The Quantum Theory of Interacting Gravitational and Spinor Fields

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A brief treatment of spinors in general coordinates is given. The mathematical results are used to derive the Hamiltonian for a spinor field interacting with the gravitational field. The Hamiltonian formulation follows a method developed by Dirac, which has been used by Pirani and Schild to treat the free gravitational field. The Dirac method is further used here to carry out a reduction of the Hamiltonian to a form suitable for quantization. The quantization procedure is actually given, and the problems arising in connection with it—such as the factor ordering ambiguity and the vacuum expectation value of the spinor stress density—are discussed.

1. INTRODUCTION

THE Hamiltonian formulation of Einstein's gravitational field equations, which has recently been accomplished by Pirani and Schild[1] and independently by Bergmann[2] and his co-workers, has enabled workers in general relativity to consider seriously the possibility of carrying out a rigorous quantization of Einstein's theory. Bergmann and his group hope to develop a quantum theory of the motions of point singularities (particles of matter) in an otherwise "free" gravitational field; i.e., a quantum version of the work of Einstein, Infeld, and Hoffmann.[3] Schild's group, on the other hand, take the more direct course of describing gravitating matter (as well as electromagnetic radiation) by means of additional fields which interact with the gravitational field. The present paper is written in the latter vein.

The addition of fields having tensor transformation properties introduces no difficulties, as is evident from the brief treatment of the electromagnetic field in [PS]. Spinor fields, however, require special handling. In this paper the Hamiltonian for a spinor field interacting with the gravitational field will be derived. In addition to the problems arising in the course of this derivation, a number of others will appear when the attempt is made to pass to the quantum theory. All these will be discussed.

In this paper Latin indices range over the values 1,2,3 and Greek indices over the values 1,2,3,4. Use of three real coordinates x^i and one imaginary coordinate x^4 will be made from the outset, with an eye toward facilitating the direct transcription of the results obtained here into an eventual quantum perturbation theory of electron-graviton interactions. With this convention the determinant g of the metric tensor $g_{\mu\nu}$ is positive; the invariant volume element is $g^{\frac{1}{2}}dx^1dx^2dx^3dx^0$, where $x^0=-ix^4$; and the conjugate of a tensor $T^{\mu\cdots}{}_{\nu\cdots}$ is defined by

$$\bar{T}^{\mu\cdots}{}_{\nu\cdots}=(-)^q(T^{\mu\cdots}{}_{\nu\cdots})^*,$$

where q is the number of times the index 4 occurs among the $\mu\cdots$, $\nu\cdots$. In the case of c-numbers the asterisk denotes the ordinary complex conjugate; in the case of operators or matrices it denotes the Hermitian adjoint.

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[1] F. A. E. Pirani and A. Schild, Phys. Rev. **79**, 986 (1950); this paper will be referred to as [PS].

[2] Bergmann, Penfield, Schiller, and Zatzkis, Phys. Rev. **80**, 81 (1950); J. Heller and P. G. Bergmann, Phys. Rev. **84**, 665 (1951). In the latter paper the Hamiltonian formulation has been carried out in spinor form. The result is entirely equivalent to the more usual formulation in terms of the metric tensor. It is to be emphasized that the work of the present authors differs from that of Heller and Bergmann in that the spinor quantities which appear here are extra quantities (added to the gravitational field quantities) and have no metric significance.

[3] Einstein, Infeld, and Hoffmann, Ann. Math. **39**, 1, 66 (1938); A. Einstein and L. Infeld, Can. J. Math. **1**, 209 (1949).

2. SPINORS IN GENERAL COORDINATES

Spinors are most conveniently treated in general coordinates after the manner of Pauli.[4] Since the details of Pauli's formalism will be of importance, we present it here in a modified version which is specially adapted to the present problem.

One generalizes the Dirac matrices γ_μ by introducing a set of matrices Γ^μ satisfying[5]

$$\{\Gamma^\mu,\Gamma^\nu\}=2g^{\mu\nu}. \tag{2.1}$$

As corollaries of (2.1) one may readily derive the identities[5]

$$[\Gamma^\mu,[\Gamma^\nu,\Gamma^\sigma]]=4(g^{\mu\nu}\Gamma^\sigma-g^{\mu\sigma}\Gamma^\nu), \tag{2.2}$$

$$\{\Gamma^\mu,[\Gamma^\nu,\Gamma^\sigma]\}=\tfrac{2}{3}[\Gamma^\mu,\Gamma^\nu,\Gamma^\sigma]. \tag{2.3}$$

The Γ^μ may be constructed so as to be differentiable[6] matrix functions of the coordinates by observing that the contravariant metric tensor will be altered by an infinitesimal amount $\delta g^{\mu\nu}$ if the Γ^μ are altered by infinitesimal amounts given by

$$\delta\Gamma^\mu=\tfrac{1}{2}\Gamma_\nu\delta g^{\mu\nu}. \tag{2.4}$$

By keeping the $\delta g^{\mu\nu}$ always differentiable the Γ^μ may be built up continuously from the Dirac matrices γ_μ, while the metric tensor is built up from its value $\delta_{\mu\nu}$ in orthonormal rectilinear coordinates in flat space-time.

[4] W. Pauli, Ann. Physik **18**, 337 (1933).

[5] The following bracket notation is used here:

$$\{A,B\}=AB+BA,\quad [A,B]=AB-BA,$$
$$[A,B,C]=A[B,C]+B[C,A]+C[A,B].$$

In Sec. 4 the notation (F,G) will be used to denote the Poisson bracket of F and G.

[6] We mean differentiable up to any order required in the discussion.

It is evident that the Γ^μ have the same group theoretical properties as the γ_μ, namely, (I) there exists only one irreducible family of equivalent representations of the Γ^μ (which is 4-dimensional); (II) if $[X, \Gamma_\mu]=0$ for all μ, then X is a multiple of the unit matrix; (III) if $\{\Gamma'^\mu, \Gamma'^\nu\}=2g^{\mu\nu}$, then there must exist a matrix function S such that

$$\Gamma'^\mu = S^{-1}\Gamma^\mu S, \quad |S|=1. \tag{2.5}$$

Equation (2.5) defines a "transformation in spin-space." It is important in what follows to keep a clear distinction between spin-space transformations and coordinate transformations. Under coordinate transformations the Γ^μ will transform like the components of a contravariant vector.

The Γ^μ are not uniquely determined by the construction (2.4) since the expression on the right is not an exact differential. The final form for the Γ^μ will depend upon the path of integration taken in the 10-dimensional space of the $g_{\mu\nu}$. If we adopt some integration convention, however, then we may write[7]

$$\Gamma^\mu = \sqrt{g^\mu}_{\nu'}\gamma_{\nu'}, \tag{2.6}$$

where the coefficients $\sqrt{g^\mu}_{\nu'}$ are certain functions of the $g_{\mu\nu}$, satisfying

$$\sqrt{g^\mu}_{\sigma'}\sqrt{g^\nu}_{\sigma'} = g^{\mu\nu}. \tag{2.7}$$

Indices which are not tensor indices have been written with a prime.

The process of choosing an explicit form (2.6) for the Γ^μ is equivalent to completing the differential form (2.4) by adding a suitable term. Such an additional term must correspond to an infinitesimal spin-space transformation $S=1+F^{\alpha\beta}\delta g_{\alpha\beta}$, where the $F^{\alpha\beta}$ are certain matrix functions of the $g_{\mu\nu}$ having vanishing traces. Using the identity

$$\partial g^{\mu\nu}/\partial g_{\alpha\beta} = -\tfrac{1}{2}(g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}), \tag{2.8}$$

we may therefore write

$$\partial\Gamma^\mu/\partial g_{\alpha\beta} = -\tfrac{1}{4}(g^{\mu\alpha}\Gamma^\beta+g^{\mu\beta}\Gamma^\alpha)-[F^{\alpha\beta}, \Gamma^\mu]. \tag{2.9}$$

The necessary identity $\partial^2\Gamma^\mu/\partial g_{\alpha\beta}\partial g_{\gamma\delta} \equiv \partial^2\Gamma^\mu/\partial g_{\gamma\delta}\partial g_{\alpha\beta}$ leads, with the aid of (2.2), to the following condition on the functions $F^{\alpha\beta}$:

$$\begin{aligned}\partial F^{\alpha\beta}/\partial g_{\gamma\delta} - \partial F^{\gamma\delta}/\partial g_{\alpha\beta} - [F^{\alpha\beta}, F^{\gamma\delta}] \\ = (1/64)(g^{\alpha\gamma}[\Gamma^\beta, \Gamma^\delta]+g^{\beta\delta}[\Gamma^\alpha, \Gamma^\gamma] \\ +g^{\alpha\delta}[\Gamma^\beta, \Gamma^\gamma]+g^{\beta\gamma}[\Gamma^\alpha, \Gamma^\delta]).\end{aligned} \tag{2.10}$$

In terms of the coefficients $\sqrt{g^\mu}_{\nu'}$, $F^{\alpha\beta}$ has the explicit form

$$F^{\alpha\beta} = \tfrac{1}{8}\sqrt{g}_{\sigma\mu'}(\partial\sqrt{g^\sigma}_{\nu'}/\partial g_{\alpha\beta})[\gamma_{\mu'}, \gamma_{\nu'}]. \tag{2.11}$$

The integration convention leading to the explicit form (2.6) may be chosen differently at each point of space-time. The functional form of the $\sqrt{g^\mu}_{\nu'}$ will therefore, in general, involve the x^μ as parameters. That is, the $\sqrt{g^\mu}_{\nu'}$, and hence the Γ^μ, will have an explicit dependence on the coordinates in addition to an implicit dependence through the $g_{\mu\nu}$. In order to discuss this type of situation it will be convenient to distinguish between "total" and "partial" coordinate differentiation. The symbol $\partial/\partial x^\mu$ will be allowed to act only on the explicitly occurring x's. If f is an explicit function of the x's as well as of a set of field variables y_A, then its total derivative with respect to x^σ is defined by

$$f_{,\sigma} \equiv (\partial f/\partial y_A)y_{A,\sigma}+\partial f/\partial x^\sigma. \tag{2.12}$$

$y_{A,\sigma}$ is the gradient of y_A. Repeated total differentiation is denoted by $f_{,\sigma\tau\rho\cdots}$, etc.

In addition to allowing the functional form of the $\sqrt{g^\mu}_{\nu'}$ in (2.6) to vary from point to point, one may also impose an arbitrary spin-space transformation S_0 which has an explicit dependence on both the g's and the x's. The Γ's and F's then take their most general forms

$$\Gamma^\mu(g, x) = \sqrt{g^\mu}_{\nu'}S_0^{-1}\gamma_{\nu'}S_0, \tag{2.13}$$

$$\begin{aligned}F^{\alpha\beta}(g, x) = \tfrac{1}{8}\sqrt{g}_{\sigma\mu'}(\partial\sqrt{g^\sigma}_{\nu'}/\partial g_{\alpha\beta})S_0^{-1}[\gamma_{\mu'}, \gamma_{\nu'}]S_0 \\ +S_0^{-1}(\partial S_0/\partial g_{\alpha\beta}).\end{aligned} \tag{2.14}$$

In any given coordinate system the dependence of the Γ's on the g's and x's will now have a particular functional form. Owing to the fact, however, that the coordinate transformation law of the Γ's is given by

$$\Gamma'^\mu = (\partial x'^\mu/\partial x^\nu)\Gamma^\nu(g, x) \tag{2.15}$$

rather than by

$$\Gamma''^\mu = \Gamma^\mu(g', x'), \tag{2.16}$$

this functional form will be altered whenever we carry out a coordinate transformation. A fixed functional form can be maintained only if every coordinate transformation $x\rightarrow x'$ is accompanied by a spin-space transformation S such that $S^{-1}\Gamma'^\mu S = \Gamma''^\mu$. For the infinitesimal coordinate transformation

$$x'^\mu = x^\mu - \delta\Lambda^\mu(x) \tag{2.17}$$

one may readily show that the corresponding spin-space transformation is given by

$$S = 1-\Delta_\mu\delta\Lambda^\mu+(2F_\mu{}^\nu+\tfrac{1}{8}[\Gamma_\mu, \Gamma^\nu])\delta\Lambda^\mu{}_{,\nu}, \tag{2.18}$$

where

$$\Delta_\mu = \tfrac{1}{8}\sqrt{g}_{\nu\sigma'}(\partial\sqrt{g^\nu}_{\tau'}/\partial x^\mu)S_0^{-1}[\gamma_{\sigma'}, \gamma_{\tau'}]S_0+S_0^{-1}(\partial S_0/\partial x^\mu), \tag{2.19}$$

satisfying

$$[\Delta_\mu, \Gamma^\nu] = -\partial\Gamma^\nu/\partial x^\mu. \tag{2.20}$$

Equation (2.18) is the generalization to general coordinates of the familiar spinor transformation law in Minkowski space. In the subsequent discussion, however, nothing will be gained by imposing (2.18), and we shall leave coordinate and spin-space transformations completely independent of one another. In this way the matrices $F^{\alpha\beta}$ will transform under $x\rightarrow x'$ like the components of a contravariant tensor and under spin-space transformations S according to

$$F'^{\alpha\beta} = S^{-1}F^{\alpha\beta}S+S^{-1}(\partial S/\partial g_{\alpha\beta}), \tag{2.21}$$

while the matrices Δ_μ will have the transformation laws

$$\left.\begin{aligned}\Delta_\mu' &= (\partial x^\nu/\partial x'^\mu)\Delta_\nu-(\partial x'^\gamma/\partial x^\beta)(\partial^2x^\alpha/\partial x'^\gamma\partial x'^\mu) \\ &\qquad\times\{\tfrac{1}{8}[\Gamma_\alpha, \Gamma^\beta]+2F_\alpha{}^\beta\} \\ \text{and}\quad \Delta_\mu' &= S^{-1}\Delta_\mu S+S^{-1}(\partial S/\partial x^\mu).\end{aligned}\right\} \tag{2.22}$$

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[7] For example, we may choose the straight-line path of integration from $\delta_{\mu\nu}$ to $g_{\mu\nu}$. We may then expand the "inverse square root" $\sqrt{g^\mu}_{\nu'}$ according to the binomial theorem

$$\sqrt{g^\mu}_{\nu'} = \delta_{\mu\nu}-\tfrac{1}{2}\varphi_{\mu\nu}+\frac{1\cdot3}{2\cdot4}\varphi_{\mu\alpha}\varphi_{\alpha\nu}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\varphi_{\mu\alpha}\varphi_{\alpha\beta}\varphi_{\beta\nu}+\cdots,$$

where $g_{\mu\nu}=\delta_{\mu\nu}+\varphi_{\mu\nu}$. The expansion converges whenever the eigenvalues of $\|\varphi_{\mu\nu}\|$ are all less than unity in absolute magnitude.

The identities $\partial^2\Gamma^\sigma/\partial x^\mu \partial g_{\alpha\beta} \equiv \partial^2\Gamma^\sigma/\partial g_{\alpha\beta}\partial x^\mu$ and $\partial^2\Gamma^\sigma/\partial x^\mu \partial x^\nu \equiv \partial^2\Gamma^\sigma/\partial x^\nu \partial x^\mu$ lead to the conditions

$$\partial\Delta_\mu/\partial g_{\alpha\beta} - \partial F^{\alpha\beta}/\partial x^\mu - [\Delta_\mu, F^{\alpha\beta}] = 0, \tag{2.23}$$

$$\partial\Delta_\mu/\partial x^\nu - \partial\Delta_\nu/\partial x^\mu - [\Delta_\mu, \Delta_\nu] = 0. \tag{2.24}$$

Pauli[4] introduces a convenient definition for the covariant derivative of Γ^μ by observing that the matrices $\Gamma'^\mu = \Gamma^\mu + \epsilon^\sigma\left(\Gamma^\mu{}_{,\sigma} + \left\{ {\mu \atop \alpha\sigma} \right\}\Gamma^\alpha\right)$, where ϵ^σ is an arbitrary infinitesimal contravariant vector, satisfy $\{\Gamma'^\mu, \Gamma'^\nu\} = 2g^{\mu\nu}$. This means that there exists a spin-space transformation $S = 1 + \epsilon^\sigma\Omega_\sigma$, such that $S^{-1}\Gamma^\mu S = \Gamma'^\mu$. Since ϵ^σ is arbitrary, this implies

$$0 = \Gamma^\mu{}_{,\sigma} + \left\{ {\mu \atop \alpha\sigma} \right\}\Gamma^\alpha + [\Omega_\sigma, \Gamma^\mu]$$
$$= (\partial\Gamma^\mu/\partial g_{\alpha\beta})g_{\alpha\beta,\sigma} + \delta\Gamma^\mu/\partial x^\sigma + \left\{ {\mu \atop \alpha\sigma} \right\}\Gamma^\alpha + [\Omega_\sigma, \Gamma^\mu]. \tag{2.25}$$

The solution of (2.25) is readily found to be

$$\Omega_\sigma = \tfrac{1}{8}g_{\sigma\mu,\nu}[\Gamma^\mu, \Gamma^\nu] + F^{\mu\nu}g_{\mu\nu,\sigma} + \Delta_\sigma. \tag{2.26}$$

Under coordinate transformations the matrices Ω_σ transform like the components of a covariant vector, while under spin-space transformations they transform according to

$$\Omega_\sigma' = S^{-1}\Omega_\sigma S + S^{-1}S_{,\sigma}. \tag{2.27}$$

Pauli calls the right-hand side of (2.25) the covariant derivative of Γ^μ. Using a subscript dot to denote the covariant derivative of any quantity, we may write

$$\Gamma_{\cdot\sigma} = 0. \tag{2.28}$$

Now, if the conjugate to Eq. (2.1) is taken in the form $2g^{\mu\nu} = \{-\bar{\Gamma}^\mu, -\bar{\Gamma}^\nu\}$, one is led to infer the existence of a matrix A with the property

$$-\bar{\Gamma}^\mu = A\Gamma^\mu A^{-1}, \quad |A| = 1. \tag{2.29}$$

Equation (2.29) and its conjugate together imply the commutability of $A^{*-1}A$ with the Γ^μ. This means $A^{*-1}A$ is equal to some multiple c of the unit matrix. Evidently $c^*c = 1$, and A may, without loss of generality, be multiplied by $c^{-\frac{1}{2}}$, thereby making it Hermitian. A is invariant under coordinate transformations, but it has the following spin-space transformation law:

$$A' = S^*AS, \tag{2.30}$$

which leaves its Hermitian character undisturbed. If the Γ^μ are constructed via (2.13) from a unitary-Hermitian set of Dirac γ's, then A has the explicit form

$$A = \pm S_0^*\gamma_4 S_0. \tag{2.31}$$

A spinor ψ is a 1-column matrix which is invariant under coordinate transformations and has the spin-space transformation law

$$\psi' = S^{-1}\psi. \tag{2.32}$$

The adjoint spinor is defined by

$$\bar{\psi} = \psi^*A, \tag{2.33}$$

and transforms according to

$$\bar{\psi}' = \bar{\psi}S. \tag{2.34}$$

Spinors may be combined with the Γ^μ, as in $\bar{\psi}\Gamma^\mu\Gamma^\nu\psi$, to form ordinary tensors which are invariant under spin-space transformations.

The covariant derivative of a spinor is defined by invoking the condition that covariant differentiation be distributive over factors in a product. Writing $(\bar{\psi}\Gamma^\mu\psi)_{\cdot\sigma} = \bar{\psi}_{\cdot\sigma}\Gamma^\mu\psi + \bar{\psi}\Gamma^\mu\psi_{\cdot\sigma}$, expanding the left-hand side according to the usual rules, and using (2.25), one finds

$$\psi_{\cdot\sigma} \equiv \psi_{,\sigma} + \Omega_\sigma\psi, \tag{2.35}$$

$$\bar{\psi}_{\cdot\sigma} \equiv \bar{\psi}_{,\sigma} - \bar{\psi}\Omega_\sigma. \tag{2.36}$$

If we write $\bar{\psi}_{\cdot\sigma} = \psi^*{}_{\cdot\sigma}A + \psi^*A_{\cdot\sigma} = (\psi^*{}_{,\sigma} + \psi^*\bar{\Omega}_\sigma)A + \psi^*A_{\cdot\sigma}$, we are led to a convenient definition for the covariant derivative of A

$$A_{\cdot\sigma} \equiv A_{,\sigma} - A\Omega_\sigma - \bar{\Omega}_\sigma A. \tag{2.37}$$

Taking the conjugate of (2.25) and using (2.27) and (2.29), we may infer $-\bar{\Omega}_\sigma = A\Omega_\sigma A^{-1} - A_{,\sigma}A^{-1}$. Multiplication of this equation on the right by A gives $A_{\cdot\sigma} = 0$. This result makes it easy to take the conjugates of tensor quantities of the form $\bar{\psi}\Gamma^\mu\psi_{\cdot\nu}$, etc. It also leads, in conjunction with the explicit form (2.26), to the identities

$$\partial A/\partial g_{\mu\nu} = AF^{\mu\nu} + \bar{F}^{\mu\nu}A, \tag{2.38}$$

$$\partial A/\partial x^\sigma = A\Delta_\sigma + \bar{\Delta}_\sigma A. \tag{2.39}$$

Spinors with tensor indices are readily introduced into the general coordinate formalism. For example, the vector-spinor ψ_μ, which transforms as a covariant vector under coordinate transformations and as a spinor under spin-space transformations, has a covariant derivative given by

$$\psi_{\mu\cdot\sigma} \equiv \psi_{\mu,\sigma} - \left\{ {\alpha \atop \mu\sigma} \right\}\psi_\alpha + \Omega_\sigma\psi_\mu. \tag{2.40}$$

This enables us to calculate the commutation relation for indices induced by repeated covariant differentiation. We find

$$\begin{aligned} \psi_{\cdot\mu\nu} - \psi_{\cdot\nu\mu} &= (\Omega_{\mu,\nu} - \Omega_{\nu,\mu} - [\Omega_\mu, \Omega_\nu])\psi \\ &= -\tfrac{1}{8}R_{\mu\nu\sigma\tau}[\Gamma^\sigma, \Gamma^\tau]\psi, \end{aligned} \tag{2.41}$$

where $R_{\mu\nu\sigma\tau}$ is the curvature tensor.[8]

3. THE LAGRANGIAN AND THE FIELD EQUATIONS

The Lagrangian density for the combined gravitational and spinor fields is conveniently taken in the form

$$\begin{aligned} \mathfrak{L} \equiv & -\beta^{-1}g^{\frac{1}{2}}g^{\mu\nu}\left(\left\{ {\alpha \atop \alpha\beta} \right\}\left\{ {\beta \atop \mu\nu} \right\} - \left\{ {\alpha \atop \mu\beta} \right\}\left\{ {\beta \atop \nu\alpha} \right\}\right) \\ & -\tfrac{1}{2}\hbar c g^{\frac{1}{2}}(\bar{\psi}\Gamma^\mu\psi_{\cdot\mu} - \bar{\psi}_{\cdot\mu}\Gamma^\mu\psi + 2\kappa\bar{\psi}\psi) \\ \equiv & -\tfrac{1}{4}\beta^{-1}g^{\frac{1}{2}}(g^{\alpha\delta}g^{\beta\epsilon}g^{\gamma\zeta} - 2g^{\alpha\delta}g^{\beta\zeta}g^{\gamma\epsilon} + 2g^{\alpha\gamma}g^{\beta\zeta}g^{\delta\epsilon} \\ & - g^{\alpha\beta}g^{\delta\epsilon}g^{\gamma\zeta})g_{\alpha\beta,\gamma}g_{\delta\epsilon,\zeta} - \tfrac{1}{2}\hbar c g^{\frac{1}{2}}\{\psi^*A\Gamma^\mu\psi_{,\mu} \\ & -\psi^*{}_{,\mu}A\Gamma^\mu\psi + \psi^*[(A\Gamma^\sigma F^{\mu\nu} + \bar{F}^{\mu\nu}\bar{\Gamma}^\sigma A)g_{\mu\nu,\sigma} \\ & + A\Gamma^\sigma\Delta_\sigma + \bar{\Delta}_\sigma\bar{\Gamma}^\sigma A + 2\kappa A]\psi\}, \end{aligned} \tag{3.1}$$

[8] Using the algebraic identities satisfied by the curvature tensor, one may prove $R_{\mu\nu\sigma\tau}\Gamma^\mu\Gamma^\nu\Gamma^\sigma\Gamma^\tau = 2R$, where R is the curvature scalar.

where

$$\beta = 16\pi G c^{-4}, \tag{3.2}$$

G being the Newtonian gravitational constant and c the velocity of light. The field equations are

$$\Gamma^\mu \psi_{\cdot\mu} + \kappa\psi = 0, \quad \bar{\psi}_{\cdot\mu}\Gamma^\mu - \kappa\bar{\psi} = 0, \tag{3.3}$$

$$g^{\frac{1}{2}}(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R) = -\tfrac{1}{2}\beta\Theta_{\mu\nu}, \tag{3.4}$$

where $\Theta_{\mu\nu}$ is the symmetric stress density of the spinor field

$$\Theta_{\mu\nu} \equiv \tfrac{1}{4}\hbar c g^{\frac{1}{2}}(\bar{\psi}\Gamma_\mu\psi_{\cdot\nu} - \bar{\psi}_{\cdot\nu}\Gamma_\mu\psi + \bar{\psi}\Gamma_\nu\psi_{\cdot\mu} - \bar{\psi}_{\cdot\mu}\Gamma_\nu\psi). \tag{3.5}$$

In virtue of Eqs. (3.3) the stress density satisfies the divergence condition[9]

$$\Theta^{\mu\nu}{}_{\cdot\nu} = 0. \tag{3.6}$$

4. THE HAMILTONIAN FORMULATION

In this section we use the notation of [PS][10] with a few minor modifications. Owing to the presence of an imaginary coordinate x^4, the Jacobian of the transformation $x^\mu \to u^i, t$ will be defined

$$J \equiv -i\partial(x)/\partial(\mathbf{u}, t). \tag{4.1}$$

The normals to the space-like surfaces

$$l_\sigma \equiv J t_{,\sigma} \tag{4.2}$$

are time-like covariant vectors having lengths il given by

$$l^2 = -g^{\alpha\beta}l_\alpha l_\beta. \tag{4.3}$$

The canonical momenta for the present problem are

$$\lambda_\mu = \partial(J\mathfrak{L})/\partial\dot{x}^\mu, \tag{4.4}$$

$$\begin{aligned}\pi^{\mu\nu} &= \partial(J\mathfrak{L})/\partial\dot{g}_{\mu\nu} \equiv (\partial\mathfrak{L}/\partial g_{\mu\nu,\sigma})l_\sigma \\ &= -\tfrac{1}{4}\beta^{-1}g^{\frac{1}{2}}(2g^{\alpha\mu}g^{\beta\nu}l^\gamma - 2g^{\alpha\mu}g^{\gamma\nu}l^\beta - 2g^{\alpha\nu}g^{\gamma\mu}l^\beta \\ &\quad + 2g^{\alpha\gamma}g^{\mu\nu}l^\beta + g^{\nu\gamma}g^{\alpha\beta}l^\mu + g^{\mu\gamma}g^{\alpha\beta}l^\nu - 2g^{\alpha\beta}g^{\mu\nu}l^\gamma)g_{\alpha\beta,\gamma} \\ &\quad - \tfrac{1}{2}\hbar c g^{\frac{1}{2}}\psi^*(A\Gamma^\sigma F^{\mu\nu} + \bar{F}^{\mu\nu}\bar{\Gamma}^\sigma A)\psi l_\sigma,\end{aligned} \tag{4.5}$$

$$\pi = (\partial\mathfrak{L}/\partial\psi_{,\sigma})l_\sigma \equiv -\tfrac{1}{2}\hbar c g^{\frac{1}{2}}\psi^* A\Gamma^\sigma l_\sigma, \tag{4.6}$$

$$\pi^* = (\partial\mathfrak{L}/\partial\psi^*_{,\sigma})l_\sigma = \tfrac{1}{2}\hbar c g^{\frac{1}{2}}A\Gamma^\sigma\psi l_\sigma. \tag{4.7}$$

The momenta $\pi^{\mu\nu}$ are here seen to differ from those in [PS] through the presence of a term involving the spinor field.

Equations (4.5)–(4.7) give rise to a number of what Dirac[11] calls "ϕ-equations" involving only coordinates and momenta

$$\phi = 0, \quad \phi^* = 0, \quad \phi^\mu = 0, \tag{4.8}$$

where

$$\phi \equiv \pi + \tfrac{1}{2}\hbar c g^{\frac{1}{2}}\psi^* A\Gamma^\sigma l_\sigma, \tag{4.9}$$

$$\phi^* \equiv \pi^* - \tfrac{1}{2}\hbar c g^{\frac{1}{2}}A\Gamma^\sigma\psi l_\sigma, \tag{4.10}$$

$$\begin{aligned}\phi^\mu \equiv\ & \pi^{\mu\nu}l_\nu + \tfrac{1}{4}\beta^{-1}g^{\mu\nu}(2g^{\alpha\gamma}l^\beta - g^{\alpha\beta}l^\gamma)T_{\alpha\beta\gamma\nu} \\ & + \tfrac{1}{2}\hbar c g^{\frac{1}{2}}\psi^*(A\Gamma^\sigma F^{\mu\nu} + \bar{F}^{\mu\nu}\bar{\Gamma}^\sigma A)\psi l_\sigma l_\nu,\end{aligned} \tag{4.11}$$

$$T_{\alpha\beta\gamma\delta} \equiv g_{\alpha\beta,\gamma}l_\delta - g_{\alpha\beta,\delta}l_\gamma \equiv g_{\alpha\beta|i}(u^i{}_{,\gamma}l_\delta - u^i{}_{,\delta}l_\gamma). \tag{4.12}$$

Dirac [D] distinguishes between "strong equations" ($\equiv$) and "weak equations" ($=$), as explained in [PS]. The ϕ-equations (4.8) are weak equations. Following [PS] we construct the strong equations

$$C^{\mu\nu}C_{\mu\nu} \equiv 0, \quad C^2 \equiv 0, \tag{4.13}$$

where $C \equiv g_{\mu\nu}C^{\mu\nu}$, from the weak equations

$$C^{\mu\nu} \equiv \pi^{\mu\nu} - \partial(J\mathfrak{L})/\partial\dot{g}_{\mu\nu} = 0. \tag{4.14}$$

The Hamiltonian may now be written[12]

$$\begin{aligned}H &\equiv \int(\dot{x}^\mu\lambda_\mu + \dot{g}_{\alpha\beta}\pi^{\alpha\beta} + \pi\dot{\psi} + \dot{\psi}^*\pi^*)d\mathbf{u} - L \\ &\equiv \int\dot{x}^\mu(\lambda_\mu + \pi^{\alpha\beta}g_{\alpha\beta,\mu} + \pi\psi_{,\mu} + \psi^*{}_{,\mu}\pi^* - l_\mu\mathfrak{L})d\mathbf{u} \\ &\quad + \int\dot{x}^\mu l_\mu\beta l^{-2}g^{-\frac{1}{2}}(C^{\alpha\beta}C_{\alpha\beta} - \tfrac{1}{2}C^2)d\mathbf{u} \\ &\equiv \int(\dot{x}^\mu\varphi_\mu - 2Jl^{-2}g^{\alpha\beta}[\alpha\beta, \mu]\phi^\mu + \phi\dot{\psi} + \dot{\psi}^*\phi^*)d\mathbf{u},\end{aligned} \tag{4.15}$$

where

$$\varphi_\mu \equiv \lambda_\mu + \mathfrak{K}_\mu = 0, \tag{4.16}$$

$$\begin{aligned}\mathfrak{K}_\mu \equiv\ & l^{-2}\pi_+{}^{\alpha\beta}l^\gamma T_{\alpha\beta\gamma\mu} - 2l^{-2}l_\mu g^{\beta\delta}\pi_+{}^{\alpha\gamma}T_{\alpha\beta\gamma\delta} \\ & + \tfrac{1}{2}\hbar c g^{\frac{1}{2}}l^{-2}l_\alpha l^\beta(\psi^*A\Gamma^\alpha T_{\mu\beta} - \bar{T}_{\mu\beta}A\Gamma^\alpha\psi) \\ & + \tfrac{1}{2}\hbar c g^{\frac{1}{2}}\psi^*(A\Gamma^\gamma F^{\alpha\beta} + \bar{F}^{\alpha\beta}\bar{\Gamma}^\gamma A)\psi T_{\alpha\beta\gamma\mu} \\ & + l_\mu(\beta G_{\alpha\beta\gamma\delta}\pi_+{}^{\alpha\beta}\pi_+{}^{\gamma\delta} + \beta^{-1}X^{\alpha\beta\gamma\delta\epsilon\zeta\eta\theta}T_{\alpha\beta\gamma\delta}T_{\epsilon\zeta\eta\theta}) \\ & + \tfrac{1}{2}\hbar c g^{\frac{1}{2}}l_\mu[l^{-2}l^\alpha(\psi^*A\Gamma^\beta T_{\alpha\beta} - \bar{T}_{\alpha\beta}A\Gamma^\beta\psi) \\ & + \psi^*(A\Gamma^\sigma\Delta_\sigma + \bar{\Delta}_\sigma\bar{\Gamma}^\sigma A + 2\kappa)\psi],\end{aligned} \tag{4.17}$$

$$\pi_+{}^{\mu\nu} \equiv \pi^{\mu\nu} + \tfrac{1}{2}\hbar c g^{\frac{1}{2}}\psi^*(A\Gamma^\sigma F^{\mu\nu} + \bar{F}^{\mu\nu}\bar{\Gamma}^\sigma A)\psi l_\sigma, \tag{4.18}$$

$$T_{\mu\nu} \equiv \psi_{,\mu}l_\nu - \psi_{,\nu}l_\mu, \quad \bar{T}_{\mu\nu} \equiv \psi^*{}_{,\mu}l_\nu - \psi^*{}_{,\nu}l_\mu, \tag{4.19}$$

$$G_{\alpha\beta\gamma\delta} \equiv \tfrac{1}{2}l^{-2}g^{-\frac{1}{2}}(g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\beta}g_{\gamma\delta}), \tag{4.20}$$

$$\begin{aligned}X^{\alpha\beta\gamma\delta\epsilon\zeta\eta\theta} \equiv\ & l^{-2}g^{\frac{1}{2}}[-\tfrac{1}{16}(g^{\alpha\epsilon}g^{\beta\zeta} + g^{\alpha\zeta}g^{\beta\epsilon} - g^{\alpha\beta}g^{\epsilon\zeta})g^{\gamma\eta}g^{\delta\theta} \\ & + \tfrac{1}{2}g^{\alpha\theta}g^{\epsilon\delta}(g^{\beta\zeta}g^{\gamma\eta} + g^{\beta\gamma}g^{\zeta\eta})].\end{aligned} \tag{4.21}$$

[9] The current density $s^\mu = ieg^{\frac{1}{2}}\bar{\psi}\Gamma^\mu\psi$ also satisfies a divergence condition $s^\mu{}_{\cdot\mu} = s^\mu{}_{,\mu} = 0$.

[10] We introduce a family of space-like surfaces labeled by a real parameter t. The points in each surface are labeled by three real parameters u^i. A stroke followed by a Latin index denotes partial differentiation with respect to a parameter u^i: $y_{A|i} \equiv \partial y_A/\partial u^i \equiv y_{A,\sigma}x^\sigma{}_{|i}$. A dot denotes partial differentiation with respect to the parameter t: $\dot{y}_A \equiv \partial y_A/\partial t \equiv y_{A,\sigma}\dot{x}^\sigma$.

[11] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); this paper will be referred to as [D].

[12] The Lagrangian functional is $L \equiv \int J\mathfrak{L}d\mathbf{u}$.

The computation leading to the last line of (4.15) is lengthy but straightforward. The ϕ-equation (4.16) could also have been obtained directly from Eq. (4.4) with the help of Eqs. (4.5)–(4.7). The function $\mathcal{H}_\mu$ is the Hamiltonian density for the interacting fields.

The Poisson bracket relations satisfied by the canonical variables are

$$(x^\mu(\mathbf{u}), \lambda_\nu(\mathbf{u}')) \equiv \delta_\nu{}^\mu \delta(\mathbf{u}-\mathbf{u}'), \tag{4.22}$$

$$(g_{\mu\nu}(\mathbf{u}), \pi^{\sigma\tau}(\mathbf{u}')) \equiv \tfrac{1}{2}(\delta_\mu{}^\sigma\delta_\nu{}^\tau + \delta_\mu{}^\tau\delta_\nu{}^\sigma)\delta(\mathbf{u}-\mathbf{u}'), \tag{4.23}$$

$$(\psi_\alpha(\mathbf{u}), \pi_\beta(\mathbf{u}')) \equiv (\psi_\alpha{}^*(\mathbf{u}), \pi_\beta{}^*(\mathbf{u}')) \equiv \delta_{\alpha\beta}\delta(\mathbf{u}-\mathbf{u}'). \tag{4.24}$$

All other Poisson brackets between pairs of the variables x^μ, $g_{\alpha\beta}$, ψ, ψ^*, λ_ν, $\pi^{\alpha\beta}$, π, π^* vanish.

5. REDUCTION OF THE HAMILTONIAN

Dirac [D] distinguishes between "first-class ϕ's" and "second-class ϕ's." The Poisson bracket of a first-class ϕ with any other ϕ vanishes either identically or in virtue of the ϕ-equations. The Poisson brackets of a second-class ϕ (or any linear combination of second-class ϕ's) with the other ϕ's do not all vanish. In the present example ϕ and ϕ^* are second-class ϕ's, for we have

$$(\phi_\alpha{}^*(\mathbf{u}), \phi_\beta(\mathbf{u}')) \equiv -\hbar c g^{\frac{1}{2}}(A\Gamma^\sigma l_\sigma)_{\alpha\beta}\delta(\mathbf{u}-\mathbf{u}'). \tag{5.1}$$

It is sometimes possible to turn a second-class ϕ into a first-class ϕ by adding supplementary conditions which Dirac calls χ-equations. This is evidently not possible in the present case, because the right-hand side of (5.1) cannot be set equal to zero.

It is shown in [D] that whenever second-class ϕ's are present one can carry out a simplification of the Hamiltonian scheme which consists essentially in removing some of the empty degrees of freedom. The simplification is carried out by introducing a modification of the ordinary Poisson bracket. We give now an outline of the theory.

Denote by ϕ_i the members of any set of second-class ϕ's of any dynamical system. Consider the matrix A having elements

$$A_{ij} \equiv (\phi_i, \phi_j). \tag{5.2}$$

Since the ϕ_i are second-class ϕ's, the matrix A must be nonsingular.[13] Otherwise, some of the ϕ_i could be combined linearly to form one or more first-class ϕ's. Introduce now the following bracket notation:

$$(F, G)_D \equiv (F, G) - (F, \phi_i)A^{-1}{}_{ij}(\phi_j, G), \tag{5.3}$$

where A^{-1} denotes the inverse of A. We shall call expression (5.3) the *Dirac bracket* of F and G. In [D] it is shown that Dirac brackets satisfy all the identities satisfied by ordinary Poisson brackets. In addition it may be observed that the Dirac bracket of a ϕ_i with anything vanishes.

The utility of the Dirac bracket is readily seen by considering the consistency conditions

$$0 = \dot{\phi}_i = \beta_A(\phi_i, \phi_A) + \beta_j(\phi_i, \phi_j), \tag{5.4}$$

where the ϕ_A are those ϕ's which are not numbered among the ϕ_i, and the β_A, β_i are Dirac's "velocity variables," the Hamiltonian being given by $H \equiv \beta_A\phi_A + \beta_i\phi_i$. Solving Eqs. (5.4) for the β_i, we may write the t derivative of any dynamical quantity F in the form

$$\dot{F} = \beta_A(F, \phi_A) + \beta_i(F, \phi_i) = \beta_A(F, \phi_A)_D. \tag{5.5}$$

It is evident that we get the same dynamical equations if we work with Dirac brackets instead of Poisson brackets. The second-class ϕ's can then be regarded as vanishing in the strong sense and may be used to eliminate some of the dynamical variables from the theory.

The Dirac brackets for the gravitational-spinor system are readily obtained. For any two dynamical quantities F and G we have, using (5.1),

$$(F, G)_D \equiv (F, G) - (\hbar c)^{-1} \int\int \|(F, \phi_\alpha(\mathbf{u})), (F, \phi_\alpha{}^*(\mathbf{u}))\| \times \left\| \begin{matrix} 0 & -iN^{-1}{}_{\alpha\beta}\delta(\mathbf{u}-\mathbf{u}') \\ iN^{-1}{}_{\beta\alpha}\delta(\mathbf{u}-\mathbf{u}') & 0 \end{matrix} \right\| \times \left\| \begin{matrix} (\phi_\beta(\mathbf{u}'), G) \\ (\phi_\beta{}^*(\mathbf{u}'), G) \end{matrix} \right\| d\mathbf{u}d\mathbf{u}', \tag{5.6}$$

where

$$N \equiv ig^{\frac{1}{2}} A\Gamma^\sigma l_\sigma, \tag{5.7}$$

$$N^{-1} \equiv ig^{-\frac{1}{2}} l^{-2} l_\sigma \Gamma^\sigma A^{-1}. \tag{5.8}$$

By straightforward computation we find

$$(x^\mu(\mathbf{u}), \lambda_\nu(\mathbf{u}'))_D \equiv \delta_\nu{}^\mu\delta(\mathbf{u}-\mathbf{u}'), \tag{5.9}$$

$$(g_{\mu\nu}(\mathbf{u}), \pi^{\sigma\tau}(\mathbf{u}'))_D \equiv \tfrac{1}{2}(\delta_\mu{}^\sigma\delta_\nu{}^\tau + \delta_\mu{}^\tau\delta_\nu{}^\sigma)\delta(\mathbf{u}-\mathbf{u}'), \tag{5.10}$$

$$(\pi^{\mu\nu}(\mathbf{u}), \pi^{\sigma\tau}(\mathbf{u}'))_D \equiv -\tfrac{1}{4}i\hbar c\psi^* \left[\frac{\partial N}{\partial g_{\mu\nu}}N^{-1}, \frac{\partial N}{\partial g_{\sigma\tau}}N^{-1}\right] \times N\psi\delta(\mathbf{u}-\mathbf{u}'), \tag{5.11}$$

$$(\psi_\alpha(\mathbf{u}), \psi_\beta{}^*(\mathbf{u}'))_D \equiv -i(\hbar c)^{-1}N^{-1}{}_{\alpha\beta}\delta(\mathbf{u}-\mathbf{u}'), \tag{5.12}$$

$$(\pi^{\mu\nu}(\mathbf{u}), \psi(\mathbf{u}'))_D \equiv \tfrac{1}{2}N^{-1}(\partial N/\partial g_{\mu\nu})\psi\delta(\mathbf{u}-\mathbf{u}'), \tag{5.13}$$

$$(\pi^{\mu\nu}(\mathbf{u}), \psi^*(\mathbf{u}'))_D \equiv \tfrac{1}{2}\psi^*(\partial N/\partial g_{\mu\nu})N^{-1}\delta(\mathbf{u}-\mathbf{u}'). \tag{5.14}$$

The coefficients multiplying the delta-functions in Eqs. (5.11)–(5.14) may be evaluated at either $\mathbf{u}$ or $\mathbf{u}'$.

At first sight the reduction of the Hamiltonian scheme seems to be of no advantage, since now the dynamical variables are somewhat mixed up. The momenta $\pi^{\mu\nu}$ are no longer dynamically independent of the spinor field vaiables nor even of each other. The Dirac brackets

[13] More generally, the matrix A is constructed out of "second-class χ's" as well as second-class ϕ's. The total number of second-class ϕ's and χ's must evidently be even.

$(\lambda_\mu\,\lambda_\nu)_D$, $(\lambda_\mu,\,\pi^{\sigma\tau})_D$, $(\lambda_\mu,\,\psi)_D$, $(\lambda_\mu,\,\psi^*)_D$, which have structures similar to (5.11), (5.13), (5.14), also do not vanish. However, the situation is really better than it appears. The Dirac brackets of all other pairs of the dynamical variables do vanish, and we shall see presently that we can redefine the field quantities ψ, ψ^*, $\pi^{\mu\nu}$, λ_μ in such a way that they assume the dynamical properties that we should expect of them.

We first study the structure of the matrix N. When the Γ^μ and A are given by (2.13) and (2.31), respectively, N has the explicit form

$$N\equiv S_0{}^*N_0S_0, \tag{5.15}$$

where

$$N_0\equiv ig^{\frac{1}{2}}l_\sigma\sqrt{g^\sigma{}_{\nu'}}\gamma_4\gamma_{\nu'}\equiv\alpha_i{}_i\ell+\ell_0, \tag{5.16}$$

$$\ell_{\mu'}\equiv g^{\frac{1}{2}}l_\sigma\sqrt{g^\sigma{}_{\mu'}},\quad (\ell_{\mu'})\equiv(\ell_i,\,-i\ell_0), \tag{5.17}$$

and $\alpha_i\equiv i\gamma_4\gamma_i$. Since l_μ is a time-like vector $\ell_0{}^2>\ell_i\ell_i$, the matrix N_0 has a differentiable Hermitian square root given by

$$N_0{}^{\frac{1}{2}}\equiv(2L)^{-\frac{1}{2}}(\alpha_i\ell_i+L),\quad L\equiv\ell_0\pm(\ell_0{}^2-\ell_i\ell_i)^{\frac{1}{2}}. \tag{5.18}$$

We may therefore express the matrix N in the form

$$N\equiv M^*M,\quad\text{where}\quad M\equiv N_0{}^{\frac{1}{2}}S_0. \tag{5.19}$$

The new dynamical variables which we seek are the following:

$$\boldsymbol{\psi}\equiv M\psi,\quad \boldsymbol{\psi}^*\equiv\psi^*M^*, \tag{5.20}$$

$$\boldsymbol{\pi}^{\mu\nu}\equiv\pi^{\mu\nu}+\tfrac{1}{2}i\hbar c\boldsymbol{\psi}^*[M^{*-1}(\partial M^*/\partial g_{\mu\nu}) -(\partial M/\partial g_{\mu\nu})M^{-1}]\boldsymbol{\psi}, \tag{5.21}$$

$$\boldsymbol{\lambda}_\mu(\mathbf{u})\equiv\lambda_\mu(\mathbf{u})+\tfrac{1}{2}i\hbar c\int\boldsymbol{\psi}^*(\mathbf{u}')\left[M^{*-1}(\mathbf{u}')\frac{\delta M^*(\mathbf{u}')}{\delta x^\mu(\mathbf{u})} -\frac{\delta M(\mathbf{u}')}{\delta x^\mu(\mathbf{u})}M^{-1}(\mathbf{u}')\right]\boldsymbol{\psi}(\mathbf{u}')d\mathbf{u}'. \tag{5.22}$$

One finds, by straightforward computation, the following Dirac brackets:

$$(x^\mu(\mathbf{u}),\,\boldsymbol{\lambda}_\nu(\mathbf{u}'))_D\equiv\delta_\nu{}^\mu\delta(\mathbf{u}-\mathbf{u}'), \tag{5.23}$$

$$(g_{\mu\nu}(\mathbf{u}),\,\boldsymbol{\pi}^{\sigma\tau}(\mathbf{u}'))_D\equiv\tfrac{1}{2}(\delta_\mu{}^\sigma\delta_\nu{}^\tau+\delta_\mu{}^\tau\delta_\nu{}^\sigma)\delta(\mathbf{u}-\mathbf{u}'), \tag{5.24}$$

$$(\boldsymbol{\psi}_\alpha(\mathbf{u}),\,\boldsymbol{\psi}_\beta{}^*(\mathbf{u}'))_D\equiv-i(\hbar c)^{-1}\delta_{\alpha\beta}\delta(\mathbf{u}-\mathbf{u}'), \tag{5.25}$$

$$\left.\begin{array}{l}(\boldsymbol{\lambda}_\mu(\mathbf{u}),\,\boldsymbol{\lambda}_\nu(\mathbf{u}'))_D\equiv0,\quad(\boldsymbol{\pi}^{\mu\nu}(\mathbf{u}),\,\boldsymbol{\pi}^{\sigma\tau}(\mathbf{u}'))_D\equiv0,\\ (\boldsymbol{\pi}^{\mu\nu}(\mathbf{u}),\,\boldsymbol{\psi}(\mathbf{u}'))_D\equiv0,\ \text{etc.}\end{array}\right\} \tag{5.26}$$

Since the right-hand sides of the above expressions do not involve any of the field quantities, it is evident that the new variables may be used directly in passing to an interaction representation for the purpose of carrying out a perturbation computation.

It is to be noted that since the spin-space transformation law of the matrix M has the form

$$M'=MS, \tag{5.27}$$

the variables $\boldsymbol{\psi}$, $\boldsymbol{\psi}^*$ remain unchanged under spin-space transformations.

6. QUANTIZATION

Passage to the quantum theory is effected by making the identifications[14]

$$(F,\,G)_D=(i\hbar)^{-1}[F,\,G] \tag{6.1}$$

or

$$(F,\,G)_D=\pm(i\hbar)^{-1}\{F,\,G\}, \tag{6.2}$$

according to the statistics required. In the present example the variables $\boldsymbol{\psi}$, $\boldsymbol{\psi}^*$ will satisfy anticommutation relations among themselves. The sign in (6.2) is chosen positive or negative according as the $\boldsymbol{\psi}^*$'s are placed to the left or the right of the $\boldsymbol{\psi}$'s in the Hamiltonian. The first alternative is customary. We therefore have the quantum dynamical relations

$$[x^\mu(\mathbf{u}),\,\boldsymbol{\lambda}_\nu(\mathbf{u}')]=i\hbar\delta_\nu{}^\mu\delta(\mathbf{u}-\mathbf{u}'), \tag{6.3}$$

$$[g_{\mu\nu}(\mathbf{u}),\,\boldsymbol{\pi}^{\sigma\tau}(\mathbf{u}')]=\tfrac{1}{2}i\hbar(\delta_\mu{}^\sigma\delta_\nu{}^\tau+\delta_\mu{}^\tau\delta_\nu{}^\tau)\delta(\mathbf{u}-\mathbf{u}'), \tag{6.4}$$

$$\{\boldsymbol{\psi}_\alpha(\mathbf{u}),\,\boldsymbol{\psi}_\beta{}^*(\mathbf{u}')\}=c^{-1}\delta_{\alpha\beta}\delta(\mathbf{u}-\mathbf{u}'), \tag{6.5}$$

$$\{\boldsymbol{\psi}_\alpha(\mathbf{u}),\,\boldsymbol{\psi}_\beta(\mathbf{u}')\}=\{\boldsymbol{\psi}_\alpha{}^*(\mathbf{u}),\,\boldsymbol{\psi}_\beta{}^*(\mathbf{u}')\}=0. \tag{6.6}$$

The commutators of all other pairs of the dynamical variables vanish.

In the reduced Hamiltonian scheme of the preceding section φ_μ and ϕ^μ become first-class ϕ's.[15] According to [D] the ϕ-equations corresponding to the first-class ϕ's become, in the quantum theory, conditions on the state vector Ψ of the system. Thus we have

$$\varphi_\mu\Psi=0, \tag{6.7}$$

$$\phi^\mu\Psi=0. \tag{6.8}$$

The commutation relation (6.3) allows us to make the identification

$$\boldsymbol{\lambda}_\mu(\mathbf{u})=-i\hbar\delta/\delta x^\mu(\mathbf{u}). \tag{6.9}$$

The state vector is actually a functional $\Psi[x]$ of the functions $x^\mu(\mathbf{u})$ which describe the space-like surface $t=$ constant. If these functions suffer variations $\delta x^\mu(\mathbf{u})$, Eq. (6.7) tells us that the corresponding variation in the state vector is given by

$$i\hbar\delta\Psi[x]=\int\mathcal{H}_\mu(\mathbf{u})\delta x^\mu(\mathbf{u})d\mathbf{u}\Psi[x], \tag{6.10}$$

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[14] It is important to realize that it is the Dirac bracket rather than the ordinary Poisson bracket which corresponds to the commutator or anticommutator in the quantum theory of any dynamical system. Dirac, himself, does not seem to have emphasized this point clearly in [D].

[15] The authors have actually carried out an explicit calculation of the Dirac bracket $(\phi^\mu(\mathbf{u}),\,\phi^\nu(\mathbf{u}'))_D$. It vanishes not in the weak sense but identically. The calculation is straightforward but tedious. One obtains an expression of the form

$$X^{\mu\nu}(\mathbf{u})\delta(\mathbf{u}-\mathbf{u}')+[Y^{\mu\nu\alpha i}(\mathbf{u}')Z_\alpha(\mathbf{u})+Y^{\nu\mu\alpha i}(\mathbf{u})Z_\alpha(\mathbf{u}')]\delta_{|i}(\mathbf{u}-\mathbf{u}')$$

which vanishes on account of the equations $(Y^{\mu\nu\alpha i}+Y^{\nu\mu\alpha i})Z_\alpha\equiv0$ and $X^{\mu\nu}+Y^{\mu\nu\alpha i}{}_{|i}Z_\alpha+Y^{\nu\mu\alpha i}Z_{\alpha|i}\equiv0$. In verifying the second equation, one must make use of the identities $l_{\mu|i}\equiv(l_\mu u^j{}_{,\nu}-l_\nu u^j{}_{,\mu})x^\nu{}_{|ij}$ and $(u^i{}_{,\mu}l_\nu-u^i{}_{,\nu}l_\mu)_{|i}\equiv0$, as well as the differential identities satisfied by the matrices Γ^μ, $F^{\alpha\beta}$, A, etc.

where

$$\mathfrak{H}_\mu = \mathcal{H}_\mu + (\lambda_\mu - \lambda_\mu). \qquad (6.11)$$

Equation (6.10) is the Schrödinger equation of the electron-graviton system. Equation (6.8) is a supplementary condition on the allowable state-vectors.

Two final remarks peculiar to the gravitational problem must be made concerning the quantization procedure. First, the Hamiltonian density $\mathfrak{H}_\mu$ must be an Hermitian operator. The intrinsically nonlinear character of the gravitational system reflects itself in expression (4.17), in which there occur products involving noncommuting factors. One does not know *a priori* how these factors should be ordered so as to symmetrize $\mathfrak{H}_\mu$. One could attempt to obtain a quantum Hamiltonian by using the simplest possible symmetrization procedure, but then one would not know whether or not an equivalent quantum theory would have been obtained with a similar symmetrization procedure, if another set of gravitational variables (such as $g^{\frac{1}{2}}g^{\mu\nu}$ for example) had been used instead of $g_{\mu\nu}$ as the fundamental field quantities. One of us[16] has shown how this ambiguity can be removed in the case of Hamiltonians which are at most quadratic in the momenta, and he has discussed the gravitational case in particular. We merely quote the result The terms linear in the momenta are symmetrized by the usual anticommutator rule. The term quadratic in the momenta is written in the symmetric form $\beta l_\mu \boldsymbol{\pi}^{\alpha\beta} G_{\alpha\beta\gamma\delta} \boldsymbol{\pi}^{\gamma\delta}$ [see (4.20)]. The factor $G_{\alpha\beta\gamma\delta}$ may be interpreted as a "contravariant" metric tensor which describes the geometry of the 10-dimensional space of the $g_{\mu\nu}$. In order that the Hamiltonian density $\mathfrak{H}_\mu$ be a truly invariant quantity under point transformations in this 10-dimensional space, a divergent term

$$(85/8)\hbar^2 \beta l_\mu l^{-2} g^{-\frac{1}{2}} [\delta(\mathbf{u}-\mathbf{u})]^2, \qquad (6.12)$$

which vanishes in the classical limit $\hbar \to 0$, must be added to expression (4.17).

The second remark concerns the vacuum expectation value of the matter stress density $\Theta_{\mu\nu}$. In discussing this quantity we may ignore the dynamical properties of the gravitational field and regard the $g_{\mu\nu}$ as given c-numbers. However, the $g_{\mu\nu}$ must be chosen so as to make space-time flat, as befits a true gravitational vacuum. The vacuum expectation value is then given by[17]

$$\langle \Theta_{\mu\nu} \rangle_0 = \tfrac{1}{2} \hbar c \kappa^2 g^{\frac{1}{2}} g_{\mu\nu} \Delta^{(1)}(0). \qquad (6.13)$$

We must not couple this physically meaningless divergent vacuum value to the gravitational field. We must instead redefine the stress density by subtracting expression (6.13). This corresponds to the subtraction of a term $\frac{1}{2}\hbar c\kappa^2 g^{\frac{1}{2}} \Delta^{(1)}(0)$ from the Lagrangian density of the matter field and hence to the addition of a term

$$\tfrac{1}{2} \hbar c \kappa^2 l_\mu g^{\frac{1}{2}} \Delta^{(1)}(0) \qquad (6.14)$$

to expression (4.17).

The necessity of adding the divergent expression (6.14) to the Hamiltonian density $\mathfrak{H}_\mu$ has been confirmed in actual calculations of the gravitational self-energies and stress renormalizations of material particles.[18] The procedure is entirely equivalent to the charge symmetrization procedure in quantum electrodynamics in which the vacuum expectation value of the current is made to vanish.

The physical necessity of adding the divergent expression (6.12), on the other hand, can be confirmed only when calculations are carried out on the interaction of the gravitational field with itself.

[16] B. S. DeWitt, Phys. Rev. **85**, 653 (1952).

[17] See, for example, J. Schwinger, Phys. Rev. **75**, 658 (1949), Eqs. (1.77, 78).

[18] B. S. DeWitt, Ph.D. thesis, Harvard University, 1950.