Structure of the Nucleon*

R. G. SACHS
University of Wisconsin, Madison, Wisconsin
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An attempt is made to determine what restrictions are imposed on the nucleon wave functional by the known values of the nucleon moments and by the neutron-electron interaction. The nucleon is assumed to consist of a core particle (nucleore) of spin $\frac{1}{2}$ surrounded by a pion field. No detailed reference is made to the interaction producing the field. Nucleore recoil is neglected. It is found that the neutron and proton moments satisfy a mirror condition

$$\mathfrak{M}_n+\mathfrak{M}_p=1-(4/3)P_1,$$

where P_1 is the probability that pions (any number or charge) occur in the field with total orbital angular momentum $L=1$. Insertion of the measured values of the moments in the equation yields $P_1=9$ percent. A model of the nucleon in which one pion plays the predominant role is not consistent with this result. This is probably the underlying reason for the failure of the weak coupling theory to give the correct ratio of neutron to proton moments. The neutron and proton moments can be accounted for if the field contains at least two pions with appreciable probability. A successful model consists of 91 percent bare nucleore, and 9 percent a pair of mesons, each in p states forming the state $L=1$.

The neutron electron interaction is shown to depend on the mean square radius of the charge distribution, $\langle r^2\rangle_{\mathrm{Av}}$, in the nucleon. If only one or two pions with $L=1$ are contained in the proper field with appreciable probability, the observed interaction combined with the above value of P_1 leads to the very reasonable value of the mean square displacement of a pion, ≈ 0.5 times the square of the Compton wavelength of the pion.

The results suggest strongly that a weak nonlinear coupling would be capable of accounting for the data, but a linear coupling of intermediate strength cannot be excluded.

An analysis of the pseudoscalar field in terms of spherical waves is given in the Appendix. Consideration is also given there to the space and time inversion properties of the field functionals.

1. INTRODUCTION

THE original proposal of Yukawa that the nuclear forces are due to an emission and absorption of mesons by nucleons implies that the nucleon is a structured system, that it consists of a core (the *nucleore*) convoyed by a cloud of mesons, known as the meson proper field. Estimates of the coupling between the nucleore and the meson field lead to the conclusion that it is not weak, so the proper field may be presumed to be rather intense, from which it may be concluded that the structural features of the nucleon have a significant effect on its physical properties.

Attempts to describe the structure of the proper field have usually been based on a meson theory which is, at least in principle, complete. A specific form of the interaction between the meson and the nucleore is assumed and an attempt is made to solve the corresponding dynamical problem to obtain the wave functional of the nucleon. The physical properties of the nucleon, such as the magnetic moment, electron-neutron interaction, and so on, may then be obtained from this functional. These theories do not yield correct quantitative results in the approximations to which the calculations have been carried out. This failure may be due to the inadequacy of the approximations or it may be due to the use of an incorrect form of the nucleore-meson coupling.

Another approach to the problem, which may shed some light on the cause of failure of the theory, is to consider the structure of the nucleon much as we might consider the structure of a nucleus, namely, to seek a wave functional which fits the data.[1] This approach implies that we accept the spirit of the original Yukawa theory in that the nucleon is described by a functional of the meson field such that there is a finite probability for the occurrence of one, two, or possibly more mesons. But rather than attempt a derivation of the functional from a specific interaction, we make use of all the general principles and of all the quantitative properties of the nucleon that are at our disposal to fix the form of the wave functional in so far as that is possible. It could turn out that our information is so limited, and the parameterization of the functional so complicated that no very enlightening results are obtained in this way. However, we will find that this is not the case; on the contrary, rather specific limitations on the functional are indicated by the available data.

Although the proposed approach is somewhat less fundamental than the direct dynamical attack, it has the advantage that it offers a procedure for giving a physical interpretation of the data on the nucleon. Furthermore, it may turn out that the nucleon has so complex a structure as to require discussion in terms of its structure rather than in terms of the solution of a simple dynamical problem. Finally, there is a fair chance that a knowledge of the wave functional may make clear the source of difficulty in the present fundamental theories. In this connection it is to be noted that most attempts at the theory have been based on a linear coupling between nucleore and field which is treated either as small in magnitude (weak coupling) or as large in magnitude (strong coupling). It is possible that a calculation for intermediate coupling would be successful. On the other hand, the possibility that the correct coupling is nonlinear certainly cannot be excluded.[2] Of course, the field equations for free mesons

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[1] A similar approach to the problem has recently been discussed by Umezawa, Takahashi, and Kamefuchi, Phys. Rev. 85, 505 (1952).

[2] See for example, L. I. Schiff, Phys. Rev. 84, 10 (1951) and R. J. Glauber, Phys. Rev. 84, 395 (1951).

may also be nonlinear as a consequence of meson-meson interactions.[3] Rather distinct differences between the wave functionals characteristic of these different possibilities are to be expected so direct information concerning the wave functional may suggest a choice between the alternative possible sources of difficulty.

It is characteristic of the presently conceived dynamical theories that they force consideration of mesons at relativistic energies and the consequent inclusion of nucleore recoil. On the grounds of simplicity, we argue that this is an undesirable feature of these theories and that, in view of the general failure of the theories, a feature which need not be included in our model of the nucleon. If we find a successful model involving only nonrelativistic mesons, we will then face the problem of constructing an interaction which is consistent with the model.

2. GENERAL RESTRICTIONS ON THE WAVE FUNCTIONAL

For the sake of definiteness we will base the discussion on the assumption that the only important structural element in the nucleon, other than the core, is the pion. The additional simplifying assumption is made that recoil of the nucleore can be neglected. Then the ground state of the system nucleore (at rest) plus pion field is that which is referred to as "the nucleon." The associated state vector is a functional of the pion field variables and a function of the spin variable of the nucleore. If an expansion of the wave functional in terms of appropriate free pion states (denoted by index s) is made, it is expected that it will have the form

$$\Psi=\Psi_0{}^{(\sigma)}+\sum_s\Psi_1{}^{(s,\sigma)}+\sum_{s,s'}\Psi_2{}^{(s,s',\sigma)}+\cdots, \qquad (1)$$

where σ is the nucleore spin variable and $\Psi_N{}^{(s,s',\cdots,\sigma)}$ describes a state of N pions in states $s, s', \cdots s^{(N)}$. The usual weak coupling approximation amounts to the neglect of all terms beyond the first two in this expression.

The form (1) implies that the nucleore is taken to have spin $\frac{1}{2}$. The fact that the nucleon has spin $\frac{1}{2}$ means that the total angular momentum of the nucleore plus the pion field associated with any term in the series (1) must be $\frac{1}{2}$. If the states of a given number of pions are classified by the total (orbital) pion angular momentum L, the requirement that $\mathbf{J}=\mathbf{L}+\frac{1}{2}\boldsymbol{\sigma}$ have the characteristic value $\frac{1}{2}$, immediately leads to the restriction

$$L=0 \text{ or } 1 \qquad (2)$$

on each term in Ψ, since we have chosen to ignore recoil of the nucleore. Furthermore, the parity of each term must be the same, so the pseudoscalar character of the pion field leads to the condition

$$(-1)^{\sum_j l_j}=(-1)^N \qquad (3)$$

for each term in (1), if l_j is the orbital angular momentum quantum number of the jth pion.

It is worth remarking here that, whereas weak coupling implies the approximation

$$\Psi\approx\Psi_0+\sum_s\Psi_1{}^{(s)}, \qquad (4)$$

linear coupling of the nucleore with the pion field implies that only terms with

$$l_j=1 \qquad (5)$$

are important. This can be established by considering the process of creation of a nucleon from an initial state consisting of a bare nucleore. Linear coupling means that the pions are emitted one at a time so conservation of angular momentum and parity at every stage of the process limits the orbital angular momentum of each pion to the above value.

Additional restrictions on the functional are introduced by conservation of charge. The nucleore is assumed to have just two charge states, 0 and 1, (isotopic spin $\frac{1}{2}$) so the total charge, C, of the pion field for any term in (1) is limited to the values 0 and 1 for the proton or 0 and -1 for the neutron. We will make the assumption that Ψ has the mirror property, namely, that the wave functional for the neutron is obtained when positive and negative pions are interchanged in the proton wave functional and, at the same time, the charge on the nucleore in each term of (1) is reversed (i.e., 1 is replaced by 0 and 0 by 1). In the language of isotopic spin this mirroring can be accomplished by the transformation in charge space corresponding to a reflection in a plane containing the axis of quantization.

Since pions occur as positive, negative, and neutral particles, it is reasonable to assign to the pion an isotopic spin $t=1$ and to consider the consequences of the assumption that the interactions are charge invariant, i.e., that they are invariant under rotations in charge space. This is the condition that leads to charge independence of nuclear forces. It implies that the total isotopic spin

$$\mathbf{T}=\sum_j\mathbf{t}_j+\tfrac{1}{2}\boldsymbol{\tau} \qquad (6)$$

is conserved. Here $\mathbf{t}_j$ is the isotopic spin vector of the jth pion and $\boldsymbol{\tau}$ is twice the isotopic spin operator of the nucleore. We take $\tau_j=\pm1$ to correspond to nucleore charges 1 and 0, respectively. The value of T for the nucleon state Ψ is $T=\frac{1}{2}$. Therefore, if

$$\mathbf{Y}=\sum_j\mathbf{t}_j \qquad (7)$$

is the total isotopic spin of the pions, its associated quantum numbers Y are restricted to the values

$$Y=0,\ 1. \qquad (8)$$

In the state $Y=0$ the net pion charge (projection of Y on the axis of quantization) is always zero so the pions occur as neutrals or in positive-negative pairs. In the state $Y=1$, the relative amplitudes of the two possible charge states are determined by the condition that

[3] Compare L. I. Schiff, Phys. Rev. **84**, 1 (1951); B. J. Malenka, Phys. Rev. **85**, 686 (1952).

$T=\frac{1}{2}$. If $Z_Y{}^C$ is the isotopic spin wave function of the pions for charge $C=Y$, $Y-1$, $\cdots -Y$, and $x^{\pm}$ is the isotopic spin function of the nucleore, then the total isotopic spin function of a neutron, when the pions have $Y=1$, has the form

$$X_{\frac{1}{2}}{}^{-\frac{1}{2}}=-\sqrt{\tfrac{2}{3}}Z_1{}^{-1}x^{+}+\sqrt{\tfrac{1}{3}}Z_1{}^{0}x^{-}. \tag{9}$$

Thus the probability, in the $Y=1$ state, for finding a net pion charge 0 is one-half the probability for finding a negative charge, -1.

More particular statements can be made concerning specific terms in Eq. (1). For one-pion states, Ψ_1, Eqs. (2) and (3) combine to impose the restriction

$$L=l=1,\quad \text{(one pion)}. \tag{10}$$

Also the total isotopic spin of the pions is just that of the single pion

$$Y=t=1,\quad \text{(one pion)}.$$

The charge in the one-pion state of the neutron is therefore distributed in accordance with Eq. (9), $\frac{2}{3}$ negative and $\frac{1}{3}$ neutral.

Simple statements concerning the two-pion states are also possible. If l_1 and l_2 are the orbital angular momenta of the two pions [not restricted by the linearity condition Eq. (5)], then $L=0$ can only be formed if $l_1=l_2$. For $L=1$ we might have $l_1=l_2$ or $l_1=l_2\pm 1$, but the parity condition, Eq. (3), eliminates all but the first possibility. Thus,

$$l_1=l_2\quad \text{(two pions)}. \tag{11}$$

The fact that the pion satisfies Einstein-Bose statistics implies that the wave function of any system composed of a fixed number of particles must be symmetric for the interchange of all coordinates (including the isotopic spin variable) of any pair of pions. This condition restricts the structure of the functions Ψ_2, Ψ_3 and so on. In particular, the two-pion states have the property that $Y=0$ is a symmetric function and $Y=1$ is an antisymmetric function. Since $l_1=l_2$ the angular parts of the $L=0$ and $L=1$ functions are also symmetric and antisymmetric, respectively [see Eqs. (25) and (26)]. Therefore if the functions are classified by the symmetry of the radial functions, we find that the radially symmetric states are restricted to the combinations

$$\left.\begin{array}{l} Y=0 \text{ for } L=0,\\ Y=1 \text{ for } L=1,\end{array}\right\}\quad \text{(two pions—radial sym)} \tag{12a}$$

and the radially antisymmetric states to the combinations

$$\left.\begin{array}{l} Y=1 \text{ for } L=0,\\ Y=0 \text{ for } L=1.\end{array}\right\}\quad \text{(two pions—radial antisym)} \tag{12b}$$

It follows that, in the symmetric states, the net pion charge is zero for $L=0$, while for $L=1$ the distribution of pion charge in the neutron is $\frac{2}{3}$ negative, $\frac{1}{3}$ zero. In the radially antisymmetric states, the charge is zero for $L=1$, and is distributed in the 2 to 1 ratio for $L=0$.

If we denote by $P_L{}^C(N)$ the probability for the occurrence of N pions in the neutron with total orbital angular momentum L and total charge C, the results of our discussion may be summarized by the statements:

$$P_0{}^C(1)=0, \tag{13a}$$

$$P_1{}^{\pm}(1)=2P_1{}^0(1), \tag{13b}$$

$$P_0{}^{\pm}(2,\mathcal{S})=0, \tag{13c}$$

$$P_1{}^{\pm}(2,\mathcal{S})=2P_1{}^0(2,\mathcal{S}), \tag{13d}$$

$$P_0{}^{\pm}(2,\mathcal{A})=2P_0{}^0(2,\mathcal{A}), \tag{13e}$$

$$P_1{}^{\pm}(2,\mathcal{A})=0, \tag{13f}$$

where the symbols $\mathcal{S}$ and $\mathcal{A}$ classify the two-pion states according to their radial symmetry. Only one sign C is admissable for a given nucleon, $C=+$ for the proton, $C=-$ for the neutron.

3. DESCRIPTION OF THE PION FIELD

The analysis of the foregoing section proceeds on the assumption that individual pions are assigned to states of definite orbital angular momentum in contrast to the more usual assignment to states of given linear momentum. The description of the quantized field in terms of these spherical waves is quite similar to the description in terms of plane waves. Since this particular form of analysis is important for the treatment of the nucleon magnetic moment, its pertinent features are described in Appendix 1. The discussion is based on the assumption that the free pions are properly described by the linear Pauli-Weisskopf theory.[4] Nonlinear theories involving pion-pion interactions[3] are excluded for reasons of simplicity.

It is shown in the Appendix that the field can be described in terms of the number $N_{klm}{}^{\gamma}$ of pions with charge $\gamma(=1, 0, \text{ or } -1)$ in a state of orbital angular momentum l with magnetic quantum number m. The energy of a neutral pion is $\hbar\omega_0$, with $\omega_0{}^2=c^2(k^2+\mu_0{}^2)$, and that of a charged pion is $\hbar\omega$, with $\omega^2=c^2(k^2+\mu^2)$, where $\hbar\mu_0/c$ and $\hbar\mu/c$ are the masses of neutral and charged pions. The quantum numbers (k, l, m) are collectively denoted by the symbol s, with the notation $-s\equiv(k, l, -m)$.

It is clear that in this representation the total angular momentum is not, in general, a good quantum number because each pion is assigned a magnetic quantum number. Since the pion field is to be assigned a total angular momentum L, it is of some interest to note the connection between the representation in terms of the $N_s{}^{\gamma}$ and that characterized by given L. The basic vectors in terms of which the functional of the field is

[4] For a discussion of the basic features of the theory in a corresponding notation see G. Wentzel, *Quantentheorie der Wellenfelder* (Franz Deuticke, Vienna, 1943); English translation (Interscience Publishers, Inc., New York, 1949).

described are the characteristic vectors $\Phi\{N_s^\gamma\}$ corresponding to fixed values of N_s^γ. For a state of a fixed number N^γ of pions of each charge, the functional $\Phi_L{}^{M_L}$ of total angular momentum L is a linear combination of those $\Phi\{N_s^\gamma\}$ for which $\sum_s N_s^\gamma = N^\gamma$ and $\sum_{s,\gamma} mN_s^\gamma = M_L$. Since the functionals $\Phi\{N_s^\gamma\}$ are just the properly symmetrized products of one particle functions,[5] the relationship between $\Phi_L{}^{M_L}$ and $\Phi\{N_s^\gamma\}$ is given by the unitary transformation

$$\langle l_1, m_1; l_2, m_2, \cdots l_N, m_N | l_1, l_2, \cdots l_N; L, M_L\rangle,$$

which can be obtained from standard sources.[6] Here, the number of (l_i, m_i) values equal to (l, m) is $\sum_{k,\gamma} N_s^\gamma$. Use is made of this transformation in calculating the two-pion contribution to the magnetic moment (Sec. 7). For more than two particles, the transformation is not, in general, uniquely determined.

Given the functionals $\Phi_L{}^{M_L}(N^+, N^0, N^-)$ of fixed numbers of pions, they can be combined with the nucleore spin function to form functionals $\Psi^M(N^+, N^0, N^-; L)$ of total angular momentum $\frac{1}{2}$ and magnetic quantum number $M(=\pm\frac{1}{2})$. Then the nucleon functional indicated by Eq. (1) is a linear combination of these:

$$\Psi^M = \sum_{(N)}\sum_L B_L(N^+, N^0, N^-)\Psi^M(N^+, N^0, N^-, L). \quad (14)$$

Now we wish to show that the coefficients $B_L(N^+, N^0, N^-)$ can always be chosen to be real numbers. The importance of this statement is that it reduces, essentially by a factor of 2, the number of parameters required to describe the functional. The proof of the statement is based on a time reversal argument of the type suggested by Wigner.[7] He has pointed out that, if the time reversal operation is represented by an operator K, then for a state of given total angular momentum j the wave function $\psi_j{}^m$ can always be chosen in such a way that

$$K\psi_j{}^m = i^{2m}\psi_j{}^{-m}. \quad (15)$$

It is shown in Appendix 2 that the choice of representation of the pion field is consistent with Eq. (15):

$$K\Phi_L{}^{M_L} = i^{2M_L}\Phi_L{}^{-M_L}. \quad (16)$$

Therefore, for an appropriate[8] choice of the coefficients used in combining the $\Phi_L{}^{M_L}$ with the nucleore spin functions, this property carries over to the $\Psi^M(N^+, N^0, N^-; L)$:

$$K\Psi^M(N^+, N^0, N^-; L) = i^{2M}\Psi^{-M}(N^+, N^0, N^-; L).$$

Now K involves taking the conjugate complex, so in any linear combination of functions the coefficients are conjugated by K. Therefore,

$$K\Psi^M = i^{2M}\sum_{(N)}\sum_L B_L{}^*(N^+, N^0, N^-) \times\Psi^{-M}(N^+, N^0, N^-; L).$$

But, unless there is an accidental degeneracy,

$$K\Psi^M = i^{2M}\Psi^{-M} = i^{2M}\sum_{(N)}\sum_L B_L(N^+, N^0, N^-) \times\Psi^{-M}(N^+, N^0, N^-; L),$$

whence it follows that the coefficients B_L are real. The possibility that the ground state of the nucleon is accidentally degenerate does not appear to warrant serious consideration.

The probability $P_L(N^+, N^0, N^-)$ for the occurrence in the nucleon of a state of N^γ pions of charge γ having total orbital angular momentum L is given by

$$P_L(N^+, N^0, N^-) = |B_L(N^+, N^0, N^-)|^2.$$

Since the B_L are real,

$$B_L(N^+, N^0, N^-) = \pm[P_L(N^+, N^0, N^-)]^{\frac{1}{2}}, \quad (17)$$

a result which will be used in Sec. 7.

4. MAGNETIC MOMENT OF THE NUCLEON

The contribution $\mathfrak{M}_0$ of the pion field to the magnetic moment operator is obtained in Appendix 1 by considering the change in energy of the pions caused by the introduction of a uniform magnetic field. There is some ambiguity about $\mathfrak{M}_0$ because it may receive a contribution from the pion-nucleore interaction term. It is well known that the electromagnetic field induces emission of pions in the pseudoscalar theory. However, any such "interaction moment" is ignored here, the justification being that for a point interaction between pion and nucleore, the static interaction moment vanishes in the approximation that nucleore recoil is negligible.

The pion moment operator may be expressed in terms of the creation and annihilation operators, $a_s{}^*$, a_s, for positive pions in states of given energy and angular momentum, and in terms of similar operators for negatives, $b_s{}^*$, b_s:

$$\mathfrak{M}_0 = \tfrac{1}{2}ec\sum_{s,s'}\delta_{kk'}\delta_{ll'}\omega^{-1}(a_s{}^*a_{s'} - b_s{}^*b_{s'} + (-1)^{m'}a_s{}^*b_{-s'}{}^* + (-1)^m b_{-s}a_{s'})(l, m|\mathbf{L}|l, m'), \quad (18)$$

where $(l, m|\mathbf{L}|l, m')$ is the matrix element of the one particle angular momentum operator, $\mathbf{L} = -i\mathbf{r}\times\text{grad}$.

In a state for which the number of pions is prescribed, only the first two terms contribute to the expectation value of the magnetic moment. Aside from the relativistic correction factor $(\mu c/\omega)$, the magnetic

[5] See, for example, H. Weyl, *The Theory of Groups and Quantum Mechanics* (Methuen & Company, Ltd., London, 1931), p. 246 ff.

[6] E. Condon and G. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1951), p. 76.

[7] E. P. Wigner, Nachr. Akad. Wiss. Göttinger, Math.-physik Kl., p. 546 (1932).

[8] Since we have chosen the functionals $\Phi_L{}^{ML}$ in such a way that they satisfy the condition Eq. (16), it follows from Wigner's general statement that one can always choose the basis in accordance with Eq. (15). The usual transformation coefficients $(J, M, L, S|L, M_L, S, M_S)$ satisfy the condition

$$(J, -M, L, S|L, -M_L, S, -M_S) = (-1)^{L+S-J}(J, M, L, S|L, M_L, S, M_S),$$

from which it follows that they are to be modified by multiplication by i^{L+S-J} in order that Eq. (15) be satisfied. This point was called to the author's attention by Dr. E. N. Adams, II.

moment contribution in such a state is just what would be expected for a system of particles with given l values, $\pm(e/2\mu)\mathbf{l}$ for each pion, the sign being positive for positive pions, negative for negative pions. But in the nucleon state, Eq. (14), the number of pions is not specified, so the pair terms $a_s{}^*b_{-s'}{}^*$ and $a_sb_{-s'}$ in Eq. (18) may play a role in determining the nucleon moment. Since they can contribute only to matrix elements between states $\Psi(N^+, N^0, N^-)$ and $\Psi(N^+-1, N^0, N^--1)$, these terms are only of importance if at least two pions occur with appreciable probability in the nucleon proper field.

$\mathfrak{M}_0$ is just the pion orbital part of the magnetic moment operator. Presumably the charged nucleore has a moment of about[9] one nuclear magneton associated with its spin, $\boldsymbol{\sigma}$. The total moment operator is therefore

$$\mathfrak{M}=(e/2\mathfrak{M})\tfrac{1}{2}(1+\tau_3)\boldsymbol{\sigma}+\mathfrak{M}_0,$$

where $\hbar\mathfrak{M}/c$ is the mass of the nucleore. The magnetic moment of the nucleon is the expectation value

$$\mathfrak{M}=\langle\Psi^{\frac{1}{2}},\mathfrak{M}_z\Psi^{\frac{1}{2}}\rangle \tag{19}$$

in the state $\Psi^{\frac{1}{2}}$ for which the z component of the total angular momentum is $\frac{1}{2}$.

5. THE MIRROR THEOREM

It seems very reasonable to assume that the mirror property described in Sec. 2 applies to the nucleon functional Ψ in good approximation. This assumption is less restrictive than the requirement of charge invariance (conservation of total isotopic spin) although it is contained implicitly in that property.

According to the mirror property, the same functional can be used to describe neutron and proton if we understand that a given variable appearing in Ψ refers to oppositely charged pions for the two different nucleons. The isotopic spin of the nucleore also has the opposite value. Then, if the operators a_s, b_s refer to positive and negative pions respectively for the neutron, they refer to negative and positive pions for the proton; so the operators describing an observable characteristic of the proton differ from those characteristic of a neutron by the interchange of the quantities a_s and b_s and by changing the sign of τ_3. In particular, if the neutron moment is obtained by taking the expectation value of (we now use units of nuclear magnetons)

$$\mathfrak{M}_n=\tfrac{1}{2}(1+\tau_3)\boldsymbol{\sigma}+\textstyle\sum_{s,s'}(\mathfrak{M}c/\omega)\delta_{kk'}\delta_{ll'}(l,m|\mathbf{L}|l,m')$$
$$\times(a_s{}^*a_{s'}-b_s{}^*b_{s'}+(-1)^{m'}a_s{}^*b_{-s'}{}^*+(-1)^{m}b_{-s}a_{s'}), \tag{20}$$

then the proton moment is to be obtained by taking the expectation value for the identical state of

$$\mathfrak{M}_p=\tfrac{1}{2}(1-\tau_3)\boldsymbol{\sigma}+\textstyle\sum_{s,s'}(\mathfrak{M}c/\omega)\delta_{kk'}\delta_{ll'}(l,m|\mathbf{L}|l,m')$$
$$\times(b_s{}^*b_{s'}-a_s{}^*a_{s'}+(-1)^{m'}b_s{}^*a_{-s'}{}^*+(-1)^{m}a_{-s}b_{s'})\epsilon_m\epsilon_{m'}.$$

A quantity of particular significance is then seen to be the sum of neutron and proton moments, which is to be found by taking the expectation value in the given state of

$$\mathfrak{M}_n+\mathfrak{M}_p=\boldsymbol{\sigma}+\textstyle\sum_{s,s'}(\mathfrak{M}c/\omega)\delta_{kk'}\delta_{ll'}(l,m|\mathbf{L}|l,m')$$
$$\times((-1)^{m'}a_s{}^*b_{-s'}{}^*+(-1)^{m}b_{-s}a_{s'}+(-1)^{m'}b_s{}^*a_{-s'}{}^*+(-1)^{m}a_{-s}b_{s'}).$$

Diligent application of Eq. (A-5) leads immediately to the vanishing of the second term[10] so the sum of neutron and proton moments is, according to Eq. (19),

$$\mathfrak{M}_n+\mathfrak{M}_p=\langle\Psi^{\frac{1}{2}},\sigma_z\Psi^{\frac{1}{2}}\rangle. \tag{21}$$

Now we have seen in Sec. 2 that $\Psi^{\frac{1}{2}}$ consists of a mixture of a state containing no pions with states containing one or more pions whose total orbital angular momentum is either $L=0$ or $L=1$. The total angular momentum for each state is $J=\frac{1}{2}$. Equation (21) clearly contains no cross terms between the states with differing numbers of pions or different L values. In the no-pion state

$$\langle\Psi_0{}^{\frac{1}{2}},\sigma_z\Psi_0{}^{\frac{1}{2}}\rangle=1$$

and in a state of given L, the expectation value may be calculated by the usual vector rule:

$$\langle\Psi^{\frac{1}{2}}(L=0),\sigma_z\Psi^{\frac{1}{2}}(L=0)\rangle=1,$$
$$\langle\Psi^{\frac{1}{2}}(L=1),\sigma_z\Psi^{\frac{1}{2}}(L=1)\rangle=-\tfrac{1}{3}.$$

If we introduce the probability P_L for the occurrence of any number of pions of any energy but having a fixed total orbital angular momentum L, each of these expectation values contributes to the moment with a weight given by the appropriate P_L. The probability of the no-pion state is $1-P_0-P_1$ so the sum of the moments is found to be

$$\mathfrak{M}_n+\mathfrak{M}_p=1-(4/3)P_1. \tag{22}$$

Thus the sum of the measured neutron and proton moments yields direct information on the probability for the occurrence of pion states with $L=1$ in the nucleon. Insertion of the experimental value $\mathfrak{M}_n+\mathfrak{M}_p=0.880$ leads to the result

$$P_1=0.090. \tag{23}$$

6. THE NEUTRON MOMENT—ONE-PION MODEL

It is now of some interest to consider the restrictions imposed on the magnetic moment of a single nucleon by the condition Eq. (23). The moment depends in a detailed way on the structure of the nucleon wave functional so rather detailed assumptions concerning the functional must be made in order to arrive at definite conclusions. The simplest assumption is that suggested by the weak coupling theory, that no more than one pion occurs with an appreciable probability.

[9] The nucleore magneton rather than the nucleon magneton should be used here, but, in the absence of knowledge concerning the nucleore mass, we assume that the difference is small.

[10] The fact that the pion contributions cancel has been observed by Y. Takahashi, Prog. Theoret. Phys. 6, 624 (1951); see also reference 1.

This will be called the one-pion model of the nucleon. For the sake of definiteness we treat the neutron.

In the one-pion model of the neutron,

$$P_1 = P_1^0(1) + P_1^-(1).$$

According to Eq. (13b), the two probabilities are then

$$P_1^0(1) = \tfrac{1}{3}P_1,$$

and

$$P_1^-(1) = \tfrac{2}{3}P_1.$$

Furthermore,

$$P_0 = P_0^0(1) = P_0^-(1) = 0.$$

The pair terms in the magnetic moment operator Eq. (20) play no role in determining the static moment in this model, so the moment is to be obtained by the usual vector rule. There is no magnetic moment contribution from the no-pion and neutral pion states, while in the negative pion state which occurs with probability $\frac{2}{3}P_1$, the contribution of the nucleore is $-\frac{1}{3}$ while that of the pion is $-\frac{2}{3}\langle \mathfrak{M}c/\omega\rangle$. Here, the angular brackets denote the average over the energy (ω) distribution of the pions. The neutron moment is therefore

$$\mathfrak{M}_n = -(2/9)[1 + 2\langle \mathfrak{M}c/\omega\rangle]P_1. \tag{24}$$

Now $\langle \mathfrak{M}c/\omega\rangle$ is clearly less than $\mathfrak{M}/\mu$, so

$$|\mathfrak{M}_n| < (2/9)(1 + 2\mathfrak{M}/\mu)P_1.$$

If we insert the nucleon to pion mass ratio, $\mathfrak{M}/\mu = 6.64$, and the value $P_1 = 0.09$, the result is

$$|\mathfrak{M}_n| < 0.29.$$

This is in direct contradiction to the experimental fact that $\mathfrak{M}_n = -1.91$. Thus we can say that, although Eq. (24) could give agreement with the neutron moment alone, agreement is not possible if we take cognizance of the condition imposed on P_1 by the sum of neutron and proton moments. The neutron and proton moments together are not in accord with the one-pion model. This conclusion is quite independent of the assumption of charge invariance which is required to obtain the particularly simple form of Eq. (24).

Since the one-pion model is substantially equivalent to weak coupling theory, our result is probably the underlying reason for the failure of that theory to give the correct ratio of neutron to proton moment. However, the usual dynamical meson theories are such that relativistic pions play a dominant role, so nucleon recoil is an important factor in the calculation of the moments.[11] Therefore, too close a correspondence between the results of the dynamical theories and our no-recoil theory is not to be made.

7. THE NEUTRON MOMENT—TWO-PION MODEL

Relatively simple results are obtained if we extend the treatment of the neutron moment to include two-pion states. The contribution of one-pion states is still given by Eq. (24). Three different kinds of contributions arise from the two-pion states: the orbital moment of the two pions in state $L=1$,

$$\langle N=2, L=1|\mathfrak{M}_n|N=2, L=1\rangle,$$

the cross term of the orbital moment between $L=0$ and $L=1$, $\langle N=2, L=1|\mathfrak{M}_n|N=2, L=0\rangle$, and the cross term between the 2-pion and 0-pion states introduced by the pair creation operators appearing in Eq. (20), $\langle N=2, L|\mathfrak{M}_n|N=0\rangle$. In calculating the cross terms explicit use must be made of the (real) radial functions $F_{L,S,l}(r_1, r_2)$ of the two-pion state with given l of each pion and with given L and S. S is the symmetry parameter which distinguishes a function symmetric for interchange of r_1 and r_2, $S=\mathcal{S}$, from an antisymmetric function, $S=\mathcal{A}$. The operator $\mathfrak{M}_n$ is independent of the distances r_1 and r_2, so the matrix elements involve simple overlap integrals of the radial functions and no cross terms arise between states of opposite symmetry. In calculating the matrix elements we will assume, for the sake of simplicity, that pions of relativistic energies do not play an important role so the factor $\mathfrak{M}c/\omega$ appearing in Eq. (20) can be replaced by $\mathfrak{M}/\mu$.

The $L=1$ to $L=1$ matrix element may be obtained by direct application of the vector rule. From Eq. (11) we see that the two pions share equally in the orbital angular momentum, hence pion pairs of opposite charge make no contribution and the pion contribution in the state containing one negative and one neutral pion ($C=-1$) is one-half of that due to a single pion with $l=1$. Since the nucleore has positive charge in the latter state, its intrinsic moment must be included. Thus

$$\langle N=2, C=0, L=1|\mathfrak{M}_n|N=2, C=0, L=1\rangle = 0,$$

$$\langle N=2, C=-1, L=1|\mathfrak{M}_n|N=2, C=-1, L=1\rangle = -\tfrac{1}{3}(1+\mathfrak{M}/\mu).$$

Determination of the $L=1$, $L=0$ cross term can also be made on the basis of simple two-particle wave functions. The $L=0$ state has no net charge in the $\mathcal{S}$ states [see Eq. (13c)] and the $L=1$ has none in the $\mathcal{A}$ states [see Eq. (13f)] so the moment must arise from positive-negative pairs. The angular part, $\Phi_L^{M_L}$, of the $L=0$ wave function has the form

$$\Phi_0^0(1,2) = (-1)^l \textstyle\sum_m (-1)^m Y_l^m(1) Y_l^{-m}(2)/(2l+1)^{\frac{1}{2}}, \tag{25}$$

while that of the $L=1$ function for $M_L=0$ (the only contributing function) is

$$\Phi_1^0(1,2) = (-1)^l \textstyle\sum_m (-1)^m m Y_l^m(1) Y_l^{-m}(2)/(\sum_m m^2)^{\frac{1}{2}}. \tag{26}$$

If $\mathbf{l}_1$ and $\mathbf{l}_2$ are the orbital angular momentum operators of the positive and negative pions, respectively, the magnetic moment operator is $\mathbf{l}_1 - \mathbf{l}_2$ and the matrix

[11] J. M. Luttinger, Helv. Phys. Acta **21**, 483 (1948); M. Slotnick and W. Heitler, Phys. Rev. **75**, 1645 (1949); K. M. Case, Phys. Rev. **76**, 1 (1949).

element of the z component is

$$\langle\Phi_1^0, (\mathbf{l}_1-\mathbf{l}_2)_z\Phi_0^0\rangle = 2(\sum_m m^2)/[(2l+1)\sum_m m^2]^{\frac{1}{2}} = 2[l(l+1)/3]^{\frac{1}{2}}.$$

The $L=1$ function is combined with the spin function of the nucleore to form total angular momentum $J=\frac{1}{2}$. The coefficient of the $M_L=0$ function is therefore $-1/\sqrt{3}$. Furthermore, in the $Y=0$ isotopic spin function the charged pair term occurs with a probability $\frac{2}{3}$. This introduces a factor $\sqrt{\frac{2}{3}}$ into the matrix element. The net result is then

$$\langle N=2, C=0, L=1|\mathfrak{M}_n|N=2, C=0, L=0\rangle = (2/3)^{\frac{3}{2}}(\mathfrak{M}/\mu)\sum_l[l(l+1)]^{\frac{1}{2}}I_{01}(S,l),$$

where $I_{LL'}(S,l)$ is the overlap integral

$$I_{LL'}(S,l) = \int_0^\infty\int_0^\infty F_{LSl}(r,r')F_{L'Sl}(r,r')r^2r'^2drdr'.$$

After the pair creation and annihilation operators in Eq. (20) have been applied, the calculation of the (two-pion)−(no-pion) cross term is very similar to the treatment of the $L=0$, $L=1$ cross term. There is no contribution from the two-pion term with $L=0$. For $L=1$ the matrix element is

$$\begin{aligned}\langle N=2, C=0, L=1|\mathfrak{M}_n|N=0\rangle &= \sqrt{\tfrac{1}{3}}(\mathfrak{M}/\mu)\sum_l(-1)^l[\sum_m m^2]I_1(S,l)/(\sum_m m^2)^{\frac{1}{2}} \\ &= (\mathfrak{M}/\mu)\sum_l\tfrac{1}{3}[l(l+1)(2l+1)]^{\frac{1}{2}}I_1(S,l),\end{aligned}$$

where $I_L(S,l)$ is the radial integral,

$$I_L(S,l) = (-1)^l\int_0^\infty F_{LSl}(r,r)r^2dr,$$

which vanishes for $S=\mathcal{A}$.

If we introduce the (real) probability amplitudes and add all contributions to the moment in the two-pion model, the result is, according to Eq. (17),

$$\begin{aligned}\mathfrak{M}_n = &-\tfrac{1}{3}(1+2\mathfrak{M}/\mu)P_1^-(1) - \tfrac{1}{3}(1+\mathfrak{M}/\mu)P_1^-(2) \\ &\pm 2(2/3)^{\frac{3}{2}}(\mathfrak{M}/\mu)\sum_S[P_1^0(2,S)P_0^0(2,S)]^{\frac{1}{2}} \\ &\times\sum_l[l(l+1)]^{\frac{1}{2}}I_{01}(S,l) \\ &\pm 2(\mathfrak{M}/\mu)[(1-P_0-P_1)P_1^0(2,\mathcal{S})]^{\frac{1}{2}} \\ &\times\tfrac{1}{3}\sum_l[l(l+1)(2l+1)]^{\frac{1}{2}}I_1(\mathcal{S},l),\end{aligned}$$

which can be reduced by means of the charge-invariance conditions Eqs. (13) to

$$\begin{aligned}\mathfrak{M}_n = &-(2/9)(1+2\mathfrak{M}/\mu)P_1(1) - (2/9)(1+\mathfrak{M}/\mu)P_1(2,\mathcal{S}) \\ &\pm(4/9)\sqrt{2}(\mathfrak{M}/\mu)\sum_S[P_1(2,S)P_0(2,S)]^{\frac{1}{2}} \\ &\times\sum_l[l(l+1)]^{\frac{1}{2}}I_{01}(S,l)\pm(2/\sqrt{3})(\mathfrak{M}/\mu) \\ &\times[(1-P_0-P_1)P_1(2,\mathcal{S})]^{\frac{1}{2}} \\ &\times\tfrac{1}{3}\sum_l[l(l+1)(2l+1)]^{\frac{1}{2}}I_1(\mathcal{S},l). \quad (27)\end{aligned}$$

Here P_L is the total probability for finding either one or two pions in state L, while $P_0(2,S)$ and $P_1(2,S)$ are the probabilities for finding two pions (any allowed charge) of given S in the states $L=0$ and $L=1$, respectively.

The linear coupling condition, Eq. (5), suggests that consideration be limited to $l=1$, in which case we drop the index l. Then

$$\begin{aligned}\mathfrak{M}_n = &-(2/9)(\mathfrak{M}/\mu)\{(2+\mu/\mathfrak{M})P_1(1) \\ &+(1+\mu/\mathfrak{M})P_1(2,\mathcal{S}) \\ &\pm 4\sum_S[P_1(2,S)P_0(2,S)]^{\frac{1}{2}}I_{01}(S) \\ &\pm 3\sqrt{2}[(1-P_0-P_1)P_1(2,\mathcal{S})]^{\frac{1}{2}}I_1(\mathcal{S})\}. \quad (28)\end{aligned}$$

Clearly for the nucleon to pion mass ratio $\mathfrak{M}/\mu=6.64$, this equation is capable of yielding the known value of the neutron moment, $\mathfrak{M}_n=-1.91$, even within the restrictions imposed by Eq. (23), $P_1=0.09$, because no conditions have been set on P_0. However, it is of particular interest that the values

$$P_1(1)=P_1(2,\mathcal{A})=P_0=0, \quad P_1(2,\mathcal{S})=P_1=0.090, \quad (29)$$

along with the relationship

$$I_1(\mathcal{S}) = \int_0^\infty F_{1\mathcal{S}}(r,r)r^2dr = 1, \quad (30)$$

and an appropriate choice of sign of the square root, gives

$$\mathfrak{M}_n = -1.93,$$

in rather remarkable (but probably fortuitous) agreement with the observed moment. This result, that the magnetic moments of both nucleons can be accounted for if the proper field contains only a pair of pions, and that with the small probability of 9 percent, is very suggestive. However, many other selections of the coefficients in Eq. (28) will give the correct moment. An even greater variety of possibilities is offered by l values different from $l=1$ or by numbers of pions greater than two.

It should be remarked that Eq. (30) is quite consistent with the normalization condition

$$\int\int F_{1S}^2(r,r')r^2r'^2drdr' = 1.$$

A square radial distribution, i.e., a sharp cutoff at some fixed value of r and r' would lead to just such a relationship.

8. THE NEUTRON-ELECTRON INTERACTION

Another source of information concerning the structure of the nucleon is the neutron-electron interaction. To calculate the interaction on the basis of our model, we note that the interaction with any external electrostatic field $V(\mathbf{r})$ is

$$W = \int[\rho(\mathbf{r})+\tfrac{1}{2}e(1+\tau_3)\delta(\mathbf{r})]V(\mathbf{r})d^3r,$$

where $\rho(\mathbf{r})$ is the charge density operator for the pion field,

$$\rho(\mathbf{r})=-(ie/\hbar)[\pi(\mathbf{r})\psi(\mathbf{r})-\pi^*(\mathbf{r})\psi^*(\mathbf{r})].$$

From Eqs. (A-3) and (A-4) we find

$$W=\tfrac{1}{2}e\{(1+\tau_3)V(0)+\sum_{ss'} \times[(\omega/\omega')^{\frac{1}{2}}((-1)^m a_s^*-b_{-s})(a_{s'}+(-1)^{m'}b_{-s'}^*)(-1)^m +(\omega'/\omega)^{\frac{1}{2}}((-1)^{m'}a_{s'}-b_{-s'}^*)(a_s^* +(-1)^m b_{-s})(-1)^{m'}](s|V|s')\},$$

where

$$(s|V|s^*)=\int \phi_{s'}^*V\phi_{s'}d^3r.$$

Note that, since $V(r)$ is a real function,

$$(-s'|V|-s)=(-1)^{m+m'}(s|V|s').$$

The factors $(\omega/\omega')^{\frac{1}{2}}$ can be dropped if relativistic pions are assumed to be unimportant. Then

$$W=e\{\tfrac{1}{2}(1+\tau_3)V(0)\sum_{s,s'}(a_s^*a_{s'}-b_s^*b_{s'})(s|V|s')\}.$$

There are no pair creation or annihilation terms remaining in this approximation to the operator W, so the interaction is made up of terms arising from states of a given number of pions, and the contribution of each such state can be calculated by use of the properly symmetrized product of Schrödinger wave functions. Since the neutron has no net charge, $\frac{1}{2}(1+\tau_3)=-(N^+-N^-)$, and the expectation value of W is

$$\bar{W}=\int \bar{\rho}(\mathbf{r})V(\mathbf{r})d^3r-V(0)\sum_{(N)} \times P(N^+,N^0,N^-)(N^+-N^-),$$

where $\bar{\rho}(\mathbf{r})$ is the mean pion charge density in the nucleon proper field.

If $V(r)$ is a slowly varying function it may be expanded about $r=0$:

$$V(\mathbf{r})=V(0)+\mathbf{r}\cdot(\text{grad}\,V)_{r=0} +\sum_{ij} x_ix_j(\partial^2V/\partial x_i\partial x_j)_{r=0}+\cdots.$$

Now

$$\int \bar{\rho}(\mathbf{r})V(0)d^3r=V(0)\sum_{(N)} P(N^+,N^0,N^-)(N^+-N^-).$$

Furthermore, $\bar{\rho}$ has even parity since all pion states have the same parity, so

$$\int \bar{\rho}(\mathbf{r})\mathbf{r}\cdot(\text{grad}V)_0 d^3r=0.$$

Finally, each pion state has $J=\frac{1}{2}$ so only the part of $x_ix_j(\partial^2V/\partial x_i\partial x_j)_{r=0}$ with the angular dependence of an S function can contribute to the integral. Therefore,

$$\bar{W}=-\tfrac{1}{3}e(\nabla^2V)_{r=0}\langle r^2\rangle_{\text{Av}},$$

where $\langle r^2\rangle_{\text{Av}}$ is the mean square radius of charge in the proper field,

$$e\langle r^2\rangle_{\text{Av}}=-\int r^2\bar{\rho}(\mathbf{r})d^3r,$$

the sign having been chosen in accordance with the fact that the net charge of the proper field is negative.

When the external field is produced by an electron at the point $\mathbf{R}$, we have

$$(\nabla^2V)_{r=0}=4\pi e\delta(\mathbf{R}),$$

so the neutron-electron interaction is simply[12]

$$\bar{W}=-(4\pi/3)e^2\langle r^2\rangle_{\text{Av}}\delta(\mathbf{R}).$$

It is customary to describe the measured values of the interaction in terms of the effective depth

$$W_0=\int \bar{W}(\mathbf{R})d^3R/(4\pi/3)(e^2/mc^2)^3,$$

which is now found to be

$$W_0=-[\langle r^2\rangle_{\text{Av}}/(e^2/mc^2)^2]mc^2.$$

The experimental value[13] $|W_0|\approx 4$ kev requires that

$$\langle r^2\rangle_{\text{Av}}/(e^2/mc^2)^2\approx 8\times 10^{-3},$$

or

$$\mu^2\langle r^2\rangle_{\text{Av}}\approx 3.2\times 10^{-2}, \tag{31}$$

where μ^{-1} is the Compton wavelength of the pion.

The order of magnitude of the displacement of a pion in the proper field of a nucleon would be expected to be μ^{-1}, so Eq. (31) may be interpreted as a measure of the probability of occurrence of charged pions. This probability, 3.2 percent, is in rough agreement with the results of our discussion of the magnetic moment. A more specific statement can be made on the basis of either the one-pion or two-pion model. The only contribution to $\langle r^2\rangle_{\text{Av}}$ in the former comes from the negative pion so we have

$$\langle r^2\rangle_{\text{Av}}=\langle N=1|r^2|N=1\rangle P_1^-(1),$$

or, by Eq. (13b)

$$\langle r^2\rangle_{\text{Av}}=(2/3)\langle N=1|r^2|N=1\rangle P_1(1),$$

where $\mathbf{r}$ is a pion coordinate. Since only one pion occurs in this model, $P_1(1)=P_1=0.09$ so Eq. (31) yields the very reasonable value for the mean square displacement of a pion:

$$\langle N=1|r^2|N=1\rangle\approx 0.5\mu^{-2}.$$

The failure of the one-pion model to account for the magnetic moments suggests that we carry the argument to the two-pion model; then the contributions to $\langle r^2\rangle_{\text{Av}}$ include a term produced by states of net negative

[12] Compare L. L. Foldy, unpublished Case Inst. of Technology Technical Report No. 15 (1952); Phys. Rev. **87**, 688 (1952). The separation of the term associated with the nuclear magnetic moments, enlightening though it be, is quite artificial for slow neutrons. See B. D. Fried, Phys. Rev. **86**, 434 (1952).

[13] Harvey, Hughes, and Goldberg, Phys. Rev. **87**, 220 (1952).

charge (one negative, one neutral pion) and a cross term between $\mathcal{S}$ and $\mathcal{A}$ states of net charge 0 (one negative, one positive pion). The charged pairs do not contribute within states of given symmetry because in them there is an equal chance for pions of opposite charge to be at a given distance from the nucleore. The expression for $\langle r^2\rangle_{\text{Av}}$ is found to be (note again that charged pairs occur with amplitude $\sqrt{\frac{2}{3}}$ in the $Y=0$ state),

$$\begin{aligned}\langle r^2\rangle_{\text{Av}}=\tfrac{2}{3}\{&P_1(1)\langle N=1|r^2|N=1\rangle\\ &\pm 2\sqrt{2}\textstyle\sum_L[P_L(2,\mathcal{S})P_L(2,\mathcal{A})]^{\frac{1}{2}}\\ &\times\langle N=2,\mathcal{S},L|r^2|N=2,\mathcal{A},L\rangle\\ &+P_1(2,\mathcal{S})\langle N=2,\mathcal{S},L=1|r^2|N=2,\mathcal{S},L=1\rangle\\ &+P_0(2,\mathcal{A})\langle N=2,\mathcal{A},L=0|r^2|N=2,\\ &\qquad\mathcal{A},L=0\rangle\}.\end{aligned}\quad(32)$$

This result is so ambiguous that the condition $P_1=0.09$ does not lead to any clear cut conclusion. However, we can again turn to the very simple conditions, Eq. (29), which are consistent with the observed nucleon moments. Then Eq. (32) becomes

$$\langle r^2\rangle_{\text{Av}}=0.06\langle 2\mathcal{S}1|r^2|2\mathcal{S}1\rangle,$$

where $\langle 2\mathcal{S}1|r^2|2\mathcal{S}1\rangle$ is the mean square displacement of the pion in the prescribed state,

$$\langle 2\mathcal{S}1|r^2|2\mathcal{S}1\rangle\equiv\langle N=2,\mathcal{S},L=1|r^2|N=2,\mathcal{S},L=1\rangle.$$

Again a very reasonable value is obtained for the pion displacement from the condition, Eq. (31), imposed by the observed neutron-electron interaction:

$$\mu^2\langle 2\mathcal{S}1|r^2|2\mathcal{S}1\rangle\approx 0.5. \quad(33)$$

The principal significance of these results abides in the support they lend to the view that the pion proper field occurs with a relatively small probability.

9. CONCLUSION

It is not surprising that just three experimental data, the two moments and the electron-neutron interaction, can be fitted by considering a sufficiently complex model of the nucleon. What is surprising is the strong indication that the model must involve more than one pion, but that otherwise a rather weak and simple pion field is adequate to account for the data. The limitation on the number of pions might be somewhat softened by the inclusion of nucleore recoil, but the failure of dynamical theories along those lines would seem to lend support to the present model. Further support is provided by the fact that the neutron-electron interaction, as expressed by Eq. (33), seems to fit so well with our estimate of P_1 based on the mirror theorem.

The simplest model which accounts for the data consists of a bare nucleore (core of nucleon) with a 91 percent probability and a nucleore plus two pions with a 9 percent probability. Each of the pions is in a p state and the two together are in a radially symmetric state and have total orbital angular momentum $L=1$. This suggests that a rather weak nonlinear coupling warrants serious consideration as the source of the pion field.

A further test of the model is available in the form of data on the cross sections for photoproduction of pions and data on the nucleon pion scattering. However, these data involve excited states of the nucleon so they may not provide very direct information.

Interesting discussions of these matters with his colleagues, Professors J. M. Luttinger and J. L. Powell, have contributed much to the author's understanding of the subject.

APPENDIX 1

Analysis of the Field in Spherical Waves

We use, as a basis for discussion, the Pauli-Weisskopf theory of a charged plus a neutral field.[4] If $\psi_0(\mathbf{r})$, $\pi_0(\mathbf{r})$ are the (real) field variables of the neutral field and $\psi(\mathbf{r})$, $\pi(\mathbf{r})$ those of the charged field, the field Hamiltonian in the absence of the nucleore is taken to be

$$H_0=\int d^3r\{\tfrac{1}{2}[\pi_0{}^2+c^2(\text{grad}\psi_0)^2+c^2\mu_0{}^2\psi_0{}^2]+\pi^*\pi+c^2(\text{grad}\psi^*\cdot\text{grad}\psi)+c^2\mu^2\psi^*\psi\},$$

where $\hbar\mu_0/c$ and $\hbar\mu/c$ are the masses of neutral and charged pions, respectively. The momentum density of the field is

$$\mathbf{G}=-\tfrac{1}{2}[\pi_0\,\text{grad}\psi_0+(\text{grad}\psi_0)\pi_0]-[\pi\,\text{grad}\psi+(\text{grad}\psi^*)\pi^*]. \quad(\text{A1})$$

The field commutation relations are

$$[\pi_0(\mathbf{r}),\psi_0(\mathbf{r}')]=[\pi(\mathbf{r}),\psi(\mathbf{r}')]=(\hbar/i)\delta(\mathbf{r}-\mathbf{r}'),$$

and all other pairs commute.

The functions ψ_0, π_0, ψ, π are expanded in terms of the spherical solutions ϕ_s of the equation

$$\nabla^2\phi_s+k^2\phi_s=0.$$

These solutions take the form

$$\phi_s=f_l(kr)Y_l{}^m,\quad s\equiv(k,l,m),$$

where the $Y_l{}^m$ are spherical harmonics and $f_l(kr)$ are the spherical Bessel functions which are regular at the origin. The ϕ_s are assumed to vanish on the surface of a sphere of very large radius, a condition which leads to discrete values for k, and the $f_l(kr)$ are taken to be normalized within this volume. The definition of the $Y_l{}^m$ is such that[6]

$$Y_l{}^{m*}=(-1)^mY_l{}^{-m}. \quad(\text{A2})$$

The field operators are expanded in terms of the ϕ_s as

$$\begin{aligned}\psi_0&=\textstyle\sum_s q_s\phi_s,\quad \pi_0=\sum_s p_s\phi_{-s},\\ \psi&=\textstyle\sum_s Q_s\phi_s,\quad \pi=\sum_s P_s\phi_{-s},\end{aligned}\quad(\text{A3})$$

where the notation $-s\equiv(k,l,-m)$ has been used. Then the nonvanishing commutators of the operators q_s, p_s, Q_s, P_s are

$$[p_s, q_{s'}]=[P_s, Q_{s'}]=(-1)^m(\hbar/i)\delta_{ss'}.$$

The reality conditions on ψ_0 and π_0 become

$$q_s^*=(-1)^m q_{-s}, \quad p_s^*=(-1)^m p_{-s}.$$

The Hamiltonian H_0 is now

$$H_0=\sum_s \{\tfrac{1}{2}(p_s^*p_s+\omega_0^2 q_s^*q_s)+P_s^*P_s+\omega^2 Q_s^*Q_s\},$$

where

$$\omega_0^2=c^2(k^2+\mu_0^2), \quad \omega^2=c^2(k^2+\mu^2).$$

Diagonalization of H_0 is accomplished in the usual way by the introduction of the creation and annihilation operators, c_s^*, c_s for neutral, a_s^*, a_s for positive and b_s^*, b_s for negative pions:

$$\begin{aligned} q_s&=(\hbar/2\omega_0)^{\frac{1}{2}}(c_s+(-1)^m c_{-s}^*),\\ p_s&=i(\hbar\omega_0/2)^{\frac{1}{2}}((-1)^m c_s^*-c_{-s}),\\ Q_s&=(\hbar/2\omega)^{\frac{1}{2}}(a_s+(-1)^m b_{-s}^*),\\ P_s&=i(\hbar\omega/2)^{\frac{1}{2}}((-1)^m a_s^*-b_{-s}). \end{aligned} \tag{A4}$$

The energy is, of course, given by

$$H_0=\sum_s[N_s{}^0\hbar\omega_0+(N_s{}^+ + N_s{}^-)\hbar\omega],$$

where $N_s{}^\gamma$ is the number of pions of charge γ in state s if the zero-point energy is ignored.

The angular momentum (in units of $\hbar$) of the pion field is $\mathbf{M}=\hbar^{-1}\int \mathbf{r}\times\mathbf{G}d^3r$, where $\mathbf{G}$ is the momentum density, Eq. (A1). Thus

$$\mathbf{M}=(i\hbar)^{-1}\int\{\tfrac{1}{2}[\pi_0\mathbf{L}\psi_0+(\mathbf{L}\psi_0)\pi_0]+\pi\mathbf{L}\psi+(\mathbf{L}\psi^*)\pi^*\}d^3r,$$

where $\mathbf{L}=-i\mathbf{r}\times\text{grad}$ is the usual one-particle orbital angular momentum operator. When $\mathbf{M}$ is expressed in terms of the p_s, q_s, P_s, Q_s, the coefficients are just the matrix elements

$$(l, m|\mathbf{L}|l, m')=\int Y_l{}^{m*}\mathbf{L}Y_l{}^{m'}d\Omega,$$

which have, via Eq. (A2), the important property

$$(l, -m'|\mathbf{L}|l, -m)=-(-1)^{m+m'}(l, m|\mathbf{L}|l, m'). \tag{A5}$$

By making use of this property the angular momentum can be expressed in terms of the creation and annihilation operators as

$$\mathbf{M}=\sum_{s,s'}\delta_{kk'}\delta_{ll'}(l, m|\mathbf{L}|l, m')\times(a_s^*a_{s'}+b_s^*b_{s'}+c_s^*c_{s'}). \tag{A6}$$

It is clear that M_z is diagonal (if z is the axis of quantization of the $Y_l{}^m$) and has the characteristic values

$$M_z=\sum_s m(N_s{}^0+N_s{}^+ + N_s{}^-).$$

Thus $N_s{}^\gamma$ is the number of charge γ-pions with z component of angular momentum m. Since the magnetic quantum number of each pion is specified, it is clear that the total angular momentum of the pion system cannot be diagonal. However, the special state of one pion, $N_s=0$, $s\neq\sigma$, and $N_\sigma=1$, where $\sigma\equiv(\kappa, \lambda, \mu)$ is easily shown to be a characteristic state of the operator $\mathbf{M}^2$ with characteristic value $\lambda(\lambda+1)$.

In order to determine the magnetic moment, we consider the change in energy of the nucleon produced by the introduction of a weak, uniform magnetic field $\mathbf{H}$. Care must be taken to include the interaction of the pion field with the nucleore in the discussion since for a pseudoscalar field an external field may induce pion emission. For this purpose we make the assumption that the pion-nucleore interaction is a point interaction of the form $\int H'\{\psi_0, \psi\}\delta(\mathbf{r})d^3r$, if the origin is taken to be at the nucleore. Then the introduction of an external field characterized by vector potential $\mathbf{A}(\mathbf{r})$ leads to a pion plus nucleore Hamiltonian of the form

$$\begin{aligned} H=\int d^3r\{&\tfrac{1}{2}[\pi_0^2+c^2(\text{grad}\psi_0)^2+c^2\mu_0^2\psi_0^2]+\pi^*\pi\\ &+c^2([\text{grad}+(ie/\hbar c)\mathbf{A}(\mathbf{r})]\psi^*\\ &\cdot[\text{grad}-(ie/\hbar c)\mathbf{A}(\mathbf{r})]\psi\\ &+c^2\mu^2\psi^*\psi+H'\{\mathbf{A}, \psi_0, \psi\}\delta(\mathbf{r})\}, \end{aligned} \tag{A7}$$

where $H'\{\mathbf{A}, \psi_0, \psi\}$ includes any modification of the interaction produced by $\mathbf{A}$. Only the linear terms in $\mathbf{A}$ of Eq. (A7) are of interest for our purpose and these are

$$\begin{aligned} H_1=\int d^3r\{&(iec/\hbar)[\psi^*(\mathbf{A}\cdot\text{grad}\psi)-(\text{grad}\psi^*\cdot\mathbf{A})\psi]\\ &+(\mathbf{A}(\mathbf{r})\cdot\mathbf{h}'\{\psi_0, \psi\})\delta(\mathbf{r})\}, \end{aligned} \tag{A8}$$

if $\mathbf{h}'$ is a vector which provides the coefficients of the linear term in $\mathbf{A}$ when $H'\{\mathbf{A}, \psi_0, \psi\}$ is expanded in powers of $\mathbf{A}$. For the uniform field, $A=-\frac{1}{2}\mathbf{r}\times\mathbf{H}$ so the pion-nucleore interaction term in Eq. (A8) makes no contribution to the magnetic moment in consequence of our assumption of a point interaction. The pion magnetic moment operator, $\mathfrak{M}_0$, is obtained by setting $H_1=-(\mathfrak{M}_0\cdot\mathbf{H})$ for the uniform field. Hence

$$\mathfrak{M}_0=(ec/2\hbar)\int d^3r\{\psi^*\mathbf{L}\psi-(\mathbf{L}\psi^*)\psi\}.$$

Substitution for $\psi(\mathbf{r})$ in terms of a_s, b_s, etc., by means of Eqs. (A3) and (A4) leads, after application of Eq. (A5), to the result Eq. (18).

APPENDIX 2

Space and Time Inversion Properties of the Functional

An amusing feature of the spherical wave expansion is the simple formulation it provides for the space-inversion transformation. The inversion is to be accomplished by means of a unitary transformation U_i of the field variables, and U_i satisfies the conditions

$$U_i\psi(\mathbf{r})U_i^{-1}=\psi(-\mathbf{r}),\quad U_i\pi(\mathbf{r})U_i^{-1}=\pi(-\mathbf{r}),$$

etc., which are easily seen, on the basis of Eqs. (A3) and (A4) and the fact that the field functions have an intrinsic odd parity, to be equivalent to

$$U_ia_sU_i^{-1}=(-1)^{l+1}a_s, \tag{A9}$$

and similar equations for the operators b_s and c_s. A transformation U satisfying the condition Eq. (A9) is

$$U_i(a)=\exp\{i\pi\sum_s(l+1)a_s{}^*a_s\},$$

as can be established by means of the commutation relations for the operators a_s and $a_s{}^*$. The complete U_i is to be obtained by supplementing $U_i(a)$ with corresponding factors $U_i(b)$ and $U_i(c)$. The associated transformation of the wave functional $\Psi\{N_s{}^\gamma\}$ for a characteristic state associated with the integers $N_s{}^\gamma$, is

$$\Psi'\{N_s{}^\gamma\}=U_i\Psi\{N_s{}^\gamma\}=(-1)^{\Sigma_{s,\gamma}(l+1)N_s{}^\gamma}\Psi\{N_s{}^\gamma\},$$

which agrees with the parity assignment of Eq. (3).

It is also of interest to construct the time reversal transformation for the pion field. We consider time reversal in Wigner's sense[7] so the operation on the functional is expected to be of the form

$$K=UK_0,$$

where K_0 means simple complex conjugation and U is a unitary transformation. The form of U can be determined from the condition that the angular momentum must change sign on time reversal:

$$K\mathbf{M}K^{-1}=-\mathbf{M}. \tag{A10}$$

When this transformation is applied to Eq. (A6), it is to be noted that the creation and annihilation operators a_s, $a_s{}^*$, etc., are taken to have real matrix elements so they are not affected by K_0. Then if use is made of Eq. (A5), we find that the condition Eq. (A10) is equivalent to

$$U(a_{-s}{}^*a_{-s'}+b_{-s}{}^*b_{-s'}+c_{-s}{}^*c_{-s'})U^{-1}$$
$$=(-1)^{m+m'}(a_s{}^*a_{s'}+b_s{}^*b_{s'}+c_s{}^*c_{s'}),$$

so a sufficient condition on U is

$$Ua_sU^{-1}=(-1)^ma_{-s},\text{ etc.} \tag{A11}$$

It is to be noted that this condition is sufficient to establish the change in sign of the magnetic moment $\mathfrak{M}_0$ (Eq. (18)) under time reversal, as is required. Furthermore, it is evident that the energy and total charge are invariant under the transformation.

A transformation which accomplishes our purpose is

$$U=U_aU_bU_c,$$

if

$$U_a=\exp\{-\tfrac{1}{2}i\pi\sum_s(2|m|+1)a_s{}^*a_{-s}\}$$
$$\times\exp\{\tfrac{1}{2}i\pi\sum_s a_s{}^*a_s\}, \tag{A12}$$

and U_b, U_c are given by similar expressions in b_s and c_s, respectively. That (A12) satisfies Eq. (A11) can easily be established by means of the commutation relations for the a_s, $a_s{}^*$.

It must now be established that our transformation yields Eq. (16). For the sake of simplicity we consider a wave functional Φ_a of just the positive pions (no nucleore), and we will temporarily drop the superscript on $N_s{}^+$ for the sake of the printer. The action of the annihilation operator a_s on the state $\Phi_a{}^{(0)}$ of no pions is to produce 0, so

$$U_a\Phi_a{}^{(0)}=\Phi_a{}^{(0)}. \tag{A13}$$

Now we wish to show by induction that

$$U_a\Phi_a\{N_s\}=(-1)^{\Sigma_s mN_s}\Phi_a\{N_{-s}\}, \tag{A14}$$

where the change of sign $N_s\rightarrow N_{-s}$ is meant to indicate that the number of pions with quantum number m in the one state is equal to the number with quantum number $(-m)$ in the other state. If Eq. (A14) is valid for N pions, we can show that it is valid for the state $\Phi_a\{N_s'\}$ of $N+1$ pions. Here $N_s'=N_s$ except for $s=\sigma\equiv(\kappa,\lambda,\mu)$ and $N_\sigma'=N_\sigma+1$. The proof is as follows: We have

$$\Phi_a\{N_s'\}=(N_\sigma+1)^{-\frac{1}{2}}a_\sigma{}^*\Phi_a\{N_s\},$$

whence

$$U_a\Phi_a\{N_s'\}=(N_\sigma+1)^{-\frac{1}{2}}U_aa_\sigma{}^*U_a{}^{-1}U_a\Phi_a\{N_s\}$$
$$=(N_\sigma+1)^{-\frac{1}{2}}(-1)^\mu(-1)^{\Sigma_s mN_s}a_{-\sigma}{}^*\Phi_a\{N_{-s}\},$$

according to Eqs. (A11) and (A14). But $a_{-\sigma}{}^*\Phi_a\{N_{-s}\}=(N_\sigma+1)^{\frac{1}{2}}\Phi_a\{N_{-s}'\}$, so

$$U_a\Phi_a\{N_s'\}=(-1)^{\Sigma_s mN_s'}\Phi_a\{N_{-s}'\}.$$

Since Eq. (A13) shows that Eq. (A14) is valid for $N=0$, it is valid for all N. The generalization to include negative and neutral pions is evident.

The magnetic quantum number of the pions is $M_L=\sum_{s,\gamma}mN_s{}^\gamma$, and hence for a pion functional $\Phi_L{}^{M_L}$ of a fixed number of pions with total orbital angular momentum L

$$U\Phi_L{}^{M_L}=i^{2M_L}\Phi_L{}^{-M_L}.$$

Our representation is such that the functionals Φ are the basic vectors, so they are real functions of the numbers $mN_s{}^\gamma$ and are therefore not affected by the operation K_0. Thus

$$K\Phi_L{}^{M_L}=i^{2M_L}\Phi_L{}^{-M_L},$$

in accord with Eq. (16).