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Interaction Moment Contributions to Magnetic Moments of Nuclei*,†

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A model of a heavy odd A nucleus is used in which the outer nucleon is in the state predicted by the Mayer scheme, and the core is in a state of zero angular momentum. It is found that the general trend of the deviations of the experimentally determined magnetic moments from the Schmidt curves is describable in terms of a linear combination of the three phenomenological interaction moment operators which can arise from a charge exchange potential in a second order meson calculation. The final expression for the interaction moment, matched to the data, is, by and large, independent of any detailed properties of the radial functions associated with these operators, of the radial function of the outer nucleon, and, except for a dependence on the relative number of neutrons and protons in the core, of the core structure. The consequent predictions that, for given l and j, the interaction moments of heavy odd neutron nuclei should be smaller in magnitude than those of heavy odd proton nuclei and that the addition of two protons to an odd proton nucleus or of two neutrons to an odd neutron nucleus should push the moments toward the Schmidt curves are in general accord with the data. Odd-odd nuclei are also considered.

I. INTRODUCTION

THE independent particle model of the nucleus advanced by Mayer[1] and by Haxel, Jensen, and Suess[2] has been extremely useful in correlating large bodies of data. In particular, the experimental values of the total angular momentum J of odd A nuclei are in definite disagreement with the strict one-particle interpretation of this model in only three cases, and the magnetic moments of almost all nuclei lie reasonably close to the Schmidt values predicted by this model. The deviations from the Schmidt values are not random; with the exception of a few very light nuclei, the experimental values lie between the Schmidt curves.

Attempts have been made to explain these deviations by modifying the model,[3] and by dropping the strict one-particle interpretation to allow for cases in which three or more particles outside of closed shells are effective in determining the spin and magnetic moment of the nucleus.[4] Other attempts have been made using the strict one-particle interpretation and modifying the magnetic moment operator instead. If the cases in which the spin predictions of the one-particle interpretation are at variance with the experimental values are excluded, it is of interest to see how closely the data can be matched on the basis of such an approach.

Siegert[5] pointed out that charge exchange forces imply the existence of currents which must modify the magnetic moment operator. While meson theoretical calculations to determine this interaction moment operator[6] suffer from the usual difficulties, it is significant that Villars[7] was able to show that the H^3-He^3 anomaly could be explained using pseudoscalar meson theory. Sachs,[8] however, showed that the longitudinal part of this interaction moment could be expressed in terms of the space exchange potential, the result being independent of meson theory. The longitudinal moment was thus given a firm basis, but calculations carried out by Sachs for H^3 and by Spruch[9] for a number of light nuclei showed that this operator alone could not account for the deviations from the Schmidt values.

Bloch and de Shalit independently pointed out[10] that the data could be qualitatively understood by assuming that the anomalous magnetic moment of the odd nucleon is quenched somewhat in the presence of other nucleons. This quenching could also be considered as an interaction contribution.[11]

Miyazawa[11] carried through the calculation of the interaction moment contribution for heavy nuclei, the core of the nucleus being approximated by a Fermi gas. The longitudinal part of the moment was treated phenomenologically, while the rest was calculated meson-theoretically. Miyazawa was able to match the general trend of the deviations by a small but arbitrary readjustment of the numerical constants in his final expression.

It is possible to proceed entirely phenomenologically. Thus, by invariance and symmetry considerations, Osborn and Foldy[12] derived the most general operators which can arise from a charge exchange potential; other than for the longitudinal operator, however, these can

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[1] M. G. Mayer, Phys. Rev. **78**, 16 (1950).
[2] Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949).
[3] J. Rainwater, Phys. Rev. **79**, 432 (1950); L. L. Foldy and E. J. Milford, Phys. Rev. **80**, 751 (1950); Y. A. Romanov, J. Exptl. Theor. Phys. (U.S.S.R.) **20**, 577 (1950); A. Bohr, Phys. Rev. **81**, 134 (1951); S. Gallone and G. Salveti, Phys. Rev. **84**, 1064 (1951); J. P. Davison, Phys. Rev. **85**, 432 (1952).
[4] M. Mizushima and M. Umezawa, Phys. Rev. **85**, 37 (1952).
[5] A. F. Siegert, Phys. Rev. **52**, 787 (1937).
[6] S. T. Ma and C. F. Yu, Phys. Rev. **62**, 118 (1942); C. Møller and L. Rosenfeld, Kgl. Danske Videnskab. Selskab. Math.-fys. Medd. **20**, No. 12 (1943); W. Pauli and S. Kusaka, Phys. Rev. **63**, 400 (1943).
[7] F. Villars, Helv. Phys. Acta **20**, 476 (1947).
[8] R. G. Sachs, Phys. Rev. **74**, 433 (1948).
[9] L. Spruch, Phys. Rev. **80**, 372 (1950).
[10] F. Bloch, Phys. Rev. **83**, 839 (1951); A. de Shalit, Helv. Phys. Acta **24**, 296 (1951).
[11] H. Miyazawa, Prog. Theoret. Phys. **6**, 801 (1951).
[12] R. K. Osborn and L. Foldy, Phys. Rev. **79**, 795 (1950).

be determined only to within arbitrary functions of the interparticle distances. Under the requirement that these operators behave properly under time reversal[13] and with the arbitrary restriction that they can arise from a second-order meson calculation, only three operators remain.

An investigation has been undertaken to determine the contributions of these three operators and to seek a combination which leads to agreement with the data. The calculation differs from that of Miyazawa in that it is completely phenomenological and in that the shell model is used to represent the core rather than a Fermi gas. The Fermi gas approach as used by Miyazawa has the mathematical advantage that the interaction contributions are independent of any details of the core of the particular nucleus under consideration. The shell model has the physical advantage that it is a closer representation of the actual situation and should give a more faithful picture of the trends and of the details of the variations from the Schmidt lines. Further, since the interaction contributions do depend upon some of the details of the core, calculations based on the shell model should ultimately be useful in the analysis of the details of the shell model, e.g., the order in which the shells fill and whether or not nucleons of high angular momentum "hide." [1]

The operators $\mathbf{M}_\alpha$ are encountered in the calculation of isomeric half-lives[14] and radiative capture and photomagnetic disintegration cross sections,[15,16] as well as magnetic moments. These other effects have not been considered.

II. MATRIX ELEMENTS

Only those interaction moment operators are considered which can arise from a charge exchange potential, in a second-order meson theory. In particular, operators which can arise from velocity dependent forces or from many-body forces are not considered. There are three operators listed in Osborn and Foldy[17] which satisfy the above conditions and which behave properly under time reversal. After the isotopic spin dependence has been removed and use has been made of the operator identity $\boldsymbol{\sigma}_\pi \times \boldsymbol{\sigma}_\nu Q_{\pi\nu} = i(\boldsymbol{\sigma}_\pi - \boldsymbol{\sigma}_\nu)$, these operators can be written in the form[18]

$$\mathbf{M}_1 = \mathbf{M}_{\text{long}} = (ie/2\hbar c)\sum_\nu \sum_\pi \mathbf{r}_\pi \times \mathbf{r}_\nu V(\mathbf{r}_{\pi\nu}, \boldsymbol{\sigma}_\pi, \boldsymbol{\sigma}_\nu) P_{\pi\nu},$$
$$\mathbf{M}_2 = (e/2\hbar c)\sum_\nu \sum_\pi r_{\pi\nu}^2 f_2(r_{\pi\nu})(\boldsymbol{\sigma}_\pi - \boldsymbol{\sigma}_\nu) P_{\pi\nu}, \tag{1}$$
$$\mathbf{M}_3 = (e/2\hbar c)\sum_\nu \sum_\pi f_3(r_{\pi\nu})(\mathbf{r}_{\pi\nu} \cdot (\boldsymbol{\sigma}_\pi - \boldsymbol{\sigma}_\nu))\mathbf{r}_{\pi\nu} P_{\pi\nu},$$

where $V(\mathbf{r}_{\pi\nu}, \boldsymbol{\sigma}_\pi, \boldsymbol{\sigma}_\nu)P_{\pi\nu}$ represents the charge exchange potential and $f_2(r_{\pi\nu})$ and $f_3(r_{\pi\nu})$ are undetermined functions of the interparticle distances, which are expected, however, to have ranges comparable with the range of nuclear forces. $P_{\pi\nu}$ and $Q_{\pi\nu}$ are the space and spin interchange operators, and it is understood that the operators (1) are to act on wave functions which have been antisymmetrized separately in neutrons and protons.

It will specifically be assumed throughout the article that

(a) an odd A nucleus consists of an odd nucleon in the state with radial, orbital angular momentum, and total angular momentum quantum numbers n, l, and j, respectively, and a core of zero total angular momentum.[19]

The contribution of the outer particle then gives the Schmidt values and the interaction magnetic moment [i.e., the sum of the expectation values of the operators (1)] is found to arise only from the interaction of the outer particle with the core. It also follows that antisymmetrization between the outer particle and the core is unnecessary, although the core wave function Φ must still be antisymmetrized in neutrons and protons separately. For the sake of notational convenience in what follows, the outer particle is taken to be a proton and is designated by the subscript π^0. The protons and neutrons in the core are designated by π' and ν', respectively.

The potential function $V(\mathbf{r}_{\pi^0\nu'}, \boldsymbol{\sigma}_{\pi^0}, \boldsymbol{\sigma}_{\nu'})$ is given as the sum of a function $f_1(r_{\pi^0\nu'})$ plus a term in $\boldsymbol{\sigma}_{\pi^0}\cdot\boldsymbol{\sigma}_{\nu'}$ plus a tensor term. If, in addition to the assumption (a) that the core has total angular momentum zero, the further assumption is made that

(b) the spin of the neutrons in the core is zero and/or the total spin of the core is zero,

then the contributions to $\mathbf{M}_1$ of the $\boldsymbol{\sigma}_{\pi^0}\cdot\boldsymbol{\sigma}_{\nu'}$ term and of the tensor term vanish and only the first term contributes. It is then found (see Appendix) that

$$\langle M_1{}^z\rangle = b\langle l_{\pi^0}{}^z\rangle I_1(n, l), \tag{2}$$

where the various quantities will be defined shortly.

Under the same assumptions (a) and (b), the contribution of $\mathbf{M}_2$ is given by

$$\langle M_2{}^z\rangle = -b\langle \sigma_{\pi^0}{}^z\rangle K_2(n, l). \tag{3}$$

If, in addition to (a) and (b), the further assumption is made that

(c) the orbital angular momentum of the neutrons in the core is zero,

then it is found, setting $j_z = j$, that

$$\langle M_3{}^z\rangle = -b(2j_{\pi^0}+2)^{-1}[K_3(n, l) + 2\langle \boldsymbol{\sigma}_{\pi^0}\cdot\mathbf{l}_{\pi^0}\rangle I_3(n, l)]. \tag{4}$$

[13] G. J. Kynch, Phys. Rev. 81, 1060 (1951).

[14] R. G. Sachs and M. Ross, Phys. Rev. 84, 379 (1951).

[15] N. Austern and R. G. Sachs, Phys. Rev. 81, 710 (1951).

[16] N. Austern, Phys. Rev. 83, 672 (1951).

[17] See reference 12. The longitudinal moment operator, sometimes called the space exchange operator, derived there has the wrong sign.

[18] There are certain formal advantages to using a linear combination of these as the basic operators. See the Appendix.

[19] The statement that the core has a certain angular momentum equal to zero will always mean that the core is in an eigenstate of that particularly angular momentum operator with eigenvalue zero, and not merely that the expectation value of that angular momentum operator is zero.

All magnetic moments are given in nuclear Bohr magnetons, $b=J_0a_1^2M/\hbar^2$, D_1J_0 and a_1 are the effective strength and range of the space exchange part of the nuclear potential, and M is the proton mass. D_1 gives the fraction of the total potential which is of the space exchange type. $\langle l_{\pi^0}{}^z\rangle$ and $\langle\sigma_{\pi^0}{}^z\rangle$ refer to the odd proton state. The factors $I_\alpha(n, l)$ and $K_\alpha(n, l)$, where $\alpha=1, 2, 3$, are given by the integrals

$$I_\alpha(n, l)=\textstyle\sum_{\nu'}\displaystyle\int d\tau F_\alpha(n, l)U(\mathbf{r}_{\pi^0}, \mathbf{r}_{\nu'}),$$
$$K_\alpha(n, l)=\textstyle\sum_{\nu'}\displaystyle\int d\tau F_\alpha(n, l)W(\mathbf{r}_{\pi^0}, \mathbf{r}_{\nu'}), \tag{5a}$$

where

$$F_\alpha(n, l)=f_\alpha(r_{\pi^0\nu'})R_{nl}(r_{\pi^0})R_{nl}(r_{\nu'})(\Phi, P_{\pi^0\nu'}\Phi),$$
$$U(\mathbf{r}_{\pi^0}, \mathbf{r}_{\nu'})=(-4\pi l(l+1)a_1^2J_0)^{-1}r_{\pi^0}r_{\nu'}(1-\mu^2)P_l'(\mu),$$
$$W(\mathbf{r}_{\pi^0}, \mathbf{r}_{\nu'})=(-4\pi a_1^2J_0)^{-1}r_{\pi^0\nu'}{}^2P_l(\mu). \tag{5b}$$

Here, $\sum_{\nu'}$ is extended over all neutrons in the core and integration is indicated over the coordinates of all nucleons. The inner product symbol refers only to the core spin coordinates, $\mu=\cos\theta_{\pi^0\nu'}$, P_l and P_l' are the ordinary Legendre polynomial of order l and its derivative with respect to the argument, respectively, and R_{nl} is the normalized radial function associated with the odd proton wave function. The presence of the factor $1/a_1^2J_0$ in the integrals is purely formal; it appears also in the definition of b and was introduced to make the integrals dimensionless. The radial function f_1 is that associated with the space exchange part of the nuclear potential V.

With the restriction noted above as to the operators to be considered, and under the assumptions (a), (b), and (c)

$$\langle M_{\text{int}}{}^z\rangle=\sum_{\alpha=1}^{3}\langle M_\alpha{}^z\rangle=b\{\langle l_{\pi^0}{}^z\rangle I_1(n, l)-\langle\sigma_{\pi^0}{}^z\rangle K_2(n, l)$$
$$-(2j_{\pi^0}+2)^{-1}[K_3(n, l)+2\langle\boldsymbol{\sigma}_{\pi^0}\cdot\mathbf{l}_{\pi^0}\rangle I_3(n, l)]\}. \tag{6}$$

III. SHELL MODEL CALCULATIONS

According to the shell model, the core of a heavy nucleus consists mainly of complete orbital angular momentum shells plus or minus a few nucleons. With the wave function for the core obtained on the basis of the shell model, the interaction moment breaks up into the sum of contributions due to each neutron shell interacting separately with the outer proton, and for each of these shells, all of the assumptions made in the previous section are valid. The contribution to the moment of the complete orbital angular momentum shells is therefore given by (6), where the integrals now reduce to

$$I_\alpha(n, l)=(1/4\pi)\sum_{n'l'}N(n', l')\int d\tau_1\int d\tau_2\tilde{F}_\alpha U(\mathbf{r}_1, \mathbf{r}_2),$$
$$K_\alpha(n, l)=(1/4\pi)\sum_{n'l'}N(n', l')\int d\tau_1\int d\tau_2\tilde{F}_\alpha W(\mathbf{r}_1, \mathbf{r}_2), \tag{7}$$

where

$$\tilde{F}_\alpha=f_\alpha(r_{12})R_{nl}(r_1)R_{nl}(r_2)R_{n'l'}(r_1)R_{n'l'}(r_2)P_{l'}(\mu)$$

and where $N(n', l')$ represents the number of neutrons in the $n'l'$ shell, namely, $2(2l'+1)$. Equation (6) is not applicable to any incomplete shells, for while these have a total angular momentum of zero, assumptions (b) and (c) are not satisfied. However, estimates made indicate that the use of Eqs. (6) and (7) with $N(n', l')$ taken to be the number of neutrons actually present in the incomplete shell is sufficiently accurate for the determination of the contribution of the incomplete shell, especially when it is considered that this is only a small part of the total contribution.

In order to obtain numerical results, it remains only to assume particular radial functions for the f_α, R_{nl}, and $R_{n'l'}$. The range and maximum depth of f_1, the radial function associated with the space exchange part of the exchange potential, are known with reasonable accuracy. For convenience of integration,[9] it is here approximated by a Gaussian well. The range is taken to be 1.5×10^{-13} cm. f_2 and f_3 are also assumed to be Gaussian wells of the same range, so that the three radial functions differ only in their respective strengths which are taken to be in the ratios D_1:D_2:D_3. Thus

$$f_\alpha(r_{\pi\nu})=-D_\alpha J_0\exp(-r_{\pi\nu}{}^2/a_1^2), \tag{8}$$

where J_0 is taken to be 55 Mev and D_1, which is certainly positive, must be of the order of unity if the neutron-proton potential is assumed to be primarily space exchange in nature. D_2 and D_3 may be positive or negative but are expected to be roughly of the same order of magnitude as D_1. The radial wave functions are chosen to be oscillator functions, also for convenience of integration.[9]

Subject to these assumptions, $K_\alpha(n, l)$ and $I_\alpha(n, l)$ are found to be roughly independent of n, l, and A. For heavy odd proton nuclei,

$$K_\alpha\approx0.25D_\alpha,\quad I_\alpha\approx0.10D_\alpha,\quad K_\alpha/I_\alpha\approx2.5, \tag{9}$$

where the ratio K_α/I_α is much more reliable than the values of either of the two integrals separately. With the values taken for a_1 and J_0, $b=3.0$ and the interaction moment contributions for odd proton nuclei are given by

$$\langle M_{\text{int}}(\text{odd } Z)\rangle=0.3\{D_1\langle l_{\pi^0}{}^z\rangle-2.5D_2\langle\sigma_{\pi^0}{}^z\rangle$$
$$-D_3(2j_{\pi^0}+2)^{-1}[2.5+2\langle\boldsymbol{\sigma}_{\pi^0}\cdot\mathbf{l}_{\pi^0}\rangle]\}. \tag{10}$$

For odd neutron nuclei the magnitudes of the integrals are somewhat diminished (owing to the preponderance

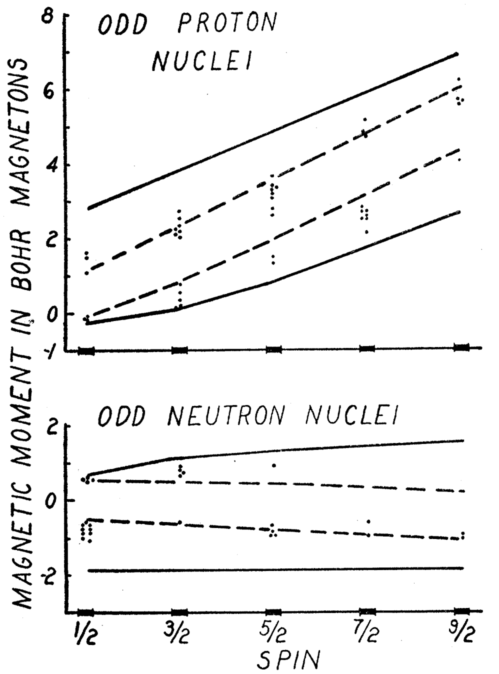

FIG. 1. The solid lines are the Schmidt curves for the magnetic moment in nuclear Bohr magnetons; the broken lines are the corrected curves taking into consideration interaction moments. The dots represent the experimental values for the magnetic moments of nuclei with A greater than 30.

of neutrons in heavy nuclei), although the ratio of the two integrals remains essentially unchanged, and

$$\langle M_{\text{int}}(\text{odd } N)\rangle = -0.25\{D_1\langle l_{\nu 0}^2\rangle - 2.5D_2\langle \sigma_{\nu 0}^z\rangle - D_3(2j_{\nu 0}+2)^{-1}[2.5+2\langle \boldsymbol{\sigma}_{\nu 0}\cdot \mathbf{l}_{\nu 0}\rangle]\}. \quad (11)$$

The minus sign is a consequence of the antisymmetry of the operators (1) in neutrons and protons. The multiplicative constant 0.25 is not expected to be accurate to two decimal places but merely indicates that the magnitude of the interaction moment of a heavy odd neutron nucleus has been found to be somewhat smaller than that of a corresponding odd proton nucleus.

The expressions (10) and (11) differ from the corresponding expressions of Miyazawa in two respects. Firstly, since he assumed the same density for neutrons as for protons, his expression for odd Z and odd N nuclei differs in sign but not in magnitude. Secondly, his expression is slightly different from the expression contained within the curly brackets of Eqs. (10) and (11). Since the nuclear model used here is the more realistic one, the results obtained here are probably somewhat better.

Reasonable agreement with the data is obtained with $D_1=0.4$, $D_2=4.6$, and $D_3=-7.0$. It is to be noted that D_1, which is a measure of the fraction of the potential which is of an exchange nature, lies between 0 and 1, as is to be expected. While the above values are not very precisely determined, it would be difficult to match the data with D_1 outside of the range 0 to 0.8, but because of the uncertainties in the numerical values of the I_α and of the K_α, and because of the scatter of the experimental data, the value of D_1 cannot be very much more restricted. The interaction moments are plotted with the above values of the D_α as corrections to the Schmidt curves in Fig. 1.

IV. MORE DETAILED BEHAVIOR OF THE INTERACTION MOMENTS

It was found, in the previous section, that the interaction moments of odd neutron nuclei are less than the interaction moments of odd proton nuclei of comparable A and with the same l and j. This prediction that odd neutron nuclei should deviate less from the Schmidt curves than do odd proton nuclei is in general agreement with the data.

In the analysis of the general trend of the deviations, it was found that the $I_\alpha(n, l)$ and the $K_\alpha(n, l)$ are roughly independent of A. This is due to the compensation of two effects; as one goes to heavier nuclei, the number of interactions is increased but the average interaction moment per pair is decreased. The decrease will be referred to as a size effect. However, on the addition of two nucleons of the odd type, i.e., two protons to an odd proton nucleus or two neutrons to an odd neutron nucleus, provided that there is no change in l or j, there is only a size effect, so that the magnitude of the interaction moment should decrease. Thus the addition of two nucleons of the odd type should push the moment toward the Schmidt values.[20]

Table I lists all pairs of nuclei which are in the same l and j state and which differ by two nucleons of the odd type. $\delta(\Delta M)$ is the difference in the magnitudes of the deviations from the Schmidt curves between the lighter and heavier nucleus, respectively, and is so defined as to be positive if the heavier nucleus lies closer to the Schmidt value. The decrease $\delta(\Delta M)$ in the magnitude of the interaction moment is estimated to be of the order of a few hundredths of a nuclear Bohr magneton, and if the observed shift is much greater than this, it is probably due primarily to other effects, making the above predictions invalid. As is seen in Table I, the predicted behavior is followed in nearly all known cases. As for the two exceptions, the magnitude of the difference in the moments of Nd^{143} and Nd^{145} indicates that it is due to other effects, while the Mayer level scheme requires the coupling of three or more nucleons to explain the angular momentum of Sm^{147} and Sm^{149}.

It is to be noted that the above prediction is valid even if the nuclear state is not of the pure Mayer type but involves some small admixture of other states,[21] provided that the percentage of the admixed state is not altered by the addition of the two odd type nucleons.

[20] de Shalit arrived at a similar conclusion for two nucleons of the odd or even type by an examination of the data. R. K. Wangness, Phys. Rev. **78**, 620 (1950); A. de Shalit, Phys. Rev. **80**, 103 (1950).

[21] See, e.g., J. P. Davidson, reference 3.

On the other hand, any theory which correlates some relevant nuclear property, such as the percentage of admixed states, to the nuclear size may also be able to explain the above effect.

Nothing can reliably be predicted regarding the effect of the addition of two neutrons to an odd proton nucleus or of the addition of two protons to an odd neutron nucleus, for in this case both effects, namely, the size effect and the increase in the number of interactions, are present, and it has already been seen that these two effects are in opposite directions. There is the further complication that the extra particles will not, in general, enter in such a way that the assumptions required for the applicability of Eqs. (10) and (11) to calculate the additional interaction moment are satisfied. Thus, for example, in computing the additional interaction moment, the tensor term contribution to $\langle M_1{}^z\rangle$ will not in general vanish.

It should be mentioned here that none of the effects obtained in this section depend on the values of the D_α.

V. GENERALIZATION OF THE MODEL

The results derived thus far presuppose not only the the validity of the general features of the one-particle interpretation of the shell model, but even depend upon the details of that model. Thus, a particular order of energy levels has been assumed and specific radial functions have been chosen for the nucleons within the core and for the outer nucleon. While the possibility exists that levels which are full for lighter nuclei may be empty for heavier nuclei (owing to level crossings), the fair degree of constancy of the $I_\alpha(n, l)$ and $K_\alpha(n,l)$ as the levels are filled indicates that the results obtained are not too dependent upon the particular order used, but the question remains as to how sensitive the results are to the specific radial functions chosen for the nucleons within the core and for the outer nucleon, and indeed to the very assumption that the core need be thought of as consisting of shells.[22]

A more serious question that must be considered is the sensitivity of the results to the assumptions made concerning the radial functions f_2 and f_3 about which nothing is known *a priori* other than that their ranges must be comparable with that of f_1.

It will be assumed, in what follows, that most of the nucleons in the core are in a state in which the spin angular momenta and orbital angular momenta of the neutrons and protons separately are all equal to zero. It will also be assumed that the particle density is roughly uniform throughout the core. (Both of these assumptions are, for heavy nuclei, certainly satisfied by the shell model.) Finally, it will be assumed that the ranges a_α of the functions f_α are small compared to the dimensions of the nucleus under consideration. Very light nuclei are thus specifically excluded.

[22] One of the authors (L.S.) would like to thank Dr. R. G. Sachs for a very helpful comment on the possibility of generalizing the model.

It follows from the last assumption that the only appreciable contributions to the integrals $I_\alpha(n,l)$ and $K_\alpha(n,l)$ come from the regions in which $\mathbf{r}_{\pi^0}\approx\mathbf{r}_{\nu'}$. Under these conditions,

$$r_{\pi^0}r_{\nu'}(1-\mu^2)\approx r_{\pi^0\nu'}{}^2\approx r_{\pi^0}{}^2\theta_{\pi^0\nu'}{}^2. \tag{12}$$

The integrands are thus vanishingly small for $r_{\pi^0\nu'}>a_\alpha$, or $\theta_{\pi^0\nu'}>a_\alpha/r_{\pi^0}$. Taking typical values for a_α and r_{π^0} and considering the the maximum values of l found in the nucleus, it is, then, a good approximation to let

$$[l(l+1)]^{-1}P_l'(\mu)\approx\tfrac{1}{2}P_l(\mu). \tag{13}$$

Using Eqs. (12) and (13), it is found that $K_\alpha(n,l)/I_\alpha(n,l)\approx 2$. This is to be compared with the value of 2.5 obtained for this ratio in Sec. III on the basis of the shell model. Since the proton interacts with only those neutrons in a volume of radius a_α, it follows, since we can take $P_l(\mu)\approx 1$, that

$$\tfrac{1}{2}K_\alpha(n,l)\approx I_\alpha(n,l)\approx CD_\alpha'(N/A), \tag{14}$$

where

$$D_\alpha'=(1/8\pi)(a_\alpha/a_1)^2(a_\alpha/r_0)^3,$$

$(4/3)\pi r_0{}^3A$ is the nuclear volume, and C is a constant between 0 and 1. Equation (6) now gives

$$\begin{aligned}\langle M_{\text{int}}{}^z(\text{odd }Z)\rangle &= bC(N/A)\{D_1'\langle l_{\pi^0}{}^z\rangle-2D_2'\langle\sigma_{\pi^0}{}^z\rangle\\ &\quad -D_3'(2j_{\pi^0}+2)^{-1}[2+2\langle\boldsymbol{\sigma}_{\pi^0}\cdot\mathbf{l}_{\pi^0}\rangle]\},\\ \langle M_{\text{int}}{}^z(\text{odd }N)\rangle &= -bC(Z/A)\{D_1'\langle l_{\nu^0}{}^z\rangle-2D_2'\langle\sigma_{\nu^0}{}^z\rangle\\ &\quad -D_3'(2j_{\nu^0}+2)^{-1}[2+2\langle\boldsymbol{\sigma}_{\nu^0}\cdot\mathbf{l}_{\nu^0}\rangle]\},\end{aligned} \tag{15}$$

where the (N/A) (which is proportional to the neutron density) of the odd proton case is replaced by $(-Z/A)$ (which is proportional to the proton density) for the

TABLE I. Variations in the deviations of the magnetic moments from the Schmidt curves for nuclei of the same l and j differing by two nucleons of the odd nucleon type.

		State	$\delta(\Delta M)_{\text{exp}}$
Odd Z			
	Cl^{37}		
	K^{39}	$d_{3/2}$	$+0.29$
	Cs^{137}		
	La^{139}	$g_{7/2}$	$+0.06$
Odd N			
	Mo^{95}		
	Mo^{97}	$d_{5/2}$	$+0.02$
	Cd^{111}		
	Cd^{113}	$s_{1/2}$	$+0.02$
	Sn^{115}		
	Sn^{117}	$s_{1/2}$	$+0.08$
	Sn^{117}		
	Sn^{119}	$s_{1/2}$	$+0.05$
	Te^{123}		
	Te^{125}	$s_{1/2}$	$+0.15$
	Ba^{135}		
	Ba^{137}	$d_{3/2}$	$+0.10$
	Nd^{143}		
	Nd^{145}	$f_{7/2}$	-0.38
	Sm^{147}		
	Sm^{149}	$?_{5/2}$	-0.05

$\delta(\Delta M)=|\Delta M_L|-|\Delta M_H|$ where ΔM_L and ΔM_H are the deviations from the Schmidt curves of the moments of the lighter nucleus and heavier nucleus, respectively.

TABLE II. Magnetic moments of odd-odd nuclei. $\Delta M = M_{\text{exp}} - M_{\text{Schmidt}}$. ΔM_{calc} is the deviation from the Schmidt value calculated from the data on odd-A nuclei.

	Proton orbital	Neutron orbital	J	M_{exp}	ΔM	ΔM_{calc}
K^{40}	$d_{3/2}$	$f_{7/2}$	4	-1.29	0.39	1
K^{40}	$s_{1/2}$	$f_{7/2}$	4	-1.29	2.17	-0.5
Co^{60}	$f_{7/2}$	$p_{3/2}$	5	3.2[a]	-0.7	0.3[b]
Rb^{86}	$f_{5/2}$	$g_{9/2}$	2	-1.68	0.46	0.3

[a] C. J. Gorter *et al.*, Physica **18**, 135 (1952); B. Bleany *et al.*, Phys. Rev. **85**, 688 (1952).
[b] Gorter *et al.* quote a value of -0.37.

odd neutron case. The only differences between Eqs. (15) and the Eqs. (10) and (11) derived using the shell model is the value 2 rather than 2.5 for the ratio K_α/I_α appearing within the curly brackets and the appearance of the factors (N/A) and $(-Z/A)$. The presence of the D_α' rather than the D_α is, except for the interest in D_1, of no significance since these quantities are chosen to match the data.

It is seen from Eqs. (15), derived under fairly broad assumptions, that any small variations in the ranges and shapes of the wells cannot critically affect the results of the previous sections, since the errors introduced in assuming them to be of the same form are by and large taken into account when the coefficients are matched to the data. It is further seen that the conclusions reached in those sections, except those concerning the magnitudes of D_2 and D_3 and, to a much lesser extent, of D_1, are essentially independent of the assumption of the shell model for the core.

The conclusions of Sec. IV concerning the moments of odd neutron *vs* odd proton nuclei and concerning nuclei differing by two nucleons of the odd type are now immediate consequences of the "density factors"‡ (N/A) and (Z/A) of Eqs. (15). Taken literally, the Eqs. (15) also predict that for given l and j, the interaction moments of odd proton nuclei should increase with increasing A, while those of odd neutron nuclei should decrease with increasing A. That this last prediction does not give good agreement with the data is not surprising, for its validity requires that the I_α and K_α remain fairly constant over a wide range of A and for those cases in which the odd nucleon is in different radial quantum number states. The former predictions, however, require only the relative constancy of the I_α and K_α for small variations of A.

VI. ODD-ODD NUCLEI

For self-conjugate nuclei, the operator given in Eq. (1) will give no contribution to the magnetic moment, to the extent of the equality of neutron-neutron and proton-proton forces. There will be contributions due to operators which are symmetric in neutrons and protons but these operators arise only in higher order meson calculations, and their contributions can therefore be expected to be small. It is then satisfying that the magnetic moments of self-conjugate nuclei can all be reasonably well accounted for by the orbital and spin contributions of the individual nucleons.[23]

For odd-odd nuclei which are not too light, for which the odd proton and odd neutron are in states $l_{\pi0}$, $j_{\pi0}$ and $l_{\nu0}$, $j_{\nu0}$, respectively, and for which the core has zero angular momentum, it is found, neglecting the contribution due to the interaction of the odd proton with the odd neutron, that[24]

$$\Delta M(l_{\pi0}, j_{\pi0}, l_{\nu0}, j_{\nu0}, j) = (\langle j_{\pi0}{}^z\rangle/j_{\pi0})\Delta M(l_{\pi0}, j_{\pi0}) + (\langle j_{\nu0}{}^z\rangle/j_{\nu0})\Delta M(l_{\nu0}, j_{\nu0}). \quad (16)$$

Here ΔM represents the difference between the experimental value and the calculated spin and orbital contributions. Thus, $\Delta M(l_{\pi0}, j_{\pi0})$ and $\Delta M(l_{\nu0}, j_{\nu0})$ can be taken from the experimental data for odd A-nuclei. (The data should, however, be weighted in favor of nuclei with comparable A.) Equation (16) is valid for any two-particle interaction moment operator; there is no restriction to the operators listed in Eq. (1).

Table II lists the experimental and calculated deviations from the Schmidt values of the measured odd-odd nuclei which are not self-conjugate. The only case in which the contributions of the odd proton and odd neutron to the calculated deviation are of the same sign is that of K^{40} with the odd proton in the $d_{\frac{3}{2}}$ state, and in this case the sign of the calculated deviation is in the right direction,[24] although the magnitude is too large. The other calculated values are therefore more questionable; in particular, ΔM of Co^{60} could be slightly negative. On the other hand, the possibility that the odd proton in K^{40} is in the $s_{\frac{1}{2}}$ state (which would eliminate the contradiction of the Nordheim spin composition rule[25]) is ruled out, for ΔM in this case cannot reasonably be taken to be greater than about zero.

VII. CONCLUSIONS

Making very few assumptions about the nature of the core of the nucleus, the radial function of the outer nucleon, or the radial functions associated with the interaction operators, it is found that the general trend of the deviations of the experimentally determined magnetic moments from the Schmidt curves is describable in terms of a linear combination of three interaction moment operators that can arise from a second order meson theoretical calculation. As is to be expected on physical grounds, the interaction moments of odd proton nuclei and of odd neutron nuclei turn out to be proportional to the neutron and proton densities, respectively. Interaction moment contributions are thus one possible explanation of the fact that the addi-

‡ *Note added in proof:*—The larger deviation of odd proton nuclei can also be explained by the assumption of a strong spin-orbit coupling term in the Hamiltonian; J. H. D. Jensen and M. G. Mayer, Phys. Rev. **85**, 1040 (1952).

[23] E. Feenberg, Phys. Rev. **76**, 1275 (1949).
[24] This result was indicated by I. Talmi, Phys. Rev. **83**, 1248 (1951), who did not, however, discuss its scope or range of validity.
[25] L. W. Nordheim, Revs. Modern Phys. **23**, 322 (1951).

tion of two neutrons to an odd neutron nucleus or of two protons to an odd proton nucleus pushes the moments towards the Schmidt curves, provided that l and j are unchanged, and of the fact that odd neutron nuclei tend to deviate less from the Schmidt curves than do odd proton nuclei with the same l and j and with comparable A.

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APPENDIX

The evaluation of the matrix elements of $M_1{}^z$ and $M_3{}^z$ is greatly facilitated by the use of the well-known theorem that in the diagonal matrix element of any vector $\mathbf{A}$ whose components satisfy certain commutation rules, $\mathbf{A}$ can be replaced by $[j(j+1)]^{-1}(\mathbf{A}\cdot\mathbf{J})\mathbf{J}$. In the case of $\mathbf{M}_3$, the theorem should be applied immediately. In the case of $\mathbf{M}_1$, the theorem should be applied after the spin of the outer particle has been eliminated, with j now replaced by l. The matrix element then involves a scalar operator and, as such, is independent of the projection of l. Equation (2) follows, finally, upon the use of the relation

$$\mathbf{r}_1\times\mathbf{r}_2\cdot\mathbf{L}_1P_l(\mu)=-i\hbar r_1r_2(1-\mu^2)P_l'(\mu).$$

Specialization to the shell model is then trivial. The shell model evaluation of $\langle M_1{}^z\rangle$ also follows from Appendix I of reference 9 by setting

$$I_{nln'l'}=\Big(\sum_{m=-l}^{l} m^2K_{nlmn'l'}\Big)\Big/\sum_{m=-l}^{l} m^2.$$

The derivation there is more general in that it includes cases in which there are three or more particles outside of the core.

There are certain formal advantages to replacing the factor $[\mathbf{r}_{\pi\nu}\cdot(\boldsymbol{\sigma}_\pi-\boldsymbol{\sigma}_\nu)]\mathbf{r}_{\pi\nu}$ of $\mathbf{M}_3$ by $[\mathbf{r}_{\pi\nu}\cdot(\boldsymbol{\sigma}_\pi-\boldsymbol{\sigma}_\nu)]\mathbf{r}_{\pi\nu}-\frac{1}{3}(\boldsymbol{\sigma}_\pi-\boldsymbol{\sigma}_\nu)r_{\pi\nu}{}^2$, which corresponds to a linear combination of $\mathbf{M}_2$ and $\mathbf{M}_3$, since this has simpler transformation properties under space rotation and Eq. (4) assumes a simpler form.

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Surface Production of Charged Mesons by Photons on Nuclei*

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An estimate has been made of the photoproduction cross section of charged mesons from nuclei, and in particular from "surface-nucleons," i.e., the weakly interacting nucleons which make up the less dense nucleon atmosphere surrounding the main body of a nucleus. Comparison with experiment indicates that the production of mesons from the core of a nucleus is appreciably suppressed, apart from the effects of the initial momentum distribution of the nucleus, and of meson absorption. It is found, for example, that apart from having the correct $A^{\frac{2}{3}}$ dependence for the $\pi^++\pi^-$ cross section, the surface production alone can account for large fractions of the observed yields, and because of differences in average binding between neutrons and protons in nuclei, gives π^-/π^+ ratios which have the same trends as a function of A as the observed ratios. A possible explanation of these results is that there occurs a large competing photodisintegration process as a result of meson exchange effects between strongly coupled nucleons in the interior of a nucleus.

1. INTRODUCTION

The main features of the experimental observations on the production of charged π-mesons from nuclei are the following:

(1) The yields are considerably less than from appropriate equal numbers of free nucleons,[1,2] and

(2) the sum of the π^+ and π^- cross sections exhibits very accurately an $A^{\frac{2}{3}}$ dependence.[1,3]

There are two well-known effects that are in the right direction for explaining these results. Firstly, because of the momentum distribution of nucleons in a nucleus, it is not energetically possible for all the protons or all the neutrons to participate in the production.[4] Secondly, some of the mesons which are produced will be absorbed before they escape from the nucleus. Each of these effects in general reduces the meson yields, and since the absorption of a meson produced in the interior of a nucleus is more probable than for one produced at the surface, the second tends to produce the observed $A^{\frac{2}{3}}$ dependence.[5]

However, on the basis of estimates[6] of the absorption mean free path for mesons in nuclear matter, obtained from the results of meson scattering experiments, the very good experimental $A^{\frac{2}{3}}$ dependence is difficult to understand as due merely to absorption of the mesons. This result might be taken to indicate, therefore, that there is in some way a further suppression of the meson

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[1] R. M. Littauer and D. Walker, Phys. Rev. 83, 206 (1951); 86, 838 (1952).
[2] J. Steinberger and A. Bishop, Phys. Rev. 78, 494 (1950).
[3] R. F. Mozley, Phys. Rev. 80, 493 (1950).
[4] M. Lax and H. Feshbach, Phys. Rev. 81, 189 (1951).
[5] Brueckner, Serber, and Watson 84, 258 (1951).
[6] J. Steinberger (private communication).