Electron-Lattice Interaction and Superconductivity

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An electron moving faster than sound would emit phonons in the form of a narrow wake behind it, analogous to the Čerenkov radiation. Two electrons would, therefore, interact strongly when one lies in the wake of the other. Based on this idea a model for the superconducting state is suggested. The electron-lattice interaction, obtained by using the Bloch-Nordsieck transformation, is examined in the coordinate space, and the "wake" character of the interaction is shown. In the appendix, using a canonical transformation, the interaction between two electrons is calculated.

I. INTRODUCTION

FRÖHLICH[1] and Bardeen[2] have, independently, developed a theory of superconductivity based on the interaction of electrons with lattice vibrations. That the basic idea of their theory is correct is proved by the experimental discovery of the isotope effect.[3]

Using the second-order perturbation theory, Fröhlich has calculated the self-energy of the electrons in the Fermi-distribution, arising as a result of the virtual emission and absorption of phonons. Because of the Pauli principle this interaction energy depends on the distribution of the electrons in the momentum space; and if the interaction is strong enough, it gives rise to a new distribution corresponding to a lower energy, in which a shell of electrons is lifted out of the Fermi distribution. Such a shell distribution of the electrons in the momentum space has been identified by Fröhlich with the superconducting state. Fröhlich derives a condition for the metal to be superconducting, and unfortunately, this also happens to be the condition for which the perturbation theory breaks down. It is, therefore, not unlikely that a more rigorous calculation of the interaction energy, which seems to be difficult at the present time, may not give the shell structure at all. Also, the energy difference between the superconducting and the normal states as calculated on the basis of Fröhlich's theory comes out to be large by a factor of one hundred. This is perhaps due to the fact that in his theory most of the contribution to the interaction energy comes from short wavelength phonons. The low energies involved in the transition may suggest that long wavelengths are more important. The primary aim of the present note is to propose an alternative model for the superconducting state. A similar model has also been proposed recently by Bohm and Staver.[4] The model is valid only when the wavelength of the phonons involved is large. However, it suffers from certain disadvantages. The experiments on thin films[5] and on colloidal mercury[6] indicate that there is very little change in transition temperature with dimensions even when the film thickness or particle size is as small as 5×10^{-6} cm, thereby indicating that it is the short wavelength phonons which are important in the electron-lattice interaction.

II. DISCUSSION OF THE MODEL

The electrons actually involved in the superconducting transition are those which lie near the top of the Fermi distribution and have velocity which in metals is nearly 10^3 times the velocity of sound in that medium. Such an electron would emit sound radiation (virtual phonons since we are considering the state of the system at $T=0$) in the form of a wake analogous to what is known as the Čerenkov[7] radiation. The angular width of the wake would be of the order of $v_{\text{sound}}/v_{\text{electron}}\approx10^{-3}$ radian. Any two electrons would, therefore, interact most strongly when one lies in the wake of the other. And, since the wake is extremely narrow, one electron would almost follow in line with the other. The interaction energy of two electrons as a result of the virtual emission and absorption of phonons is of the order of 10^{-4} ev and corresponds to the energy with which the two electrons are bound in the wake. This picture, therefore, leads to the formation of localized groups of co-moving electrons. These clusters are stable in the sense that it would require energy to dissolve them. Let us now examine qualitatively how far the concept of a wake helps to understand the basic property of a superconducting state.

It is now recognized that the basic property of a superconducting state is the Meissner effect, which is contained in the phenomenological equations of London.[8] The London equations can be derived from nonrelativistic quantum mechanics if one makes the assumption that the wave functions of the electrons are not altered in a magnetic field. In the normal state the wave functions of the electrons in the presence of a magnetic field coil up and are, therefore, strongly modified. However, the electrons in the superconducting state are certainly not to be considered free in the

[1] H. Fröhlich, Phys. Rev. **79**, 845 (1950).
[2] J. Bardeen, Phys. Rev. **79**, 167 (1950).
[3] E. Maxwell, Phys. Rev. **78**, 477 (1950). Reynolds, Serin, Wright, and Nesbitt, Phys. Rev. **78**, 487 (1950).
[4] D. Bohm and T. Staver, Phys. Rev. **84**, 836 (1951).
[5] E. T. S. Appleyard and A. D. Misener, Nature **142**, 474 (1938); Appleyard, Bristow, and Misener, Proc. Roy. Soc. (London) **A172**, 540 (1939).
[6] D. Schoenberg, Proc. Roy. Soc. (London) **A175**, 49 (1940).
[7] P. A. Čerenkov, Phys. Rev. **52**, 378 (1937).
[8] F. London and H. London, Physica **2**, 341 (1935).

sense of the ordinary conducting electrons. We have remarked in the foregoing paragraph that electrons in the superconducting state are bound up in the wake with an energy of the order of 10^{-4} ev, and if the strength of the resultant magnetic field is less than a certain critical value, it would not be able to throw the electron out of the wake. The electron would, therefore, continue to follow the path which it had in the absence of the field, and the wave functions would remain unaltered. Or one might say in the terminology of London that the wave functions behave as if they were rigid. A quantitative justification of these ideas is desirable.

In view of what has been said above it would be interesting to investigate the nature of the electron-lattice interaction in the coordinate space in the long wavelength limit of the lattice waves. The "wake" character of this interaction, using the Bloch-Nordsieck transformation, is shown explicitly in Sec. III. Also, using a canonical transformation, we have calculated, in the appendix, what may be called the Møller interaction of two electrons as a result of the virtual emission and absorption of phonons. Though formally this method is equivalent to the usual perturbation theory method used by Fröhlich, it has the advantage of giving a better understanding of the basic ideas of Fröhlich's theory. In addition, this method enables one to calculate the contribution of the nondiagonal elements of the interaction matrix. Fröhlich considers only the contribution of the diagonal terms.

III. "WAKE" CHARACTER OF THE INTERACTION

Using Bloch-Nordsieck transformation Bardeen[9] has obtained an expression for the interaction energy between electrons and lattice vibrations. The Bloch-Nordsieck transformation is valid for lattice waves of wavelength large compared to that of the electrons near the top of the Fermi distribution, and does not involve the approximation of a small coupling constant. The expression for the interaction energy as given by Bardeen[9] [see Eqs. (6–13) and (6–14) of reference 9] is

$$U(r)=-\sum_q \frac{|m_q|^2 \cos\mathbf{q}\cdot(\mathbf{r}_1-\mathbf{r}_2)}{\omega_q^2-(\mathbf{v}\cdot\mathbf{q})^2} + \frac{1}{2}\frac{(3(\mathbf{v}\cdot\mathbf{q})^2-\omega_q^2)\{m_q^2 e^{i\mathbf{q}\cdot(\mathbf{r}_1+\mathbf{r}_2)}+m_q^{*2}e^{-i\mathbf{q}\cdot(\mathbf{r}_1+\mathbf{r}_2)}}{(\omega_q^2-(\mathbf{v}\cdot\mathbf{q})^2)^2}, \quad (1)$$

where

$$|m_q|^2=(2\omega_q/\hbar)|M_q|^2. \quad (2)$$

M_q is the matrix element corresponding the the transition $\mathbf{k}'=\mathbf{k}\pm\mathbf{q}$ and is given by Eq. (10A) of the appendix. ω_q is the angular frequency of lattice waves corresponding to the wave number $\mathbf{q}$ and $\mathbf{v}$ denotes the velocity of the electrons near the top of the Fermi distribution. $\mathbf{r}_1$ and $\mathbf{r}_2$ denote, respectively, the position vectors of electrons 1 and 2. Since we are concerned only with the interaction between two electrons, we have omitted the summation, which occurs in Bardeen's expression, over all electrons.

The quantity inside the curly brackets in the second term on the right-hand side of Eq. (1) can be put in the form

$$|m_q|^2\{e^{i\mathbf{q}\cdot(\mathbf{r}_1+\mathbf{r}_2)}e^{i\varphi_q}+e^{-i\mathbf{q}\cdot(\mathbf{r}_1+\mathbf{r}_2)}e^{-i\varphi_q}\},$$

where φ_q is the phase of the wave with wave vector $\mathbf{q}$. When one averages over-all possible phases of different q's, the second term in (1) would give zero contribution. Hence, it is only the first term which contributes to the interaction energy. Also, this term when transformed into momentum space gives the E_2 term of the Fröhlich theory [see Eq. (2–15) of reference 1] for the case of two electrons provided q^2 is neglected compared to k^2, where k is the wave number of the electron.

Using (2) in (1) and remembering that $\omega_q^2=S^2q^2$, we have

$$U(r)=-A\sum_q \frac{S^2q^2\cos\mathbf{q}\cdot\mathbf{r}}{S^2q^2-(\mathbf{v}\cdot\mathbf{q})^2}, \quad (3)$$

since the second term gives zero contribution. In (3)

$$A=(8C^2/9nVMS^2), \quad (4)$$

$\mathbf{r}_1-\mathbf{r}_2=\mathbf{r}$, and S denotes the velocity of sound. Equation (3) can be written as

$$U(r)=\Re A\nabla_r^2\sum_q \frac{S^2e^{i\mathbf{q}\cdot\mathbf{r}}}{S^2q^2-(\mathbf{v}\cdot\mathbf{q})^2}, \quad (5)$$

where $\Re$ means that we have to take the real part of the expression and the suffix r denotes that the derivative is with respect to r. Choosing the direction of $\mathbf{v}$ along the z axis and performing the following change of variables

$$q_x'=Sq_x,\quad q_y'=Sq_y,\quad q_z'=(S^2-v^2)^{\frac{1}{2}}q_z$$

and

$$x'=x/S,\quad y'=y/S,\quad z'=z/(S^2-v^2)^{\frac{1}{2}}, \quad (6)$$

and replacing the sum in (5) by an integration we have

$$U(r)=\Re A\frac{V}{(2\pi)^3}\cdot\frac{1}{(S^2-v^2)^{\frac{1}{2}}}\nabla_r^2\int\frac{e^{i\mathbf{q}'\cdot\mathbf{r}'}}{q'^2}d\mathbf{q}' = \Re A\frac{V}{(S^2-v^2)^{\frac{1}{2}}}\nabla_r^2\left(\frac{1}{4\pi|\mathbf{r}'|}\right),$$

where V is the volume of the assembly.

Going back to the old coordinates we have

$$U(r)=\Re\frac{AV}{4\pi}\cdot\frac{\beta}{i}\nabla^2\left(\frac{1}{x^2+y^2-\beta^2z^2}\right)^{\frac{1}{2}}, \quad (7)$$

where

$$\beta^2=(v^2/S^2-1)^{-1}. \quad (8)$$

[9] J. Bardeen, Revs. Modern Phys. **23**, 268 (1951).

From (7) it is clear that the interaction $U(r)$ is real only within the cone (defined by $x^2+y^2-\beta^2z^2=0$) whose semivertical angle with the z axis, which is also the direction of $\mathbf{v}$, is $\theta\approx S/v\approx 10^{-3}$ radian. The interaction is attractive and is singular on the sound-cone. The singularity arises because the upper limit of q in the integration was taken as $+\infty$ for the sake of mathematical simplicity. As a matter of fact this upper limit for the theory to be valid would be much less than $q=q_{\max}$, corresponding to the Debye limit. Nevertheless, the "wake" character of the interaction would essentially remain the same as expressed by (7).

In order that the foregoing considerations be valid the distance between two electrons should not be less than one wavelength of the phonons, i.e., $r\geq 1/q$. Assuming that the theory is valid for wave numbers $\leq(1/100)q_{\max}$, we get for $r_{\min}$ the value 10^{-6} cm, since $q_{\max}=10^8$ cm^{-1}. Let us now make a rough estimate of the magnitude of $U(r)$ for $r=r_{\min}$. From (7),

$$U(r_{\min})\approx -\frac{AV}{4\pi}\frac{1}{r_{\min}^3}=-\frac{8C^2}{9nMS^2}\frac{1}{4\pi r_{\min}^3},$$

using (4), or

$$U(r_{\min})=-F\left(\frac{8\xi_0}{3n}\right)\cdot\frac{1}{4\pi r_{\min}^3}, \tag{9}$$

where F has been put equal to $(C^2/3\xi_0MS^2)$ and is a dimensionless constant of the order of unity [see Eq. (2.9) of reference 1] and ξ_0 is the Fermi energy which has the magnitude 10 ev. Substituting the numerical values of $n=10^{22}$, $r_{\min}=10^{-6}$ cm and $\xi_0=10$ ev in (9) we have

$$U_{\max}\approx 10^{-4}\text{ ev}. \tag{10}$$

Thus we see that an electron moving faster than sound would emit radiation in the form of a narrow wake; and that there would be no interaction between two electrons if one lies outside the wake of the other. An electron lying on the surface of the wake of another electron is sucked inside the wake with a force given by the gradient of (7). The maximum interaction between two electrons in this model is roughly 10^{-4} ev. Such a model would lead to clusters of co-moving electrons. However, the clusters would have all possible directions giving zero net current. Our model may provide a basis for superconductivity in the sense that the wave functions of the electrons in this model, as discussed qualitatively in Sec. II, do not change in a magnetic field and may thus lead to the quantum mechanical derivation of the phenomenological equations of London. This point needs a further quantitative investigation.

The author takes this opportunity to express his thanks to Professors R. Peierls and J. Bardeen for many stimulating discussions.

APPENDIX

In this section, using a canonical transformation, we shall calculate the Møller interaction of two electrons as a result of the virtual emission and absorption of phonons. The Hamiltonian of our system consisting of two electrons and lattice vibrations can be expressed in the form

$$\mathbf{H}=H_0+H+e^2/|\mathbf{r}_{12}|, \tag{1A}$$

where

$$\left.\begin{aligned} H_0&=H_{01}+H_{02}+H_p,\\ \text{and}\quad H&=H_1+H_2 \end{aligned}\right\}. \tag{2A}$$

H_0 denotes the unperturbed Hamiltonian of the electron-phonon system. The interaction Hamiltonian H consists of two terms H_1 and H_2, corresponding, respectively, to the interaction of electrons 1 and 2 with the phonon field. $e^2/|\mathbf{r}_{12}|$ represents the Coulomb interaction of two electrons, which we shall neglect in the subsequent discussion. We transform $\mathbf{H}$ with the help of a transformation matrix S, which is expanded according to the perturbation method so that terms containing powers of S higher than the second are neglected. Then

$$\begin{aligned}\mathbf{H}'&=e^{iS}\mathbf{H}e^{-iS}\\ &=\mathbf{H}-i[\mathbf{H},S]-\tfrac{1}{2}[[\mathbf{H},S],S]+\cdots\\ &=H_0+H_1+H_2-i[H_{01}+H_p,S_1]-i[H_{02}+H_p,S_2]\\ &\quad -i[H_1,S]-i[H_2,S]-\tfrac{1}{2}[[H_{01},S],S]\\ &\quad -\tfrac{1}{2}[[H_{02},S],S]-\tfrac{1}{2}[[H_p,S],S]+\cdots,\end{aligned} \tag{3A}$$

since $[H_{02},S_1]=[H_{01},S_2]=0$, and S has been put equal to S_1+S_2.

Let S_1 and S_2 be defined by

$$\left.\begin{aligned} H_1&=i[H_{01}+H_p,S_1]\\ \text{and}\quad H_2&=i[H_{02}+H_p,S_2] \end{aligned}\right\}. \tag{4A}$$

Equation (3) then becomes

$$\begin{aligned}\mathbf{H}'=H_0-i/2\{&[H_1,S_1]+[H_2,S_2]\\ &+[H_1,S_2]+[H_2,S_1]\}+\cdots.\end{aligned} \tag{5A}$$

Now since H_{01} and H_p are diagonal, and using (4A),

$$\begin{aligned}\langle m|H_1|n\rangle&=i\langle m|H_{01}+H_p|l\rangle\langle l|S_1|n\rangle\\ &\quad -i\langle m|S_1|l\rangle\langle l|H_{01}+H_p|n\rangle\\ &=i\langle m|S_1|n\rangle\{E_{1m}-E_{1n}+\hbar(\omega_m-\omega_n)\}.\end{aligned}$$

Therefore,

$$\langle m|S_1|n\rangle=\frac{-i\langle m|H_1|n\rangle}{E_{1m}-E_{1n}+\hbar(\omega_m-\omega_n)}, \tag{6A}$$

and a similar expression is possible for $\langle m|S_2|n\rangle$.

The matrix elements of the commutators $[H_1,S_2]$ and $[H_2,S_1]$ correspond to the emission of a phonon by one electron and the absorption of the same phonon by the other electron. On the other hand, the matrix

elements of the commutators $[H_1, S_1]$ and $[H_2, S_2]$ correspond to the emission and absorption of a phonon by the same electron. Now

$$\langle \mathbf{k}_1', \mathbf{k}_2'[H_1, S_2]\mathbf{k}_1, \mathbf{k}_2\rangle = i\frac{\langle \mathbf{k}_1', \mathbf{k}_2'|H_1|\mathbf{k}_1, \mathbf{k}_2'\rangle\langle \mathbf{k}_1, \mathbf{k}_2'|H_2|\mathbf{k}_1, \mathbf{k}_2\rangle}{E_{k_2}-E_{k_2'}-\hbar\omega_q} - i\frac{\langle \mathbf{k}_1', \mathbf{k}_2'|H_2|\mathbf{k}_1', \mathbf{k}_2\rangle\langle \mathbf{k}_1', \mathbf{k}_2|H_1|\mathbf{k}_1, \mathbf{k}_2\rangle}{E_{k_2}-E_{k_2'}+\hbar\omega_q}, \quad (7A)$$

where use has been made of Eq. (6A). In Eq. (7A) $\mathbf{k}_1$ and $\mathbf{k}_2$ are, respectively, the wave-vectors of electrons 1 and 2 in the initial state, and $\mathbf{k}_1'$ and $\mathbf{k}_2'$ are the corresponding wave-vectors in the final state. $\mathbf{q}$ denotes the wave-vector of the phonon. The conservation of momentum demands that

$$\mathbf{k}_2=\mathbf{k}_2'+\mathbf{q}, \qquad \mathbf{k}_1=\mathbf{k}_1'+\mathbf{q}. \quad (8A)$$

The diagonal elements of the matrix (7A) are those for which $\mathbf{k}_1'=\mathbf{k}_2$ and $\mathbf{k}_2'=\mathbf{k}_1$, i.e., those for which the two electrons have interchanged their roles after transition.

The evaluation of the matrix element (7A) has been given by Bethe and Sommerfeld.[10] We shall only give the final result for the diagonal term.

$$\langle \mathbf{k}_2, \mathbf{k}_1[H_1, S_2]\mathbf{k}_1, \mathbf{k}_2\rangle = -i\sum_q \frac{|M_q|^2 2\hbar\omega_q}{(E_{k_1}-E_{k_2})^2-(\hbar\omega_q)^2}, \quad (9A)$$

where

$$|M_q|^2=\left(\frac{4C^2\hbar\omega_q}{9nVMS^2}\right)\begin{pmatrix} n_\omega \\ 1+n_\omega \end{pmatrix}. \quad (10A)$$

[10] A. Sommerfeld and H. Bethe, Handbuch der Physik **24/2**, 517 (1933).

C is the interaction constant and has the dimensions of energy, V is the volume, M is the atomic mass, n is the number of atoms per unit volume, and S is the velocity of sound. n_ω denotes the number of phonons of angular frequency ω and is zero at the absolute zero of temperature. The other matrix element $\langle \mathbf{k}_2, \mathbf{k}_1[H_2, S_1]\mathbf{k}_1, \mathbf{k}_2\rangle$ has the same value as given by (9A). Substituting in (5A) we get for the exchange interaction energy of two electrons the expression

$$U\equiv \mathbf{H}'-H_0 = -2\sum_q \frac{|M_q|^2\hbar\omega_q}{(E_{k_1}-E_{k_2})^2-(\hbar\omega_q)^2}, \quad (11A)$$

where we have left out the self-energy terms $[H_1, S_1]$ and $[H_2, S_2]$.

Expression (11A) corresponds to the E_2 term of Fröhlich [see Eq. (2.15) of reference (1)]. It is this interaction which is interesting from the point of view of superconductivity and arises as a result of the Pauli principle. The E_1 term of Fröhlich corresponds to our $[H_1, S_1]$ and $[H_2, S_2]$. It has been suggested by Bohm and Staver[5] that the chief difference between their theory and that of Fröhlich and Bardeen is that whereas the former attribute the cause of superconductivity to the interaction of two neighboring electrons through the phonon field, the latter attribute it to the interaction of an electron with itself through the phonon field. This criticism by Bohm and Staver is, however, unjustified because there is an effective interaction between electrons through the Pauli principle. The interesting term in the Fröhlich theory is the E_2 term which is identical with the exchange interaction of two electrons corresponding to our expression (11A). The E_1 term in the self-energy expression is discarded by Fröhlich as being unimportant.