function. Hence

$$\sum_\lambda \gamma_{\lambda s}{}^2 = \infty, \tag{23}$$

because the square integral of the δ function is infinite. On the other hand, the integral of $\psi_s\psi_t$ vanishes over S and one infers from this that

$$\sum_\lambda \gamma_{\lambda s}\gamma_{\lambda t}=0 \quad \text{if} \quad s\neq t. \tag{23a}$$

This equation means that the signs of the $\gamma_{\lambda s}$ fluctuate irregularly with λ so that (23a) may be true for all s and t. It is this property of the $\gamma_{\lambda s}$ which enabled us to approximate Q_μ of (15) by the diagonal matrix $\bar{Q}$. The argument leading to (23), (23a) was given in more detail elsewhere,[7] where the derivation of (32) was also given on a similar basis.

It will be useful for the subsequent developments to investigate the dependence of the E_λ and X_λ on b_s in some more detail. We shall keep all b_t fixed, except one, b_u, and look for the energy values at which b_u assumes a given value b. The calculation will be carried out by means of the formulas of the first section which in principle permit the solution of every boundary value problem. We shall have, therefore, a basic set of X_λ, E_λ which obey (2) with fixed b_t, and which define the R to be used in the calculation. There will be another set of X, E and γ (but not of R) for which b_t are all equal to the corresponding b_t of the basis set, except the $b_u=b$ which will be variable. The b_u of the basic set will be chosen zero for convenience. The characteristic values E_λ and characteristic function X_λ of the set with variable b_u will be called $E_\lambda(b)$ and $X_\lambda(b)$ and a similar notation will be adopted for the $\gamma_{\lambda s}$.

A comparison of (5), (5a), with (2) shows that we are looking for a $\varphi[=X_\lambda(b)]$ for which $b_t\alpha_t+\beta_t=0$ for $t\neq u$ and $b\alpha_u+\beta_u=0$. As a result of the former relations, only the u term will remain in the summations in (6) and since $b_u=0$ in the basic set, we have

$$-\beta_u/b=\alpha_u=R_{uu}\beta_u. \tag{24}$$

The last equation shows that the $E_\lambda(b)$ is that value of E between E_λ and $E_{\lambda+1}$ at which R_{uu} becomes $-1/b$. There is always one and only one such E which we shall call $E_\lambda(b)$; the R_{tu} in (24) is supposed to be taken at this E. It then gives all the β in terms of β_u. The corresponding wave function, when normalized, is called $X_\lambda(b)$. From (7) and (7a) we have

$$X_\lambda(b)=\sum_\mu \frac{\gamma_{\mu u}X_\mu}{E_\mu-E_\lambda(b)}(\dot{R}_{uu})^{-\frac{1}{2}}. \tag{25}$$

Again, $\dot{R}_{uu}$ should be taken at $E=E_\lambda(b)$. From (6b) we further have

$$\gamma_{\lambda u}(b)=R_{uu}(\dot{R}_{uu})^{-\frac{1}{2}}. \tag{26}$$

This is actually the α_t for a normalized φ satisfying the above boundary conditions. The corresponding β will be denoted by $\beta_{\lambda t}(b)$. They are

$$\beta_{\lambda u}(b)=-b\gamma_{\lambda u}(b)=(\dot{R}_{uu})^{-\frac{1}{2}}. \tag{27}$$

Hence, the qualitative picture which emerges is as follows (see Fig. 1): As one increases $b_u=b$, the $E_\lambda(b)$ will also increase. At the same time $X_\lambda(b)$ and hence also the $\gamma_{\lambda u}$ will change. $E_\lambda(b)$ will remain finite as b becomes infinite and in order to further increase E_λ, one must let b jump to $-\infty$ and let it increase again. By the time b has reached its original value, $E_\lambda(b)$ will have shifted to where the next higher E_λ was originally.

Concurrent with the change of $E_\lambda(b)$, both $\gamma_{\lambda u}(b)^2$ and $\beta_{\lambda u}(b)^2$ will change, their ratio remaining, of course, $1/b^2$. For $b=0$, the $\beta_{\lambda u}(0)$ vanishes and $\gamma_{\lambda u}(0)$ is close to its maximum value. As b increases, $\beta_{\lambda u}(b)^2$ will increase and come close to its maximum value for $b=\pm\infty$. At this point $\gamma_{\lambda u}(b)^2$ is zero. As b starts to increase again from $-\infty$, $\gamma_{\lambda u}(b)^2$ becomes again finite and, when b has again reached its original value, both $\gamma_{\lambda u}(b)$ and $\beta_{\lambda u}(b)$ will have assumed the values of the $\gamma_{\lambda+1\,u}$ and $\beta_{\lambda+1\,u}$, respectively. At the same time $X_\lambda(b)$ will have changed to $\pm X_{\lambda+1}$. The same picture holds for every u, except that one may arrive at different signs for $X_{\lambda+1}$ by changing different b_u. Also, if ψ_u is a very highly excited state, practically all the increase of $E_\lambda(b)$ will take place in a very narrow range of b. It may be remarked finally that all the other $\gamma_{\lambda t}$ will change concurrently with $\gamma_{\lambda u}$. One can see, in fact, that if (23a) is valid, some of them must have gone through zero and changed their sign during the above process.

VIII. THE SUM RULES

If S is chosen close to the nuclear surface, the integrands of

$$(\hbar^2/2M)\int|X_\lambda(b)|^2dS \quad \text{and}$$

$$(\hbar^2/2M)\int|\text{grad}X_\lambda(b)|^2dS \quad (28)$$

will be just about as large on the surface as inside the nucleus on the average over a reasonably large range of $E_\lambda(b)$. The latter averages are, however, clearly $\hbar^2/2MV^A$ and T/V^A, where V is the volume of the nucleus, T the total kinetic energy, and A the number of nucleons in the nucleus. On the other hand, the sums

$$\sum_s\gamma_{\lambda s}(b)^2 \quad \text{and} \quad \sum_s\beta_{\lambda s}(b)^2, \quad (28a)$$

extended over all those states s which correspond to the same nuclear products, irrespective of their state of excitation, are equal to the integrals (28) over the corresponding part of the surface S. The area of this part of S is $V^{A-1}4\pi a_s^2$, the first factor arising from the integration over the internal coordinates of the residual nucleus, the second factor from integration over Ω_s. Hence on the average over $E_\lambda(b)$

$$\sum_s\gamma_{\lambda s}(b)^2=4\pi a_s^2V^{A-1}\hbar^2/2MV^A=3\hbar^2/2Ma_s. \quad (29)$$

$$\sum_s\beta_{\lambda s}(b)^2=(1/3A)4\pi a_s^2V^{A-1}T/V^A=T/Aa_s. \quad (29a)$$

The factor $\frac{1}{3}$ in this last expression originates from the fact that (5a) contains only one component of the grad, that perpendicular to S. On the other hand, all three components enter into the expression for the kinetic energy. T/A is, of course, the kinetic energy per nucleon.

The above formulas were derived without regard to the identity of the particles in the nucleus. If one takes this into account, one obtains a further factor N or Z in (29) and (29a), depending on the nature of the particle (neutron or proton) that constitutes the extra particle in the pair s. This extra factor is undoubtedly present in (29), (29a), if one extends the summation over all states of excitation of the residual nucleus, including those in which a neutron from one of the very low levels is missing. If one restricts the summation to the not too highly excited levels, the sum rule as given above should be valid. There is a similar phenomenon in the sum rule of atomic spectra. Strictly speaking, one can only claim that the sum of the f values is Z if one considers the transitions to all possible levels from the level in question. However, if one only considers transitions of a single electron, the sum of the f is close to 1 and, in particular, the f of a single transition from the normal state is probably never greater than 1. Similarly, no single $\gamma_{\lambda s}(b)^2$ can be expected[13] [on the average over $E_\lambda(b)$] to be larger than the right side of (29). Hence $\gamma^2/(3\hbar^2/2Ma_s)$ will play a similar role for the reduced width as the f value dues for optical transitions. This quantity is given in the sixth column of Table I.

Considering the accuracy with which the nuclear density remains constant, the writers believe that (29) is accurate within about 25 percent if it is averaged over a sufficiently wide range of energy. Because of the fluctuations which even the maxima of $\gamma_{\lambda s}(b)$ show, the range may have to extend over several resonances. However, what is needed for the earlier formulas is the value of $\gamma_{\lambda s}$ for a definite value of b, such as $b=0$ for slow neutron resonances. This can be only estimated from (29). If the energy of the nuclei which constitute s is below E_λ, one would expect that $\gamma_{\lambda s}(b)^2$, as function of $E_\lambda(b)$, is a function similar to the sine, as illustrated in Fig. 1, with the maxima and minima about equally broad. One can assume for the energetically possible s, and hence also for their sum, that the maxima are about twice as high as the averages given in (29). Together with $b\gamma_{\lambda s}(b)+\beta_{\lambda s}(b)=0$ this then gives

$$\sum_s{}'\gamma_{\lambda s}(b)^2<\frac{6\hbar^2T/Aa_s}{3\hbar^2b^2+2MT/A} \quad (30)$$

as the upper limit of the sum of $\gamma_{\lambda s}(b)^2$ for energetically possible s. We call (30) the first sum rule.

One can describe the first sum rule in less mathematical language. According to (6b), the reduced width $\gamma_{\lambda s}^2$ is (apart from the factor $\hbar^2/2M$) equal to the probability, for the state X_λ, that (a) the distance r_s of the pair of nucleons constituting s be in a unit interval at a_s, and (b) that the state of the products be described by ψ_s. Hence the sum of all $\gamma_{\lambda s}^2$ is simply the probability that r_s be between a_s and a_s+1. This probability depends on the boundary condition for X_λ, that is, on b. It is zero for infinite b; for $b=0$ it is twice as great as the ratio of the volume of a spherical shell of thickness 1 to the volume of the whole sphere of radius a_s. This is the meaning of (30). If s consists of a residual nucleus and a single extra particle, the left side of (30) will contain only one significant term if the total wave function X_λ is the product of the wave function for the stationary state s of the residual nucleus and of a function of the distance of the extra particle from the residual nucleus, i.e., if the single particle picture applies for X_λ. Hence in this case $\gamma_{\lambda s}^2$ will be close to the maximum value given by the right side of (30). It will not be exactly equal to this value because the probability distribution of the extra particle is not uniform throughout the volume of the nucleus if the single particle picture is valid. However, (30) will be valid in this case approximately with the summation sign omitted on the left side and the $<$ replaced by the $\sim$ sign.

In the opposite extreme case of a many particle picture we can still expand the compound state wave function

$$X_\lambda=\sum\sum_{tn}c_{\lambda tn}\psi_t u_n(r_t). \quad (31)$$

[13] R. de L. Kronig and H. Kramers, Z. Physik **48**, 174 (1928).

TABLE I. Widths and spacing of nuclear reaction resonance lines.

Reaction	E_λ kev	$\Gamma_{\lambda s}$ kev	$\gamma_{\lambda s}^2$ 10^{-13} kev cm	D kev	$\gamma_{\lambda s}^2/D$ 10^{-13} cm	$2Ma\gamma_{\lambda s}^2/3\hbar^2$ %	ev	References
He^4+n	1150	1550	9810			49		a
Li^7+n	270	45	3500	1000	3.5	17		b
$Li^7(p,\gamma)$	440	12	1300	1000	1.3	6.3	20	c, d, e
$Li^7(p,n)$	2230	200	410	1000	0.41	2.0		f, g
Be^9+n	625	35	290	1000	0.29	1.5		h, i
	2600	800	1180	1000	1.2	6.2		j
Be^9+p	305	164	1200	1000	1.2	6.3		d
	988	94	500	1000	0.50	2.6		e
	1077	4	21	1000	0.02	0.11		e
	2420	150	310	1000	0.31	1.6		k
$B^{11}+n$	430	45	1000	1000	1.0	5.6		i
$B^{11}+p$	162	5	1420	1000	1.4	7.9		d, l
$C^{12}+p$	453	35	3180	1000	3.2	18		d, e
$C^{13}+p$	550	40	2060	1000	2.1	12		d, e
$N^{14}(n,\alpha)$	2200	260	340	500	0.67	2.0		m
$O^{16}+n$	440	45	760	500	1.5	4.9		i
	1000	100	660	500	1.3	4.3		i
	1300	40	170	500	0.34	1.1		i
	1900	30	90	500	0.18	0.5		n
	2370	120	180	500	0.36	1.0		n
$O^{16}+p$	550		~5000	3000	1.7	33		o
	3110		48	3000	0.01	0.32		o
$F^{19}+n$	100	15	90	100	0.90	0.60		i
$Na^{23}+n$	3	0.17	7.1	100	0.07	0.05		p
$Mg^{24}+n$	85	23	1780	400	4.4	13		q
	275	145	2360	400	5.9	17		q
	430	62	590	400	1.5	4.2		q
	860	50	230	400	0.58	1.6		q
	2540	150	210	400	0.52	1.5		l
$Si^{28}+n$	195	66	340	400	0.85	2.5		q
	570	38	250	400	0.63	1.9		q
$S^{32}+n$	111	18	130	300	0.41	0.98		r
	375	12	45	300	0.15	0.35		r
	585	2	12	300	0.04	0.11		r
	700	12	30	300	0.10	0.24		r
$S^{32}(n,p)$	2390	80	120	300	0.39	1.0		s
	2800	80	110	300	0.36	0.90		s
	3100	200	270	300	0.89	2.2		s
	3460	200	240	300	0.81	1.9		s
$S^{32}(n,\alpha)$	3020	390	390	300	1.3	3.1		m
	ev	ev	10^{-13} kev cm	kev	10^{-13} cm	%		
$Cl^{35}+n$	−75	2.63	1.38	10	0.10	0.011		t
$V^{51}+n$	2700	780	68.4	100	0.68	0.62		u
	76,000	19,000	155	100	1.6	1.4		u
$Mn^{55}+n$	345	5	0.66	2	0.33	0.006	2	v
	2400	300	13.9	2	6.9	0.13		w
$Co^{59}+n$	126	5	1.0	2	0.50	0.008		x
$Ni+n$	15,000	4000	74	50	1.48	0.70		y
$Zn+n$	500	1.2	0.12	2	0.06	0.001	1.7	z
$Ge+n$	95	0.37	0.085	0.1	0.85	0.0008		aa
$Br+n$	36	0.16	0.118	0.1	1.18	0.0009		x
	ev	Mev	10^{-13} ev/cm	ev	10^{-13} cm	10^{-4} %	ev	
$Ru+n$	9.0	0.4	0.3	20	0.02	0.03		bb
$Rh^{103}+n$	1.3	0.33	0.65	20	0.03	0.08	0.14	cc
$Pd^{108}+n$	24	49	23	50	0.46	2.7	0.14	dd
$Ag^{109}+n$	5.1	11	12	50	0.24	1.4	0.12	ee, ff
	13	4.8	3	50	0.06	0.35		ee
$Ag^{107}+n$	45	13	4.5	50	0.09	0.52		ee, z
$Cd^{113}+n$	0.18	0.6	3.2	50	0.06	0.38	0.11	v, gg
$In^{115}+n$	1.4	2.7	5.2	10	0.52	0.62		hh
	3.8	3.7	5.0	10	0.50	0.60		hh
	8.6	21	16	10	1.6	1.9		hh
$Sb+n$	5.8	1.1	1.0	15	0.06	0.12		v
	15	8.1	4.8	15	0.32	0.58		v
	21	11.3	5.7	15	0.38	0.60		v
$Te^{123}+n$	2.2	15	2.3	10	0.23	0.28		ii
$I^{127}+n$	20	2.3	1.2	20	0.06	0.15		jj
$Sm^{149}+n$	0.096	0.024	5.5	40	0.14	0.72	0.03	cc
$Eu+n$	−0.011	0.008	0.17	10	0.02	0.02	0.08	cc
	0.54	1.25	3.8	10	0.38	0.49	0.07	cc
$Gd+n$	0.03	0.29	3.8	10	0.38	0.50	0.05	cc
$Dy+n$	−1.01	2.9	6.7	10	0.68	0.88		hh

TABLE I.—(*Continued*).

Reaction	E_λ ev	$\Gamma_{\lambda s}$ ev	$\gamma_{\lambda s}^2$ 10^{-13} ev cm	D ev	$\gamma_{\lambda s}^2/D$ 10^{-13} cm	$2Ma\gamma_{\lambda s}^2/3\hbar^2$ %	ev	References
$Ta^{181}+n$	4.1	1.4	1.5	10	0.14	0.21		hh
	10	1.9	1.4	10	0.14	0.20		hh
	13	0.3	0.19	10	0.02	0.03		hh
	22	1.4	0.66	10	0.07	0.09		hh
$W+n$	4.0	2.4	2.7	10	0.26	0.38		hh
	7.4	1.7	1.4	10	0.14	0.20		hh
$Re+n$	2.4	0.18	2.6	10	0.26	0.36		bb
$Os+n$	6.5	3.1	2.8	20	0.14	0.39		kk
	8.8	15	11	20	0.55	1.5		**kk**
	20	24	12	20	0.60	1.7		kk
$Ir+n$	0.64	0.29	0.79	10	0.08	0.11	0.07	v
	1.28	0.56	1.1	10	0.11	0.15	0.07	v
	5.2	4.4	4.4	10	0.44	0.62		v
	8.7	6.7	5.1	10	0.51	0.72		v
$Pt+n$	11.5	13.9	8.9	20	0.44	1.3		hh
	18.2	12	6.4	20	0.32	0.91		hh
$Au^{197}+n$	4.8	15	16	40	0.40	2.3		hh, dd
$Hg^{199}+n$	−2.0	27	31	50	0.62	4.4		ll
$U^{238}+n$	11	8.6	5.9			0.89		mm
	kev	kev	10^{-13} kev cm	kev	10^{-13} cm	%		
$Pb^{208}+n$	250	10	70	400	0.18	0.029		nn
	525	10	50	400	0.13	0.021		nn
	720	10	40	400	0.10	0.017		nn

[a] Bashkin, Mooring, and Petree, Phys. Rev. **82**, 379 (1951).
[b] R. K. Adair, Phys. Rev. **79**, 1018 (1950).
[c] Fowler, Lauritsen, and Lauritsen, Revs. Modern Phys. **20**, 263 (1948).
[d] Hole, Holtsmark, and Tangen, Z. Physik **118**, 48 (1941).
[e] E. R. Cohen and R. F. Christy, Phys. Rev. **74**, 1560 (1948).
[f] R. Taschek and A. Hemmendinger, Phys. Rev. **74**, 373 (1948).
[g] G. Breit and I. Bloch, Phys. Rev. **74**, 397 (1948).
[h] Adair, Barschall, Bockelman, and Sala, Phys. Rev. **75**, 1124 (1949).
[i] C. K. Bockelman, Phys. Rev. **80**, 1011 (1950).
[j] Allen, Burcham, and Wilkinson, Proc. Roy. Soc. (London) **A192**, 114 (1947).
[k] K. E. Davis and E. M. Hafner, Phys. Rev. **73**, 1242 (1948).
[l] M. R. MacPhail, Phys. Rev. **57**, 669 (1940).
[m] A. Stebler and P. Huber, Helv. Phys. Acta **21**, 59 (1948).
[n] Bockelman, Miller, Adair, and Barschall, Phys. Rev. **84**, 69 (1951).
[o] R. A. Laubenstein and M. J. W. Laubenstein, Phys. Rev. **84**, 18 (1951). There are also higher resonance levels with characteristics similar to the $O^{16}+n$ levels. This article, together with that of R. K. Adair, Phys. Rev. **81**, 310 (1951), and that of H. L. Jackson and A. I. Galonsky, Phys. Rev. **84**, 401 (1951), gives a very neat and practical method for analyzing angular distributions. See also H. Marschall and D. Wiskott, Z. Physik **129**, 619 (1951).
[p] Hibdon, Muelhause, Selove, and Woolf, Phys. Rev. **77**, 730 (1949).
[q] R. E. Fields and M. Walt, Phys. Rev. **83**, 479 (1951).
[r] Barschall, Bockelman, and Petersen, Phys. Rev. **79**, 593 (1950).
[s] E. Bleuler, Helv. Phys. Acta **20**, 519 (1947).
[t] C. T. Hibdon and C. O. Muelhause, Phys. Rev. **79**, 44 (1950).
[u] A. W. McReynolds and R. J. Weiss, Phys. Rev. **83**, 171 (1951).
[v] Rainwater, Havens, Wu, and Dunning, Phys. Rev. **71**, 65 (1947).
[w] Harris, Hibdon, and Muelhause, Phys. Rev. **80**, 1014 (1950).
[x] L. J. Rainwater and W. W. Havens, Phys. Rev. **83**, 1123 (1951).
[y] Barschall, Bockelman, and Seagondollar, Phys. Rev. **73**, 659 (1948).
[z] Coster, Groendijk and de Vries, Physica **14**, 1 (1948).
[aa] L. J. Rainwater, Washington Meeting, American Physical Society (1949).
[bb] Goldsmith, Ibser, and Feld, Revs. Modern Phys. **19**, 259 (1947).
[cc] Borst, Ulrich, Oxbourne, and Hasbrouck, Phys. Rev. **70**, 557 (1946).
[dd] M. Goldhaber and L. Lowry, private communication.
[ee] L. J. Rainwater and W. W. Havens, Phys. Rev. **70**, 154 (1946).
[ff] W. Selove, Phys. Rev. **77**, 557 (1950).
[gg] W. W. Beeman, Phys. Rev. **72**, 986 (1947).
[hh] Rainwater, Havens, Wu, and Meaker, Phys. Rev. **71**, 165 (1947).
[ii] C. Heindl and I. W. Ruderman, Phys. Rev. **83**, 660 (1951).
[jj] W. B. Jones, Phys. Rev. **72**, 362 (1947).
[kk] Rainwater, Havens, and Wu, Phys. Rev. **71**, 174 (1947).
[ll] Harris, Hibdon, Ringo, and Robillard, Phys. Rev. **82**, 560 (1951).
[mm] H. L. Andersen, Phys. Rev. **80**, 499 (1950).
[nn] Peterson, Adair, and Barschall, Phys. Rev. **79**, 935 (1950).

In the sum, it is necessary to consider only those t which correspond to the separation of the compound state into the same constituents though in different states of excitation. We shall consider the case of the separation into a residual nucleus and a single extra particle; the summation has to be extended then over all the stationary states of the residual nucleus and t in (31) enumerates these stationary states. In contrast to the single particle picture, many of the terms are now of about equal significance in (31). The $u_n(r_t)$ denotes the various single particle wave functions of the extra nucleon. The $\gamma_{\lambda s}$ now becomes

$$\gamma_{\lambda s}=(\hbar^2/2M)^{\frac{1}{2}}\sum_n c_{\lambda sn}u_n(a_s). \tag{31a}$$

If a_s is the nuclear radius, the amplitude of the single particle wave functions $u_n(r_t)$ may be taken at least roughly constant over the volume of the nucleus. Hence, because of the normalization condition, the integral of the square of $u_n(a_s)$ over Ω_s will be of the order of $3/a_s$, its magnitude also depending on the boundary condition at a_s, i.e., on b. If (31) consists of substantially only one term, the corresponding c_{sn} will

be unity and (31a) leads us back to the formula valid in the case of the single particle picture. However, if many of the c_{sn} are finite, the square of (31a) can be estimated to be

$$\gamma_{\lambda s}^2=(3\hbar^2/2Ma_s)\sum_n c_{\lambda sn}^2. \tag{31b}$$

In (31b), the cross terms have been neglected because they are as likely to be positive as negative. The sum in (31b) must be expected to be considerably smaller than 1 because the normalization of (31) demands that the sum of $c_{\lambda tn}^2$ over n and t be 1 and the terms with $t=s$ constitute only a small fraction of all the terms. We must therefore expect that the values in the seventh column of Table I drop from a value close to unity, in the region in which the single particle picture is at least approximately valid for the compound states, to a very small fraction of this value in the region where the many particle picture is appropriate for the compound nucleus. This is in fact what is observed, the $\gamma_{\lambda s}^2$ which amounts to several percent of $3\hbar^2/2Ma_s$ for light elements, dropping to a few parts per million of this quantity in the slow neutron resonances of heavier nuclei. It is interesting to note in this connection that, at low A, γ^2 generally decreases with increasing excitation of the compound nucleus. This is what would be expected from our picture, because the single particle model becomes less accurate with increasing excitation. We may remark parenthetically, though it does not have much to do with our subject proper, that there are indications of a decrease of level density as one passes the magic numbers, and that this is particularly evident at the end of the table.

In fact, (31b) permits a somewhat closer estimate of $\gamma_{\lambda s}^2$. Because of the orthonormal property of the X_λ, the sum of the $c_{\lambda sn}^2$ over λ must be unity. There is, however, nothing special about any particular state λ and it will have to share about evenly in the finite sum of $c_{\lambda sn}^2$ with all other resonances nearby. Let us assume that $c_{\lambda sn}^2$ is about evenly distributed over the λ for which E_λ is within the energy range W_n. The average value of one $c_{\lambda sn}^2$ will then be D/W_n where D is the level spacing around E_λ. Naturally, we know very little about W_n. The smallest length that we can give it will be the difference between the energy values of the single particle levels n and $n+1$. This gives $W_n\sim n\hbar^2\pi^2/Ma_s$. Under this assumption, only one n contributes to the sum (31b) and we obtain the estimate

$$\gamma_{\lambda s}^2\simeq 3a_sD/2\pi^2n=3D/2\pi K', \tag{32}$$

where $K'=n\pi/a_s$ is the average wave number of the particle in the nucleus. One recognizes in (32) the well-known estimate of Bethe, and Feshbach, Peaslee, and Weisskopf.[11] If W_n is larger than the above estimate, every term in (31b) will be correspondingly smaller, but several terms will be finite and the result will not be much altered.

In the derivation above, the origin of the estimate (32) is the sum rule for $\sum_\lambda c_{\lambda sn}^2$ while the sum $\sum\sum_{sn} c_{\lambda sn}^2$ was significant for (30). One may call (32), therefore, the second sum rule. The present interpretation thereof uses the wave functions in configuration space rather than the semiclassical picture or the single particle picture employed in earlier derivations. Its result, however, differs in only one important respect from the ones given before. One would expect from our picture that the $\gamma_{\lambda s}^2$ show considerable fluctuations from resonance to resonance. While the earlier work is not outright inconsistent with the existence of such fluctuations, one would at first rather expect, on the basis of the picture on which they are predicated, that the $\gamma_{\lambda s}^2$ are all very closely alike. The present experimental evidence, summarized in the sixth column of Table I seems to indicate that individual $\gamma_{\lambda s}^2$ may deviate considerably from the average given in (32). It also indicates that the average value of $\gamma_{\lambda s}^2/D$ shows a considerable decrease from its value of around 2×10^{-13} cm at the light elements, where the single particle picture is approximately valid, to about a tenth of this value for the resonances of heavier elements where an opposite picture may be more appropriate. While such a behavior does not obviously follow from our picture, it can be easily interpreted on the basis of the point of view here adopted. It is possible, for instance, that one relatively low-lying level, for which the single particle picture still applies, is closely approximated by $\psi_s u_s$ so that relatively little of the sum $\sum_\lambda c_{\lambda sn}^2$ remains for the states responsible for slow neutron resonances. The aforementioned decrease of the $\gamma_{\lambda s}^2/D$ along Table I becomes even more pronounced if one multiplies the right side of (32) by a further factor $2(1+3\hbar^2b^2A/2MT)^{-1}$ in order to take into account the dependence on b at least approximately. The γ^2/D is particularly small at the slow neutron resonances for which $b=0$.

A more mathematical derivation of the second sum rule is implicit in Eqs. (27). Written out in more detail, it reads

$$\beta_{\lambda s}(b)^{-2}=\sum_\mu \frac{\gamma_{\mu s}^2}{[E_\mu-E_\lambda(b)]^2}. \tag{33}$$

In the sum on the right side, the terms $\mu=\lambda$ and $\mu=\lambda+1$ will be, in general, considerably larger than the other terms. If b has a value which places $E_\lambda(b)$ midway between E_λ and $E_{\lambda+1}$, the sum on the right side will be close to $\pi^2(\gamma_{\lambda s}^2+\gamma_{\lambda+1 s}^2)/2D^2$ with $D\sim E_{\lambda+1}-E_\lambda$. If we assume in the sense of the above developments that $\beta_{\lambda s}(b)^2\sim(2MT/A\hbar^2)\gamma_{\lambda s}^2$ the above equation becomes

$$\tfrac{1}{2}(\gamma_{\lambda s}^2+\gamma_{\lambda+1 s}^2)\simeq(D/\pi\hbar)(2MT/A)^{-\frac{1}{2}}\simeq D/\pi K, \tag{32a}$$

with K the average wave number of a nucleon. This argument seems also to show that there is a connection between individual values of the level spacing D and the strengths of the resonances at the two sides there of. Such a connection also seems to follow from earlier work.[7,11]

IX. DISCUSSIONS OF THE APPROXIMATIONS

Throughout the considerations of the last section the boundary of the internal region was placed quite close to the nuclear surface. Since the nuclear forces have a finite range, the wave function will not have the form (3) in the external region. This circumstance may impair the results of the preceding section to an extent which is not easy to estimate. If the sole effect were a departure of the external region interaction from Coulomb's law, it would only affect b which does not enter any of the formulas in the crucial fashion. This then means that the $\gamma_{\lambda s}^2$ which occur in the preceding section and which refer to an internal region of the size of the nucleus cannot be calculated accurately in practice from the observed width.

The impossibility of writing the wave functions in the external region in the form (3) with any F and G, i.e., the possibility of a nuclear reaction outside the very small internal region of the last section, might limit the validity of the formulas of the preceding section more severely. Reactions of the nature just mentioned, such as the Oppenheimer-Phillips process, the stripping and pick-up processes[14] do in fact play a dominant role under certain conditions, and they certainly invalidate the sum rule of the preceding section. Somewhat fortunately, if these processes become important, the considerations of the first five sections, although perfectly valid with a sufficiently large internal region, cease to give significant results. Hence there is little temptation to apply either the dispersion theory or the sum rules to interpret the aforementioned processes.

A second point which should be brought up is the fact that the γ, in terms of which the widths were calculated in the third and fourth sections, referred to the reduced matrix $\mathfrak{R}$ rather than the original R. As far as the sum rule (32) is concerned, this circumstance is without significance. The average value of $\gamma_{\lambda s}^2/D$ is equal[12] to $1/\pi$ times the imaginary part of $R_{ss}(E+iE')$ where E' is large as compared with the level spacing but small enough so that D and the average $\gamma_{\lambda s}^2$ do not change appreciably in an energy interval E'. Because of (23a), the value of $R_{st}(E+iE')$ becomes zero for $s \neq t$ for such an E'. It then follows from (9a) that $\mathfrak{R}$ and R_{rr} are equal at $E+iE'$ so that the average $\gamma_{\lambda s}^2/D$ has the same value, no matter whether the γ occur in the expansion (6a) of R or in the similar expansion of $\mathfrak{R}$.

It is not difficult to convince oneself that the level spacing of $\mathfrak{R}$ is smaller than that of R. The reason is that the boundary conditions in the closed channels become more favorable[15] with increasing energy. Since, according to the preceding paragraph γ^2/D is the same for $\mathfrak{R}$ and for R_{rr}, it follows that the average of the $\gamma_{\mu s}^2$ which occurs in the expansion of R is smaller than the similar quantity in the expansion of R_{rr}. However, the effect is not very large. It is in the direction to make (30) more valid.

Table I gives the relevant data for all reactions in which the width of a resonance line has been measured. We intended to include all data published up to the fall of 1951. The first column gives the colliding particles, i.e., the s of our notation. The position of the level above the normal state of the constituents of s is given in the second column, the third gives the observed partial width $\Gamma_{\mu s}$. The $\gamma_{\mu s}^2$ of the fourth column were calculated by the first of the Eqs. (18a), using as radius a_s (the variable in $G_s^2+F_s^2$) the radius of the residual nucleus augmented by the "radius of the neutron (or proton)," 1.4×10^{-13} cm. This convention was adopted to conform with the calculations of Christy and Latter[16] and with our own earlier work on the subject. As was emphasized above, the sum rules refer to the γ^2 for a position of S which hugs the residual nucleus so that the "proton radius" should not have been added to that of the residual nucleus. In the case of a neutron with zero angular momentum, $G_s^2+F_s^2=1/k_s$ is independent of r_s, so that for most of the Table the choice of a_s is irrelevant. This is not so in the case of the proton resonances at the beginning of the Table. In these cases the γ^2 calculated at the smaller radius would be considerably higher than the values given in the fourth column. Such a change would further increase the values in the sixth and seventh columns. However, as was mentioned before, the Coulomb potential is probably not unaffected in so close a neighborhood of the nucleus and the calculation of the penetration factor in the region in which specifically nuclear forces are significant is at present impossible.

The fifth column gives the level spacing. It was attempted to take into consideration the isotopic constitution of the elements involved and the possible spin values of the compound nucleus. The values given refer to the average spacing of the levels with a definite J of a particular isotope. However, we found it very difficult to estimate this quantity from existing data. The significance of the sixth and seventh columns has been mentioned before. The radiation width is given in the eighth column in the cases in which it has been measured.

[14] J. R. Oppenheimer and M. Phillips, Phys. Rev. **43**, 500 (1935); R. Serber, Phys. Rev. **72**, 1008 (1947); S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951).

[15] P. M. Morse and H. Feshback, *Methods of Theoretical Physics* (Technology Press, Cambridge, Massachusetts, 1946), Chap. VI.

[16] R. F. Christy and R. Latter, Revs. Modern Phys. **20**, 185 (1948).