Nuclear Matrix Elements in the Theory of Beta-Decay*

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An approach has been developed whereby the nuclear matrix elements can be calculated for the odd-A and half of the even-A beta-radioactive nuclei. The designation of nuclear states is made in accordance with the Mayer shell model and its extension by Nordheim to the even-A nuclei. The transforming nucleon is assigned the wave functions of a Dirac particle in a "square-well" potential representing the nuclear "core." Expressions for the matrix elements have been determined and compared with the data. Consistent agreement is found for the tensor interaction.

I. INTRODUCTION

ACCORDING to the Fermi theory of beta-radioactivity,[1,2] the probability P per unit time that a radioactive nucleus will emit an electron of energy (in units of the rest-energy of the electron) between W and $W+dW$ is

$$PdW=(G^2/2\pi^3)|M|^2F(Z,W)W \times(W^2-1)^{\frac{1}{2}}(W_0-W)^2dW, \quad (1)$$

where G is the interaction constant, F the Fermi function expressing the distortion, due to the electric field of the nucleus, of the statistical distribution, W_0 the maximum (end-point) energy of the emitted electron, and $|M|^2$ the square of the nuclear matrix element. More precisely, $|M|^2$ is a sum $\sum_{\Omega\Omega'} M_\Omega^* M_{\Omega'} C_{\Omega\Omega'}$ of terms quadratic in the matrix elements and each multiplied by a spectral correction factor $C_{\Omega\Omega'}(Z, W, W_0, R)$.

The Fermi function has been explicitly calculated and investigated in considerable detail.[3] The various spectral correction factors have also been explicitly determined.[4–7] The nuclear matrix elements, however, have heretofore only been crudely estimated on the basis of rather loose qualitative arguments (except for the allowed transitions for which angular factors alone determine M_Ω[8]). It is the purpose of this paper to remedy that lacuna for the majority of ground-to-ground transitions.

The matrix elements are formed by irreducible tensor operators compounded out of the Dirac operators and the position vector.[4,5] On the basis of their transformation properties, these operators fall into five categories or interactions: scalar, vector, axial vector, tensor, and pseudoscalar (S, V, A, T, and P). For each interaction, an $|M|^2$ is constructed from the M_Ω's having the desired transformation property and the appropriate correction factors. A knowledge of the matrix elements permits us to predict spectral distributions for the five interactions (or combinations of them) and thus, by comparison with the experimental data, to determine the correct interaction form. Angular correlation measurements give further clues in the same direction.

Integration of Eq. (1) over the whole range of electron energies (from 1 to W_0) yields $1/\tau$, the reciprocal of the theoretical lifetime. The M_Ω's are functions of nuclear quantities only and are strictly independent of W. In order to reduce greatly the calculations necessary, we can replace the C's by average values, thus taking all of $|M|^2$ outside the integral; the error introduced is not significant to the accuracy at present required. Thus,

$$(1/\tau)=(G^2/2\pi^3)|M|^2f(Z,W_0), \quad (2)$$

where $f(Z, W_0)$ is the energy integral. Substituting into this equation the experimental values of the mean life t and the end-point energy W_0, the theory predicts that

$$(G^2/2\pi^3)|M|^2ft=1. \quad (3)$$

Since the experimental ft values (now easily and quickly obtainable from the data[9]) vary over a very wide range, it is customary to list the common logarithm of the ft value instead. Then

$$\log ft+\log|M|^2=\text{constant}, \quad (4)$$

for all transitions for the proper choice of interaction.

The determination of the matrix elements presents two difficulties: the identification of the nuclear states, and the formation of appropriate nuclear wave functions. To resolve these two problems we have to postulate a model or formalism, and for each problem the approach involves elements not essential for the other. Such a model was evolved for the odd-A nuclei and for half the even-A. For those categories, our survey covers all allowed, first-forbidden, and second-forbidden transitions which occur ground-to-ground (and a few involving isomeric states where the spin assignment is clear) and for which the data (decay scheme, half-life, end-

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point energy, branching ratio if the decay is complex, optionally spectral shape) is complete and reliable,[10,11] up to neutron number 100. Heavier isotopes are omitted because they lie in the region of alpha-radioactivity where the competition of the latter process masks beta-decay in all but a very few cases, and where consequently spin assignments are very uncertain, especially in view of the extended choice of possible states for high particle number. The superallowed group does not fall within the model. Higher forbidden transitions were not dealt with because only three scattered cases were found with sufficient information; where the spin difference between the initial and final nuclei is high, the decay usually does not proceed ground-to-ground but instead via excited states and gamma-rays.

II. THE MODEL

Treating the nucleus strictly as a many-body problem, Wigner found that as a result of space, spin, and isotopic spin orthogonalities only the superallowed group of beta-transitions should occur to first order.[12] No quantitative information about the unfavored transitions has been obtained by this method. Since most decays are not superallowed, the Wigner theory is not wholly satisfactory. We are led to consider the Wigner approach as a first approximation, and to postulate some mechanism on a different basis whose contribution will yield the unfavored allowed and the forbidden transitions.

Such a mechanism is provided by the nuclear shell model.[7,13–19] We shall use the Mayer version of this model.[18,19] We consider each nucleon to behave as a particle in an attractive potential due to the other nucleons, all confined to a sphere of radius R. In the ground state, like nucleons are assumed to pair off to form an inert core as far as beta-decay is concerned; the whole interaction is ascribed to the remaining odd nucleon (for odd-A) or nucleons (for even-A). The spin and parity of an odd-A nucleus are those of the odd nucleon; the nucleon—and thus the nucleus—is assigned a spectroscopic term in analogy to atomic structure. For even-A, even-Z nuclei, there is only the core, and we take the nuclear spin as zero and all resultant nucleonic angular momenta also as zero. For even-A, odd-Z, the spins of the two odd nucleons add vectorially: If $J_1=l_1-\frac{1}{2}$ and $J_2=l_2+\frac{1}{2}$ (J and l are the total and orbital angular momenta of the nucleon), the resultant spin is the minimum $|J_1-J_2|$; if both are $l-\frac{1}{2}$ or $l+\frac{1}{2}$, the resultant spin is high (probably the maximum).[20] Experimental agreement with the spin predictions has been found in a number of cases.[21] In the ground state, a nucleon occupies the lowest available state of a single particle in the potential well due to all the other nucleons, subject to the Pauli principle. In order to make the energy levels break at the empirical "magic numbers," Mayer introduces spin-orbit coupling in such a manner that the $l+\frac{1}{2}$ term falls below the $l-\frac{1}{2}$ term. The order of levels in the Mayer scheme is shown in Appendix A; within a shell, this order is not rigid, rather there is quite a bit of crossing-over. While the spin-orbit coupling is essential for shell assignments, its effect on the energy levels is neglected in our calculations.

The shell model, which singles out the odd nucleon(s) and disregards the structure of the core, cannot of course account for the superallowed transitions since the latter are clearly related to correspondence of the proton and neutron number in the core. From the point of view of the shell model, superallowed behavior is a kind of resonance phenomenon. We shall not concern ourselves further with the superallowed transitions, but direct our attention to a comparison of the ordinary allowed and the forbidden decays.

For the odd-A nuclei, we take for the nuclear wave functions simply the wave functions (initial and final) of the transforming nucleon viewed as a Dirac particle in a three-dimensional "square-well" potential. The square well is selected for simplicity, though any other sufficiently steep well would do. For the even-A-minimum-spin-coupling nuclei, we adhere strictly to the same formalism, disregarding any limitation on the operators due to consideration of the resultant spin of the nucleus. No consistently satisfactory extension of the model has been found which will cover the even-A-high spin-coupling group.

III. DIRAC PARTICLE IN A SQUARE-WELL POTENTIAL

For a potential which is a function of r alone, the Dirac wave equation for the motion of a single particle[22] is separable, and yields the wave functions[23]

$$\psi_a=\begin{bmatrix} D_\kappa & Y_\kappa^{m-\frac{1}{2}} & if_{-\kappa} \\ -I_\kappa & Y_\kappa^{m+\frac{1}{2}} & if_{-\kappa} \\ I_{\kappa-1} & Y_{\kappa-1}^{m-\frac{1}{2}} & g_{-\kappa} \\ D_{\kappa-1} & Y_{\kappa-1}^{m+\frac{1}{2}} & g_{-\kappa} \end{bmatrix}$$

and

$$\psi_b=\begin{bmatrix} I_{\kappa-1} & Y_{\kappa-1}^{m-\frac{1}{2}} & if_\kappa \\ D_{\kappa-1} & Y_{\kappa-1}^{m+\frac{1}{2}} & if_\kappa \\ D_\kappa & Y_\kappa^{m-\frac{1}{2}} & g_\kappa \\ -I_\kappa & Y_\kappa^{m+\frac{1}{2}} & g_\kappa \end{bmatrix}, \tag{5}$$

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[21] J. E. Mack, Revs. Modern Phys. **22**, 64 (1950).
[22] E. L. Hill and R. Landshoff, Revs. Modern Phys. **10**, 87 (1938).
[23] M. E. Rose, Phys. Rev. **51**, 484 (1937).

where

$$I_\kappa=[(\kappa+m+\tfrac{1}{2})/(2\kappa+1)]^{\frac{1}{2}},\ D_\kappa=[(\kappa-m+\tfrac{1}{2})/(2\kappa+1)]^{\frac{1}{2}},$$

and $\kappa=|K|$, where K is the Dirac quantum number such that

$$\begin{aligned}&\text{for } \psi_a\text{:}\quad K=-(J+\tfrac{1}{2})=-l-1;\\&\text{for } \psi_b\text{:}\quad K=(J+\tfrac{1}{2})=l.\end{aligned}\tag{6}$$

In the Dirac representation, the orbital angular momentum l is no longer a good quantum number (i.e., it is not an eigenvalue of the problem); instead there is the new good quantum number K.

The spherical harmonics Y appearing above have been defined in agreement with Condon and Shortley[24] (leading to some sign differences from Rose). The radial functions $f(r)$ and $g(r)$ are real.

The Dirac radial equations for a single particle moving in a central field are[25]

$$\begin{aligned}&(E+Mc^2-V)rf_K-\hbar c(d/dr)rg_K-\hbar cKg_K=0,\\&(E-Mc^2-V)rg_K-\hbar c(d/dr)rf_K-\hbar cKf_K=0,\end{aligned}\tag{7}$$

where M is the rest mass, E and V the total and potential energy. For a square well of depth V_0, we substitute for the interior solution $e=(E+V_0)/Mc^2$ and $x=(Mc/\hbar)r(e^2-1)^{\frac{1}{2}}$ to recast the equations into

$$\begin{aligned}&[(d/dx)+(K/x)](e-1)^{\frac{1}{2}}xg_K-(e+1)^{\frac{1}{2}}xf_K=0,\\&[(d/dx)-(K/x)](e+1)^{\frac{1}{2}}xf_K+(e-1)^{\frac{1}{2}}xg_K=0.\end{aligned}\tag{8}$$

The solutions of these equations which are regular at the origin are spherical Bessel functions.[26] For K positive or negative, they are

$$\begin{aligned}&f_\kappa=A_\kappa(e+1)^{-\frac{1}{2}}j_{\kappa-1}(x),\quad f_{-\kappa}=A_{-\kappa}(e+1)^{-\frac{1}{2}}j_\kappa(x),\\&g_\kappa=A_\kappa(e-1)^{-\frac{1}{2}}j_\kappa(x),\quad g_{-\kappa}=-A_{-\kappa}(e-1)^{-\frac{1}{2}}j_{\kappa-1}(x),\end{aligned}\tag{9}$$

where A_κ and $A_{-\kappa}$ are normalization factors.

For the external solution, we can let $w=E/Mc^2$ and obtain equations of exactly the same form. However, while $e>1$, $w<1$, so that the argument of the Bessel functions is now imaginary. The solutions that vanish at ∞ and are real are spherical Macdonald functions.[26] Letting $y=(Mc/\hbar)r(1-w^2)^{\frac{1}{2}}$, we obtain

$$\begin{aligned}f_\kappa&=B_\kappa(1+w)^{-\frac{1}{2}}k_{\kappa-1}(y),\\g_\kappa&=-B_\kappa(1-w)^{-\frac{1}{2}}k_\kappa(y),\\f_{-\kappa}&=B_{-\kappa}(1+w)^{-\frac{1}{2}}k_\kappa(y),\\g_{-\kappa}&=-B_{-\kappa}(1-w)^{-\frac{1}{2}}k_{\kappa-1}(y).\end{aligned}\tag{10}$$

If we match the inner and outer solutions for f and g at the walls of the well, the ratio of the two equalities yields the boundary value equation. Using the rather good approximation $e+1\approx 2\approx 1+w$ and applying the recursion relations, the boundary value equation takes on the same form as in the nonrelativistic treatment.[14] If by each function we understand its value for $r=R$, we have

$$\begin{aligned}&\text{for } \psi_a\text{:}\quad j_{\kappa-2}/j_\kappa=-k_{\kappa-2}/k_\kappa,\\&\text{for } \psi_b\text{:}\quad j_{\kappa-1}/j_{\kappa+1}=-k_{\kappa-1}/k_{\kappa+1}.\end{aligned}\tag{11}$$

The consecutive solutions of the equation as x and y increase correspond to the successive nodal quantum numbers n.

For the normalization factors, the same approximation yields

$$B_\kappa/A_\kappa=j_{\kappa-1}/k_{\kappa-1},\quad B_{-\kappa}/A_{-\kappa}=j_\kappa/k_\kappa.\tag{12}$$

Using for the well range the usual nuclear radius $3.5\times10^{-3}A^{\frac{1}{3}}$ in units of the Compton wavelength of the electron, we find consistently a binding energy of the order of 8 Mev (as it should be) and a well depth of the order of 40 Mev. The results of numerical solution for the boundary values are listed in Appendix A. The spherical Bessel functions have been tabulated[27] and a polynomial expansion exists for the Macdonald functions.[26]

TABLE I. Transitions $K'-K=\pm1$ ($\Delta J=\pm1$, $\Delta l=\pm1$).—By the coupling rule, this can occur only for odd A.

Isotope	Decay	Spin assignment	log ft	S	log($\lvert M\rvert^2ft$) V	A	T
35 Br 87	—	$2p_{3/2}-2d_{5/2}$	6.8	4.4	5.4	4.4	5.4
36 Kr 87	—	$2d_{5/2}-2p_{3/2}$*	7.0	4.4	5.4	4.4	5.4
37 Rb 89	—	$2p_{3/2}-2d_{5/2}$	6.6	4.2	5.2	4.2	5.2
57 La 141	—	$2d_{5/2}-2f_{7/2}$	7.5	5.2	6.0	5.2	6.0
58 Co 141	—	$2f_{7/2}-2d_{5/2}$*	7.7	5.3	6.0	5.3	6.0
59 Pr 143	—	$2d_{5/2}-2f_{7/2}$	7.6	5.3	6.1	5.3	6.1
59 Pr 145	—	$2d_{5/2}-2f_{7/2}$	7.8	5.5	6.3	5.5	6.3
61 Pm 147	—	$2d_{5/2}-2f_{7/2}$	7.6	5.3	6.1	5.3	6.1

IV. THE MATRIX ELEMENTS

Instead of irreducible tensors, the nuclear operators can be expressed in terms of solid spherical harmonics $\mathcal{Y}_{LM}(\Omega)$, where L is the order of the tensor and M is an integer such that $|M|\leq L$;[28] this leads to considerable simplification for $L>1$.

Making use of group-theoretical theorems,[28] we can prove that the square of the nuclear matrix element reduces to

$$|M_\Omega|^2=\frac{2L+1}{2J+1}\sum_{m'=-J'}^{J'}|(nKm'|\,\mathcal{Y}_{LO}(\Omega)\,|n'K'm')|^2.\tag{13}$$

Cross terms vanish unless both Ω's have the same L,

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[25] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949).
[26] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, London, 1944), second edition.
[27] *Tables of Spherical Bessel Functions*, Mathematical Tables Project, National Bureau of Standards (Columbia University Press, New York, 1947).
[28] D. L. Falkoff and G. E. Uhlenbeck, Phys. Rev. **79**, 323 (1950).

TABLE II. Transitions $K'+K=0$ ($\Delta J=0$, $\Delta l=\pm 1$).

Isotope	Decay	Spin assignment	log ft	log($\|M\|^2ft$) S	V	A	T	P
46 Pd 111	−	$3s_{1/2}-2p_{1/2}$	6.8	3.1	4.9	5.2	5.4	3.6
47 Ag 111	−	$2p_{1/2}-3s_{1/2}$*	7.3	3.6	5.4	5.7	5.9	4.1
47 Ag 113	−	$2p_{1/2}-3s_{1/2}$*	7.0	3.3	5.1	5.4	5.6	3.8
47 Ag 115	−	$2p_{1/2}-3s_{1/2}$	6.4	2.7	4.5	4.8	5.0	3.2
48 Cd 117	−	$3s_{1/2}-2p_{1/2}$	6.1	2.4	4.2	4.5	4.7	2.9
49 In 115^m	−	$2p_{1/2}-3s_{1/2}$*	6.4	2.7	4.5	4.8	5.0	3.2
49 In 117	−	$2p_{1/2}-3s_{1/2}$*	6.2	2.5	4.3	4.6	4.8	3.0
49 In 119	−	$2p_{1/2}-3s_{1/2}$*	6.2	2.5	4.3	4.6	4.8	3.0
54 Xe 137	−	$2f_{7/2}-1g_{7/2}$*	6.3	1.7	3.7	4.7	4.7	3.0
56 Ba 139	−	$2f_{7/2}-1g_{7/2}$*	6.7	2.1	4.1	5.1	5.1	3.4
66 Dy 165	−	$2f_{7/2}-1g_{7/2}$*	6.2	1.6	3.6	4.6	4.6	2.9
7 N 16	−	$2s_{1/2}-1p_{1/2}$	6.8	1.5	4.8	4.8	5.0	3.8
36 Kr 88	−	$2d_{5/2}-1f_{5/2}$	6.8	0.0	3.8	5.1	5.2	2.3
37 Rb 88	−	$2d_{5/2}-1f_{5/2}$	7.1	0.3	4.1	5.4	5.5	2.6
54 Xe 138	−	$2f_{7/2}-1g_{7/2}$	6.5	1.9	3.9	4.9	4.9	3.2
58 Ce 144	−	$2f_{i/2}-1g_{7/2}$	7.2	2.6	4.6	5.6	5.6	3.9

in which case

$$M_\Omega M_{\Omega'}{}^*=\frac{2L+1}{2J+1}\sum_{m'=-J'}^{J'}(nKm'|\,\mathcal{Y}_{LO}(\Omega)\,|n'K'm')\times(nKm'|\,\mathcal{Y}_{LO}(\Omega')\,|n'K'm')^*. \quad (14)$$

A great simplification is introduced by neglecting the difference between the initial and final energy levels in computing the matrix elements, i.e., by assigning the Bessel functions the same argument. The error introduced is not too large because of the smallness of the binding energy and well depth relative to the rest-energy of a nucleon. With this approximation, numerical integration for each individual case is replaced by the use of analytic expressions for the integrals. These expressions, based on the properties given by Watson,[26] were computed and are listed in Appendix B.

The procedure for calculating the matrix elements is as follows: Form $\psi'^*\mathcal{Y}_{LO}(\Omega)\psi$ by matrix multiplication for each of the four possibilities ($a\to a$, $a\to b$, $b\to a$, $b\to b$). Next integrate over the region (i.e., inside or outside the well); the angular integration will yield zero except possible for some special values of $\kappa'-\kappa$. The sum of the inner and outer integrals for one such value is the $(nKm'|\,\mathcal{Y}_{LO}(\Omega)\,|n'K'm')$ corresponding to the particular set of quantum numbers. Substitute this into Eq. (13) or (14) and carry out the indicated multiplications and summation.

It should be noted that the first two components of ψ are pure imaginary, the last two real. The operators acting on ψ either leave this order unchanged or reverse the groups of two; beyond the matrix operation, the factors introduced by the operators are either real or pure imaginary. Thus the product is pure imaginary or real, and consequently so is $(nKm'|\,\mathcal{Y}_{LO}(\Omega)\,|n'K'm')$. As a result, we find automatic agreement with the independent theoretical prediction that all cross terms of matrix elements are real[29]—the i factors being always present in the correction factors when the product inside the summation in Eq. (14) is pure imaginary, and not otherwise. Cross terms play no significant role till the second-forbidden transitions.

The values of the matrix elements appear in Appendix C.

V. EXAMINATION OF THE DATA

We proceed to tabulate the forbidden transitions, assigning nuclear states[30] and computing log($|M|^2ft$) for each of the five interactions, $|M|^2$ being normalized to $|M_1|^2=1$ (ordinary allowed, not superallowed, matrix elements of magnitude 1). For comparison, it should be noted that for odd-A nuclei the ordinary allowed transitions have logft values in the range 4.9–6.1, while the superallowed are 3.1–3.8; for even-A, there appears to exist only one allowed category with intermediate values 3.9–5.3, though there is some meager evidence for superallowed logft values. Allowed matrix elements can occur with the $K'-K=0$ ($\Delta J=0$, $\Delta l=0$) group for S, V, A, or T; with the $K'-K=-1$ ($\Delta J=\pm 1$, $\Delta l=0$) group only for A or T.

In the tabulation (Tables I–V), the nonoccurrence of an interaction means that its matrix elements are not comparable in magnitude with those entered. Spins marked with asterisks indicate measured values. In the "decay" column, − or + identifies negatron or

TABLE III. Transitions $K'+K=-2$ ($\Delta J=\pm 2$, $\Delta l=\pm 1$).—This group has its significant matrix elements for A or T only, these two being in fact equal. Here we first expect to find a definitely forbidden spectral shape, of which there is indeed considerable experimental corroboration.[31]

Isotope	Decay	Spin assignment	log ft	log (Cft)	log ($\|M\|^2ft$)
16 S 37	−	$1f_{7/2}-1d_{3/2}$*	7.0	7.6	4.0
17 Cl 38	−	$1f_{7/2}-1d_{3/2}$	7.4	8.1	4.5
18 A 41	−	$1f_{7/2}-1d_{3/2}$*	8.6	8.8	5.2
19 K 42	−	$1f_{7/2}-1d_{3/2}$	8.0	8.4	4.8
33 As 72	+	$1f_{5/2}-1g_{9/2}$	8.2	8.6	5.0
33 As 76	−	$1g_{9/2}-1f_{5/2}$	8.4	8.7	4.9
36 Kr 85	−	$1g_{9/2}-1f_{5/2}$*	9.2	8.4	4.6
37 Rb 86	−	$1g_{9/2}-1f_{5/2}$	8.6	8.5	4.7
38 Sr 89	−	$2d_{5/2}-2p_{1/2}$*	8.6	8.4	4.3
38 Sr 90	−	$2d_{5/2}-2p_{1/2}$	9.2	8.3	4.2
38 Sr 91	−	$2d_{5/2}-2p_{1/2}$	8.1	8.4	4.3
39 Y 90	−	$2d_{5/2}-2p_{1/2}$	8.0	8.1	4.0
39 Y 91	−	$2p_{1/2}-2d_{5/2}$*	8.7	8.5	4.9
45 Rh 102	+	$2p_{1/2}-2d_{5/2}$	8.4	7.9	4.3
45 Rh 102	−	$2d_{5/2}-2p_{1/2}$	9.4	9.0	4.9
50 Sn 123^m	−	$1h_{11/2}-1g_{7/2}$*	9.1	8.8	5.1
50 Sn 125	−	$1h_{11/2}-1g_{7/2}$	8.9	9.0	5.3
51 Sb 122	−	$1h_{11/2}-1g_{7/2}$	8.0	8.0	4.3
51 Sb 125	$\underline{m}$	$1g_{7/2}-1h_{11/2}$	9.4	8.6	5.1
53 I 126	−	$1h_{11/2}-1g_{7/2}$	8.5	8.2	4.5
55 Cs 137	$\underline{m}$	$1g_{7/2}$*$-1h_{11/2}$	9.6	8.7	5.2

[29] C. L. Longmire and A. M. L. Messiah, Phys. Rev. **83**, 464 (1951).

[30] Mayer, Moszkowski, and Nordheim, Revs. Modern Phys. **23**, 315 (1951); L. W. Nordheim, Revs. Modern Phys. **23**, 322 (1951)—with minor deviations.

[31] C. S. Wu, Revs. Modern Phys. **22**, 386 (1950).

TABLE IV. Transitions $K'+K=1$ ($\Delta J=\pm1$, $\Delta l=\pm2$).—This is the l-forbidden group. Nuclear shell assignments are a bit shaky for this group, in that the state for which the transition is l-forbidden and the state for which it is allowed both lie in the same shell, so that they are in competition.

Isotope	Decay	Spin assignment	log ft	log ($\|M\|^2ft$)
8 O 19	−	$1d_{3/2}-2s_{1/2}$*	5.6	1.9
14 Si 31	−	$1d_{3/2}-2s_{1/2}$*	5.9	2.2
15 P 32	−	$1d_{3/2}-2s_{1/2}$	7.9	4.2
28 Ni 63	−	$1f_{5/2}-2p_{3/2}$*	7.1	3.4
28 Ni 65	−	$1f_{5/2}-2p_{3/2}$*	6.6	2.9
29 Cu 64	−	$1f_{5/2}-2p_{3/2}$	5.3	1.6
29 Cu 64	+	$2p_{3/2}-1f_{5/2}$	4.9	1.3
31 Ga 66	+	$2p_{3/2}-1f_{5/2}$	7.9	4.3
32 Ge 69	+	$1f_{5/2}-2p_{3/2}$*	6.0	2.3

positron decay. A superscript m indicates that the initial state is isomeric, an m in the "decay" column that the final state is isomeric.

The last five interactions in the $K'+K=0$ ($\Delta J=0$, $\Delta l=\pm1$) tabulation are even-A nuclei treated on a strict one-particle model. From the point of view of the nucleus as a whole, these are $J=0\rightarrow0$ (pari$^+$ change) cases. However, this does not restrict the spin of the transforming nucleon, but merely requires that the two odd nucleons at the odd-Z end of the transition have equal spins coupling to zero resultant. Hence, it is consistent to ignore the limitations usually imposed on the matrix elements as a result of the $0\rightarrow0$ identification. These matrix elements correspond to the spin-orbit coupling implicit in the Dirac equation; for the larger empirical spin-orbit coupling (from shell breaks), they would presumably be considerably larger.

Nonrelativistically, the l-forbidden transitions can only occur with a forbidden shape. In the Dirac representation, however, we find that the largest possible matrix elements are $\boldsymbol{\sigma}$ and $\beta\boldsymbol{\sigma}$ (for A and T), with allowed shape. These do not occur nonrelativistically, when l is a good quantum number, but here K is the good quantum number and l is not; the contributions are from the small components of the Dirac wave functions. Satisfactory log($|M|^2ft$) values can be obtained by considering the states involved in these transitions to be an admixture, in varying proportions, of allowed and l-forbidden; for A and T, the spectral shape is allowed (as found experimentally for P^{32}, Cu^{64}, and Ga^{66}).

Finally, for the second-forbidden transitions (Table V), our theoretical expressions yield an $|A_{ij}/T_{ij}|$ ratio of about 9 (or $|A_{ij}/R_{ij}|$ about 18) with the S_{ijk} term negligible—to very little precision because the calculational approximations used became serious for the small overlap of initial and final wave functions here encountered. The cross term is predicted positive for Tc^{99} ($a\rightarrow a$) and negative for the other two ($b\rightarrow b$). Experimentally, the Tc^{99} spectrum is fitted with $|A_{ij}/T_{ij}|=6.65$ for T (or $|A_{ij}/R_{ij}|=13.30$ for V) with the sign uncertain.[32] For Cs^{137}, $|A_{ij}/T_{ij}|=7.43$ with cross term negative has been reported,[33] though the fit with Konopinski's correction factors appears to be imperfect for any ratio.[32]

The P^{32} electron-neutrino angular correlation,[34] reinterpreted in the light of our spin assignment and of an improved theory,[35] is consistent only with T.

The beta-gamma angular correlation of Rb^{86} agrees with T only,[36] and this is probably also true for I^{126}.[37]

VI. CONCLUSIONS

The theoretical classification of the degree of forbiddenness of a transition was based upon a qualitative estimation of the magnitude of the various matrix elements; the values of $|M|^2$ fell into groups differing by a factor of the order of 100. Later, some refinements were made, mainly on an empirical basis. According to the calculations just presented, this classification loses much of its validity. Within the same order of forbiddenness, there are values differing by several orders of magnitude. On the other hand, there is no sharp demarcation in size as we go from allowed to first-forbidden to l-forbidden matrix elements. This roughly continuous distribution of $|M|^2$ values, and thus also of ft values, agrees with the data. A break does occur before the second-forbidden group. The finer subdivision we obtain eliminates much of the "straggling" observed in the experimental ft values; the range of logft values found for a particular assignment of initial and final configurations is at most 1 and usually less.

In earlier discussions of the nuclear matrix elements, it has always been assumed that there is no appreciable error in replacing β by 1 in the matrix expressions. This assumption turns out to be correct in most cases, but not for $\Delta J=0$. In that category, the matrix element of $\beta\gamma_5$ is very much smaller than that of γ_5; the matrix element of $\boldsymbol{\alpha}$ is less than that of $\beta\boldsymbol{\alpha}$, and decreases with increasing κ.

Another simplification can lead to totally wrong results. As hitherto unsuspected, for l-forbidden transitions the dominant role is played by the small components of the Dirac wave functions—an effect ignored in nonrelativistic approximations.

TABLE V. Transition $K'-K=\pm2$ ($\Delta J=\pm2$, $\Delta l=\pm2$).—For the first time, in this group, A and T differ considerably in spectral shape, thus allowing discrimination (differences in log($|M|^2ft$) have to be quite large to be decisive).

Isotopes	Decay	Spin assignment	log ft	log($\|M\|^2ft$) S	V	A	T
43 Tc 99	−	$1g_{9/2}-2d_{5/2}$	12.6	4.9	5.9	4.9	5.9
55 Cs 135	−	$1g_{7/2}$*$-2d_{3/2}$*	13.1	5.1	5.1	5.1	5.1
55 Cs 137	−	$1g_{7/2}$*$-2d_{3/2}$*	12.2	5.3	5.1	5.3	5.1

[32] F. Wagner, Jr., and M. S. Freedman, private communication.

[33] L. M. Langer and R. J. D. Moffat, Phys. Rev. **82**, 635 (1951).
[34] C. W. Sherwin, Phys. Rev. **82**, 52 (1951).
[35] E. Greuling and M. L. Meeks, Phys. Rev. **82**, 531 (1951).
[36] D. T. Stevenson and M. Deutsch, Phys. Rev. **83**, 1202 (1951).
[37] D. T. Stevenson and M. Deutsch, Phys. Rev. **84**, 1071 (1951).

The experimental evidence (logft values, spectral shapes, angular correlations) is always consistent with the theoretical predictions for the tensor interaction, but not so for the other interactions. Agreement is not achieved by any linear combination of interactions unless T predominates in it.

There is still need for considerable experimental information. Careful determination of the spectral shapes of the more highly forbidden decays should be carried out; all but the most recent such work is unreliable because of thick sources and other uncertainties. For first-forbidden, $\Delta J=0, \pm 1$ decays, the small deviation from the allowed spectrum is very difficult to detect, but angular correlation experiments could serve the same end. Many decay schemes bear reinvestigation because of lacking or inadequate gamma-ray search, or failure to ascertain whether a gamma-ray is in series with a beta-ray; this applies to essentially all the older data. Of particular theoretical interest are the even-A-high-spin-coupling nuclei, which do not lend themselves to our formalism; too few are now known, especially as to shape. Further verification of our interpretation of the l-forbidden ($\Delta J=\pm 1$, $\Delta l=\pm 2$) group is desirable. Most of the data for high-Z nuclei is too old and sketchy. Despite the difficulties, more electron-neutrino angular correlations should be undertaken. The shell assignments would be more secure and many equivocal interpretations could be eliminated if more spins of nuclei were known.

I wish to express my grateful appreciation to Professor E. Greuling for his essential and untiring guidance and for his friendly encouragement in the course of this work.

APPENDIX A

Boundary Values

Term	K	x	$j_\kappa(x)$	$j_{\kappa-1}(x)$	y	$\bar{k}_\kappa(y)$	$\bar{k}_{\kappa-1}(y)$
$1s_{1/2}$	-1	2.1	0.44	0.41	0.89	2.1	1.0
→							
$1p_{3/2}$	-2	3.4	0.31	0.26	1.5	4.3	1.7
$1p_{1/2}$	1	3.4	0.26	-0.073	1.5	1.7	1.0
→							
$1d_{5/2}$	-3	4.7	0.24	0.19	2.1	9.2	3.1
$1d_{3/2}$	2	4.7	0.19	-0.039	2.1	3.1	1.5
$2s_{1/2}$	-1	5.1	-0.11	-0.18	2.3	1.4	1.0
→							
$1f_{7/2}$	-4	5.9	0.20	0.15	2.7	19.0	6.0
→							
$1f_{5/2}$	3	5.9	0.15	-0.027	2.7	6.0	2.5
$2p_{3/2}$	-2	6.6	-0.11	-0.14	2.9	2.4	1.3
$2p_{1/2}$	1	6.6	-0.14	0.046	2.9	1.3	1.0
$1g_{9/2}$	-5	7.1	0.17	0.12	3.2	42.0	13.0
→							
$1g_{7/2}$	4	7.1	0.12	-0.020	3.2	13.0	5.0
$2d_{5/2}$	-3	8.0	-0.10	-0.11	3.6	4.2	2.1
$2d_{3/2}$	2	8.0	-0.11	0.031	3.6	2.1	1.3
$3s_{1/2}$	-1	8.3	0.060	0.11	3.7	1.3	1.0
$1h_{11/2}$	-6	8.3	0.15	0.10	3.7	90.0	27.0
→							
$2f_{7/2}$	-4	9.3	-0.095	-0.096	4.1	7.8	3.5

The function

$$\bar{k}_\kappa=(2/\pi)ye^y k_\kappa$$

here tabulated can be used instead of k_κ, since the k_κ's always occur in ratios in which the common factor $(2/\pi)ye^y$ cancels out. It is more convenient to compute $\bar{k}_\kappa$ because it can be expressed as a polynomial in $1/y$ of order κ.[23]

The arrows set off nuclear shells.

APPENDIX B

The Radial Integrals

It should be understood that by the expressions on the right side of the equations below we mean their value at $r=R$ (listed in Appendix A).

$$(2/R^3)\int_0^R j_\kappa^2 r^2 dr = j_\kappa^2 + j_{\kappa-1}^2 - (2\kappa+1)j_\kappa j_{\kappa-1}/x$$

$$(2/R^3)\int_R^\infty k_\kappa^2 r^2 dr = -k_\kappa^2 + k_{\kappa-1}^2 + (2\kappa+1)k_\kappa k_{\kappa-1}/y$$

$$(2/R^3)\int_0^R j_{\kappa-1}^2 r^2 dr = j_\kappa^2 + j_{\kappa-1}^2 - (2\kappa-1)j_\kappa j_{\kappa-1}/x$$

$$(2/R^3)\int_R^\infty k_{\kappa-1}^2 r^2 dr = k_\kappa^2 - k_{\kappa-1}^2 - (2\kappa-1)k_\kappa k_{\kappa-1}/y$$

$$(2/R^3)\int_0^R j_{\kappa+1} j_\kappa r^3 dr = (R/2x)[(2\kappa+1)j_\kappa^2 + (2\kappa+3)j_{\kappa-1}^2 - (2\kappa+1)(2\kappa+3)j_\kappa j_{\kappa-1}/x]$$

$$(2/R^3)\int_R^\infty k_{\kappa+1} k_\kappa r^3 dr = (R/2y)[-(2\kappa+1)k_\kappa^2 + (2\kappa+3)k_{\kappa-1}^2 + (2\kappa+1)(2\kappa+3)k_\kappa k_{\kappa-1}/y]$$

$$(2/R^3)\int_0^R j_\kappa j_{\kappa-1} r^3 dr = (R/2x)[(2\kappa+1)j_\kappa^2 + (2\kappa-1)j_{\kappa-1}^2 - (2\kappa+1)(2\kappa-1)j_\kappa j_{\kappa-1}/x]$$

$$(2/R^3)\int_R^\infty k_\kappa k_{\kappa-1} r^3 dr = (R/2y)[(2\kappa+1)k_\kappa^2 - (2\kappa-1)k_{\kappa-1}^2 - (2\kappa+1)(2\kappa-1)k_\kappa k_{\kappa-1}/y]$$

$$(2/R^3)\int_0^R j_{\kappa-1} j_{\kappa-2} r^3 dr = (R/2x)[(2\kappa-3)j_\kappa^2 + (2\kappa-1)j_{\kappa-1}^2 - (2\kappa-1)(2\kappa-3)j_\kappa j_{\kappa-1}/x]$$

$$(2/R^3)\int_R^\infty k_{\kappa-1} k_{\kappa-2} r^3 dr = (R/2y)[-(2\kappa-3)k_\kappa^2 + (2\kappa-1)k_{\kappa-1}^2 + (2\kappa-1)(2\kappa-3)k_\kappa k_{\kappa-1}/y]$$

$$(2/R^3)\int_0^R j_{\kappa+2} j_\kappa r^4 dr = (R^2/3x^2)\{[-x^2 + (2\kappa+1)(2\kappa+3)]j_\kappa^2 + [-x^2 + (2\kappa+3)(2\kappa+5)]j_{\kappa-1}^2 - [-(2\kappa+3)x^2 + (2\kappa+1)(2\kappa+3)(2\kappa+5)]j_\kappa j_{\kappa-1}/x\}$$

$$(2/R^3)\int_R^\infty k_{\kappa+2} k_\kappa r^4 dr = (R^2/3y^2)\{[-y^2 - (2\kappa+1)(2\kappa+3)]k_\kappa^2 + [y^2 + (2\kappa+3)(2\kappa+5)]k_{\kappa-1}^2 - [-(2\kappa+3)y^2 - (2\kappa+1)(2\kappa+3)(2\kappa+5)]k_\kappa k_{\kappa-1}/y\}$$

$$(2/R^3)\int_0^R j_{\kappa+1} j_{\kappa-1} r^4 dr = (R^2/3x^2)\{[-x^2 + (2\kappa+1)(2\kappa+3)]j_\kappa^2 + [-x^2 + (2\kappa-1)(2\kappa+3)]j_{\kappa-1}^2 - [-(2\kappa-3)x^2 + (2\kappa-1)(2\kappa+1)(2\kappa+3)]j_\kappa j_{\kappa-1}/x\}$$

$$(2/R^3)\int_R^\infty k_{\kappa+1} k_{\kappa-1} r^4 dr = (R^2/3y^2)\{[y^2 + (2\kappa+1)(2\kappa+3)]k_\kappa^2 + [-y^2 - (2\kappa-1)(2\kappa+3)]k_{\kappa-1}^2 - [(2\kappa-3)y^2 + (2\kappa-1)(2\kappa+1)(2\kappa+3)]k_\kappa k_{\kappa-1}/y\}$$

$$(2/R^3)\int_0^R j_\kappa j_{\kappa-2} r^4 dr = (R^2/3x^2)\{[-x^2 + (2\kappa+1)(2\kappa-3)]j_\kappa^2 + [-x^2 + (2\kappa-1)(2\kappa-3)]j_{\kappa-1}^2 - [-(2\kappa+3)x^2 + (2\kappa+1)(2\kappa-1)(2\kappa-3)]j_\kappa j_{\kappa-1}/x\}$$

$$(2/R^3)\int_R^\infty k_\kappa k_{\kappa-2} r^4 dr = (R^2/3y^2)\{[-y^2 - (2\kappa+1)(2\kappa-3)]k_\kappa^2 + [y^2 + (2\kappa-1)(2\kappa-3)]k_{\kappa-1}^2 - [-(2\kappa+3)y^2 - (2\kappa+1)(2\kappa-1)(2\kappa-3)]k_\kappa k_{\kappa-1}/y\}$$

$$(2/R^3)\int_0^R j_{\kappa-1} j_{\kappa-3} r^4 dr = (R^2/3x^2)\{[-x^2 + (2\kappa-3)(2\kappa-5)]j_\kappa^2 + [-x^2 + (2\kappa-1)(2\kappa-3)]j_{\kappa-1}^2 - [-(2\kappa-3)x^2 + (2\kappa-1)(2\kappa-3)(2\kappa-5)]j_\kappa j_{\kappa-1}/x\}$$

$$(2/R^3)\int_R^\infty k_{\kappa-1} k_{\kappa-3} r^4 dr = (R^2/3y^2)\{[y^2 + (2\kappa-3)(2\kappa-5)]k_\kappa^2 + [-y^2 - (2\kappa-1)(2\kappa-3)]k_{\kappa-1}^2 - [(2\kappa-3)y^2 + (2\kappa-1)(2\kappa-3)(2\kappa-5)]k_\kappa k_{\kappa-1}/y\}.$$

APPENDIX C

The Matrix Elements

This table contains all matrix elements from allowed through second-forbidden, except those for $\gamma_5\mathbf{r}$, $\beta\gamma_5\mathbf{r}$, $\boldsymbol{\alpha}\cdot\mathbf{r}$, $\beta\boldsymbol{\alpha}\cdot\mathbf{r}$, $\boldsymbol{\alpha}\times\mathbf{r}$, and $\beta\boldsymbol{\alpha}\times\mathbf{r}$, which have been omitted because they have no practical significance inasmuch as they only occur in competition with much larger less forbidden matrix elements.

The following abbreviations are used:

$$M = \int_0^R j_{\kappa'}^2 r^2 dr, \qquad m = \lambda \int_R^\infty k_{\kappa'}^2 r^2 dr,$$

$$N = \int_0^R j_{\kappa'-1}^2 r^2 dr, \qquad n = \lambda \int_R^\infty k_{\kappa'-1}^2 r^2 dr,$$

$$O = \int_0^R j_{\kappa'+1} j_{\kappa'} r^3 dr, \qquad o = \lambda \int_R^\infty k_{\kappa'+1} k_{\kappa'} r^3 dr,$$

$$P=\int_0^R j_{\kappa'}j_{\kappa'-1}r^3dr, \qquad p=\lambda\int_R^\infty k_{\kappa'}k_{\kappa'-1}r^3dr,$$

$$Q=\int_0^R j_{\kappa'-1}j_{\kappa'-2}r^3dr, \quad q=\lambda\int_R^\infty k_{\kappa'-1}k_{\kappa'-2}r^3dr,$$

$$S=\int_0^R j_{\kappa'+2}j_{\kappa'}r^4dr, \qquad s=\lambda\int_R^\infty k_{\kappa'+2}k_{\kappa'}r^4dr,$$

$$T=\int_0^R j_{\kappa'+1}j_{\kappa'-1}r^4dr, \quad t=\lambda\int_R^\infty k_{\kappa'+1}k_{\kappa'-1}r^4dr,$$

$$U=\int_0^R j_{\kappa'}j_{\kappa'-2}r^4dr, \qquad u=\lambda\int_R^\infty k_{\kappa'}k_{\kappa'-2}r^4dr,$$

$$V=\int_0^R j_{\kappa'-1}j_{\kappa'-3}r^4dr, \quad v=\lambda\int_R^\infty k_{\kappa'-1}k_{\kappa'-3}r^4dr,$$

$$\lambda=(B_KB_{K'})/(A_KA_{K'}).$$

We have neglected the discrepancy between the values of e and w for the initial and final states, and set $e\approx 1\approx w$ so that

$$\begin{aligned} 1/(e+1)&=\tfrac{1}{2}, & 1/(1+w)&=\tfrac{1}{2},\\ 1/(e-1)&=\tfrac{1}{2}H^2, & 1/(1-w)&=\tfrac{1}{2}h^2,\\ 1/(e^2-1)^{\frac{1}{2}}&=\tfrac{1}{2}H, & 1/(1-w^2)^{\frac{1}{2}}&=\tfrac{1}{2}h. \end{aligned}$$

Since $1/(e-1)\approx 33$ and $1/(1-w)\approx 164$, we neglect terms in $1/(e+1)$ and $1/(1+w)$ in comparison with the former.

We follow the Konopinski nomenclature:[2]

$$\begin{aligned} |B_{ij}|&\leftrightarrow|\,\mathcal{Y}_{2M}(\boldsymbol{\sigma},\mathbf{r})\,|=|\,\mathcal{Y}_{2M}(\beta\boldsymbol{\sigma},\mathbf{r})\,|\\ |A_{ij}|&\leftrightarrow|\,\mathcal{Y}_{2M}(\boldsymbol{\alpha},\mathbf{r})\,|=|\,\mathcal{Y}_{2M}(\beta\boldsymbol{\alpha},\mathbf{r})\,|\\ |R_{ij}|&\leftrightarrow\tfrac{1}{2}|\,\mathcal{Y}_{2M}(\mathbf{r},\mathbf{r})\,|=\tfrac{1}{2}|\,\mathcal{Y}_{2M}(\beta\mathbf{r},\mathbf{r})\,|\\ |T_{ij}|&\leftrightarrow|\,\mathcal{Y}_{2M}(\boldsymbol{\sigma}\times\mathbf{r},\mathbf{r})\,|=|\,\mathcal{Y}_{2M}(\beta\boldsymbol{\sigma}\times\mathbf{r},\mathbf{r})\,|\\ |S_{ijk}|&\leftrightarrow|\,\mathcal{Y}_{3M}(\boldsymbol{\sigma},\mathbf{r},\mathbf{r})\,|=|\,\mathcal{Y}_{3M}(\beta\boldsymbol{\sigma},\mathbf{r},\mathbf{r})\,|. \end{aligned}$$

For convenience, we define

$$|M_\Omega'|_{KK'}{}^2=4|M_\Omega|^2/(A_K{}^2A_{K'}{}^2).$$

Normalization consists in setting $|M_1|^2=1$. Then

$$|M_1'|_K{}^2=4/A_K{}^4,$$

and we have the working relation

$$|M_\Omega|_{KK'}{}^2=|M_\Omega'|_{KK'}{}^2/(|M_1'|_K|M_1'|_{K'}).$$

The quantum number n is not explicit here, but is present via the boundary values.

Ω	ψ	ψ'	ΔJ	$\lvert M_\Omega'\rvert^2$
			(a) $K'-K=0$ ($\Delta J=0$, $\Delta l=0$)	
$1,\ \beta$	a	a		$(H^2N+h^2n)^2$
	b	b		$(H^2M+h^2m)^2$
$\boldsymbol{\sigma},\ \beta\boldsymbol{\sigma}$	a	a		$[(2\kappa'+1)/(2\kappa'-1)](H^2N+h^2n)^2$
	b	b		$[(2\kappa'-1)/(2\kappa'+1)](H^2M+h^2m)^2$
			(b) $K'+K=-1$ ($\Delta J=\pm 1$, $\Delta l=0$)	
$\boldsymbol{\sigma},\ \beta\boldsymbol{\sigma}$	a	b		$[4\kappa'/(2\kappa'+1)](H^2M-h^2m)^2$
	b	a		$[4\kappa'/(2\kappa'-1)](H^2N-h^2n)^2$

			(c) $K'-K=\pm1$ ($\Delta J=\pm1$, $\Delta l=\pm1$)	
α, $\beta\alpha$	a	a	1	$[4\kappa'/(2\kappa'+1)](HN+hn)^2$
	a	a	-1	$[4\kappa'/(2\kappa'-1)](HM+hm)^2$
	b	b	1	$[4\kappa'/(2\kappa'-1)](HN-hn)^2$
	b	b	-1	$[4\kappa'/(2\kappa'+1)](HM-hm)^2$
$\mathbf{r}$, $\beta\mathbf{r}$, $\sigma\times\mathbf{r}$, $\beta\sigma\times\mathbf{r}$	a	a	1	$[\kappa'/(2\kappa'-1)](H^2Q+h^2q)^2$
	a	a	-1	$[\kappa'/(2\kappa'+1)](H^2P+h^2p)^2$
	b	b	1	$[\kappa'/(2\kappa'-1)](H^2P+h^2p)^2$
	b	b	-1	$[\kappa'/(2\kappa'+1)](H^2O+h^2o)^2$
B_{ij}	a	a	1	$\{2\kappa'(2\kappa'+1)/[(2\kappa'-1)(2\kappa'-3)]\}(H^2Q+h^2q)^2$
	a	a	-1	$\{2\kappa'(2\kappa'+3)/[(2\kappa'-1)(2\kappa'+1)]\}(H^2P+h^2p)^2$
	b	b	1	$\{2\kappa'(2\kappa'-3)/[(2\kappa'-1)(2\kappa'+1)]\}(H^2P+h^2p)^2$
	b	b	-1	$\{2\kappa'(2\kappa'-1)/[(2\kappa'+1)(2\kappa'+3)]\}(H^2O+h^2o)^2$
			(d) $K'+K=0$ ($\Delta J=0$, $\Delta l=\pm1$)	
γ_5				$[H(M+N)-h(m-n)]^2$
$\beta\gamma_5$				$[H(M-N)-h(m+n)]^2$
α				$\{[(2\kappa'-1)/(2\kappa'+1)]^{\frac{1}{2}}(HM-hm)-[(2\kappa'+1)/(2\kappa'-1)]^{\frac{1}{2}}(HN+hn)\}^2$
$\beta\alpha$				$\{[(2\kappa'-1)/(2\kappa'+1)]^{\frac{1}{2}}(HM-hm)+[(2\kappa'+1)/(2\kappa'-1)]^{\frac{1}{2}}(HN+hn)\}^2$
$\mathbf{r}$, $\beta\mathbf{r}$				$\{1/[(2\kappa'-1)(2\kappa'+1)]\}(H^2P-h^2p)^2$
$\sigma\cdot\mathbf{r}$, $\beta\sigma\cdot\mathbf{r}$				$(H^2P-h^2p)^2$
$\sigma\times\mathbf{r}$, $\beta\sigma\times\mathbf{r}$				$\{4\kappa'^2/[(2\kappa'-1)(2\kappa'+1)]\}(H^2P-h^2p)^2$
B_{ij}				$\{8(\kappa'-1)(\kappa'+1)/[3(2\kappa'-1)(2\kappa'+1)]\}(H^2P-h^2p)^2$
			(e) $K'+K=-2$ ($\Delta J=\pm2$, $\Delta l=\pm1$)	
B_{ij}	a	b		$\{16\kappa'(\kappa'+1)/[(2\kappa'+1)(2\kappa'+3)]\}(H^2O-h^2o)^2$
	b	a		$\{16\kappa'(\kappa'-1)/[(2\kappa'-1)(2\kappa'-3)]\}(H^2Q-h^2q)^2$
			(f) $K'+K=1$ ($\Delta J=\pm1$, $\Delta l=\pm2$)	
σ, $\beta\sigma$	a	b		$[4\kappa'/(2\kappa'-1)](N+n)^2$
	b	a		$[4\kappa'/(2\kappa'+1)](M+m)^2$
			(g) $K'-K=\pm2$ ($\Delta J=\pm2$, $\Delta l=\pm2$)	
A_{ij}	a	a	2	$\{16\kappa'(\kappa'-1)/[(2\kappa'-1)(2\kappa'-3)]\}(HQ+hq)^2$
	a	a	-2	$\{16\kappa'(\kappa'+1)/[(2\kappa'+1)(2\kappa'+3)]\}(HO+ho)^2$
	b	b	2	$\{16\kappa'(\kappa'-1)/[(2\kappa'-1)(2\kappa'-3)]\}(HQ-hq)^2$
	b	b	-2	$\{16\kappa'(\kappa'+1)/[(2\kappa'+1)(2\kappa'+3)]\}(HO-ho)^2$
R_{ij}	a	a	2	$\{\kappa'(\kappa'-1)/[(2\kappa'-1)(2\kappa'-3)]\}(H^2V+h^2v)^2$
	a	a	-2	$\{\kappa'(\kappa'+1)/[(2\kappa'+1)(2\kappa'+3)]\}(H^2T+h^2t)^2$
	b	b	2	$\{\kappa'(\kappa'-1)/[(2\kappa'-1)(2\kappa'-3)]\}(H^2U+h^2u)^2$
	b	b	-2	$\{\kappa'(\kappa'+1)/[(2\kappa'+1)(2\kappa'+3)]\}(H^2S+h^2s)^2$
T_{ij}	a	a	2	$\{4\kappa'(\kappa'-1)/[(2\kappa'-1)(2\kappa'-3)]\}(H^2V+h^2v)^2$
	a	a	-2	$\{4\kappa'(\kappa'+1)/[(2\kappa'+1)(2\kappa'+3)]\}(H^2T+h^2t)^2$
	b	b	2	$\{4\kappa'(\kappa'-1)/[(2\kappa'-1)(2\kappa'-3)]\}(H^2U+h^2u)^2$
	b	b	-2	$\{4\kappa'(\kappa'+1)/[(2\kappa'+1)(2\kappa'+3)]\}(H^2S+h^2s)^2$
S_{ijk}	a	a	2	$\{12\kappa'(\kappa'-1)(2\kappa'+1)/[(2\kappa'-1)(2\kappa'-3)(2\kappa'-5)]\}(H^2V+h^2v)^2$
	a	a	-2	$\{12\kappa'(\kappa'+1)(2\kappa'+5)/[(2\kappa'-1)(2\kappa'+1)(2\kappa'+3)]\}(H^2T+h^2t)^2$
	b	b	2	$\{12\kappa'(\kappa'-1)(2\kappa'-5)/[(2\kappa'+1)(2\kappa'-1)(2\kappa'-3)]\}(H^2U+h^2u)^2$
	b	b	-2	$\{12\kappa'(\kappa'+1)(2\kappa'-1)/[(2\kappa'+1)(2\kappa'+3)(2\kappa'+5)]\}(H^2S+h^2s)^2$
			(h) $K'+K=2$ ($\Delta J=\pm2$, $\Delta l=\pm3$)	
B_{ij}	a	b		$\{16\kappa'(\kappa'-1)/[(2\kappa'-1)(2\kappa'-3)]\}(Q+q)^2$
	b	a		$\{16\kappa'(\kappa'+1)/[(2\kappa'+1)(2\kappa'+3)]\}(O+o)^2$

(i) $K'+K=-3$ ($\Delta J=\pm 3$, $\Delta l=\pm 2$)

S_{ijk}	a	b	$\{144\kappa'(\kappa'+1)(\kappa'+2)/[(2\kappa'+1)(2\kappa'+3)(2\kappa'+5)]\}(H^2S-h^2s)^2$
	b	a	$\{144\kappa'(\kappa'-1)(\kappa'-2)/[(2\kappa'-1)(2\kappa'-3)(2\kappa'-5)]\}(H^2V-h^2v)^2$.

Where the cross terms occur, their magnitude is given simply by

$$|M_\Omega M_{\Omega'}{}^*|^2=|M_\Omega|^2|M_{\Omega'}|^2$$

and the sign of the cross term, i.e., the sign of each of the real expressions $(-iM_\tau M_\alpha{}^*)$, etc. . . ., is as follows:

Group	ψ	ψ'	ΔJ	Sign
$K'-K=\pm 1$			1	+
			-1	−
$K'+K=0$	a	b		+
	b	a		−
$K'-K=\pm 2$	a	a		+
	b	b		−

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Hyperfine Structure Anomalies in the $^2P_{\frac{1}{2}}$ State of Tl^{205} and Tl^{203}†

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The atomic beam magnetic resonance method has been used to make a precision measurement of the hyperfine structure separation of the ground state of Tl^{205} and Tl^{203}. The experimentally determined ratio $\Delta\nu^{205}/\Delta\nu^{203}=1.00974\pm 0.00003$ is to be compared with the ratio of the magnetic moments, determined by nuclear induction techniques, of $g_I{}^{205}/g_I{}^{203}=1.00986\pm 0.00005$. It is shown that the difference between the two ratios can be accounted for by consideration of effects predicted by Bohr and Weisskopf and by Crawford and Schawlow. In particular, the agreement between theory and experiment can be construed as evidence of the reality of the effects postulated by Crawford and Schawlow, which have not been heretofore directly observed.

INTRODUCTION

A PREVIOUS determination[1] of the ratio of the hyperfine structure separations ($\Delta\nu$) of the $^2P_{\frac{1}{2}}$ ground states of Tl^{205} and Tl^{203} differed by about 40 parts in 10^5 from the ratio of the magnetic moments of the two isotopes. This discrepancy was approximately four times as large as could be accounted for on the basis of any extant theories of hfs anomalies. Unfortunately, the combined experimental uncertainties of the two ratios was 25 parts in 10^5, with the principal contribution to the uncertainties arising from the ratio of the $\Delta\nu$'s. Since an ability to account for a hfs anomaly in Tl would be a stringent test, of a type not heretofore made, of the validity of certain aspects of current theories on the subject, a new and more precise determination of the ratio of the hfs separations has been made by use of atomic beam techniques.

THEORY

A direct determination of the $\Delta\nu$'s of Tl^{203} and Tl^{205}, from observation of the transitions $\Delta F=\pm 1$, would require frequencies that are inconveniently high ($\sim$21,300 Mc). The value of the $\Delta\nu$'s may, however, be determined from measurements of the $\Delta F=0$, $\Delta m_F=\pm 1$ transitions (1, $1\leftrightarrow 1$, 0) and (1, $0\leftrightarrow 1$, -1). The expressions for the frequencies of these lines at an arbitrary magnetic field are[2]

$$f_1=\tfrac{1}{2}\Delta\nu[(1+x)-(1+x^2)^{\frac{1}{2}}+2x(g_J/g_I'-1)^{-1}] \quad \text{for } (1,\ 1\leftrightarrow 1,\ 0), \quad (1)$$

$$f_2=\tfrac{1}{2}\Delta\nu[(1+x^2)^{\frac{1}{2}}-(1-x)+2x(g_J/g_I'-1)^{-1}] \quad \text{for } (1,\ 0\leftrightarrow 1,\ -1), \quad (2)$$

where $x=(g_J-g_I')\mu_0 H/h\Delta\nu$, and g_I' differs slightly from the nuclear g value as determined by nuclear resonance methods because of a partial decoupling of the **L** and **S** vectors in the $^2P_{\frac{1}{2}}$ state.

If the transitions are observed at a fixed magnetic field, then a measurement of the frequencies of the two lines will be sufficient to determine both x and $\Delta\nu$, provided that the value of the ratio g_J/g_I' is known.

The value of $\Delta\nu$ deduced from the Breit-Rabi formula will not be the true hfs separation. Nonvanishing matrix elements of the electron-nucleus interaction operator, which are not diagonal in J, lead to a per-

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[1] Berman, Kusch, and Mann, Phys. Rev. **77**, 140 (1950).
[2] Millman, Rabi, and Zacharias, Phys. Rev. **55**, 384 (1938).