amplitude:

$$\theta_0=(2m/e)(4\pi R^3B/\mu_0)(1/I\delta\pi)=3.87\times10^{-2}B \text{ radian},$$

where B is expressed in webers/m^2.

This angular momentum is that of the superconducting electrons, but since the total angular momentum around the vertical axis does not change, the angular momentum of the positive ions and remaining electrons composing the sphere must change in just the opposite sense and by the same amount. It is this latter change that is observed.

Another way of looking at this phenomenon was suggested by Meissner.[3] The magnetic field penetrates only a short distance below the surface of the sphere, but as it changes, an electric field is present, which acts both on the superconducting electrons and on the remaining positive ions. Since the superconducting electrons do not drag the ions with them, the two systems move independently with equal and opposite angular momenta. The motion of the positive ions is the one observed by the experimental arrangement used.

IV. DISCUSSION OF ERRORS

In spite of much care the tin sphere had a slightly ellipsoidal character, the field was not exactly uniform over the volume of the sphere, and the switching of the magnetic field did not always occur precisely at the same point. In zero field a certain rest point or zero point about which the oscillator swung was noted. With a steady upward vertical field of 10^{-2} weber/m^2, the zero point shifted 0.3 mm to the right on the scale corresponding to 1.2×10^{-5} radian of angle. With a steady downward vertical field of the same amount, the zero point shifted 0.9 mm to the right. These errors might be further reduced with greater effort should greater accuracy justify such action, but differences between the experimental and expected values may be attributed to these experimental errors.

V. CONCLUSION

The experiments gave rough agreement with the theory as to the magnitude of the effect. The result was 15 percent low for the 10^{-2} weber/m^2 driving field and 4 percent low for the 0.51×10^{-2} weber/m^2 driving field. The direction of driving torque was such as to produce an angular impulse on the superconducting sphere just as if the Faraday induction were operating on the positive ion lattice.

The authors are greatly indebted to the constant assistance of Professor C. F. Squire. One of us (R.H.P.) held the Shell Fellowship during this work.

PHYSICAL REVIEW VOLUME 86, NUMBER 6 JUNE 15, 1952

Angular Momentum Distributions in the Thomas-Fermi Model

J. H. D. JENSEN* AND J. M. LUTTINGER
Department of Physics, University of Wisconsin, Madison, Wisconsin

(Received January 31, 1952)

The Hartree and Thomas-Fermi models of the atom and nucleus are compared in their predictions for the angular momentum distributions. It is found that the most natural quantity to compare is not the "first appearance" of a given orbital angular momentum, but the mean squared angular momentum.

I. INTRODUCTION

RECENTLY the subject of the distribution of angular momentum in nuclei has been given considerable impetus by the discovery of certain "magic numbers"[1] which indicate a shell structure in nuclei. The interpretation of the magic numbers in terms of a "shell model"[2] enables us to make detailed statements about the angular momentum distribution in nuclei. Several authors[3] have attempted to find an interpretation of this shell structure in terms of a Thomas-Fermi model of the nucleus. Now, apart from objections to their detailed handling of the Thomas-Fermi model, it seems to us that these authors have also interpreted the appearance of closed shells in an incorrect way. They have identified the closing of a shell with the first appearance of a particle with a higher orbital angular momentum than was previously present. It is very easy to see that even in the atomic case these two phenomena are not related. In atoms the closed shells (noble gases) occur at $Z=2$, 10, 18, 36, 54, and 86, while the first appearance of electrons with orbital angular momentum 1, 2, 3 occurs at $Z=5$, 21, and 58, respectively. In the nuclear shells, say for neutrons, the shells close at $N=2$, 8, 20, 28, 50, 82, and 126 while the first appearance of $l=1$, 2, 3, 4, and 5 occur at $N=3$, 9, 21, 41, ~64, ~100. The closing of a shell is a phenomenon which is connected with the filling up of a group of energy levels which are relatively isolated

* On leave of absence from the University of Heidelberg, Heidelberg, Germany.

[1] M. Goeppert-Mayer, Phys. Rev. **74**, 235 (1948).

[2] M. Goeppert-Mayer, Phys. Rev. **75**, 1969 (1949); **78**, 16 (1950); Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949); Z. Physik **121**, 259 (1950).

[3] L. M. Yang, Proc. Phys. Soc. (London) **A64**, 632 (1951); D. Ivanenko and W. Rodichew, Dokl. Akad. Nauk, SSSR **70**, 605 (1951).

from the rest, and in general has little directly to do with the emergence of new orbital angular momenta.

While we believe that the Thomas-Fermi model can give no information about the location of the closed shells, still it does give some information about the distribution of orbital angular momentum. It was thought that it might be of interest to compare its predictions with those of the shell model. The first question which arises is exactly what one should compare. Usually—in the atomic case—one has compared the "first appearance" of a certain l. This question is, however, a very poorly defined one in the Thomas-Fermi model as shown in Sec. III, since the orbital angular momentum has a continuous range of values instead of the discrete values which one obtains in a rigorous quantum mechanical theory. We prefer therefore to make the comparison between something which is well defined in the Thomas-Fermi model, and the predictions of the shell model. For this purpose we have chosen to compare the mean value of the square of the orbital angular momentum. Since the Thomas-Fermi model gives a unique expression for the number of particles with orbital momentum between L and $L+dL$ —which we will denote by $n(L)$—the mean value in question is given by

$$\langle L^2\rangle_{\text{Av}}=\int L^2 n(L)dL/Z. \tag{1}$$

Here Z is the number of particles of the type we are considering.

As soon as we have decided on a level scheme in the shell model, we then know how many particles there are with each l, and we have for comparison:[4]

$$\langle L^2\rangle_{\text{Av}}=\sum_i l_i(l_i+1)/Z,$$

where the sum goes over all the Z particles.

In Sec. II we shall make this comparison for the atomic as well as the nuclear case, and shall show that for a reasonable density distribution in the latter it is possible to obtain excellent agreement. In Sec. III we shall return to the question of the "first appearance" of a particle of given orbital angular momentum.

II. THE MEAN SQUARED ORBITAL ANGULAR MOMENTUM

Let us define $dN(\mathbf{p},\mathbf{r})$ as the number of particles with momentum $\mathbf{p}$ at the point $\mathbf{r}$ in a small element of volume of phase space. Then the basic assumption of the Thomas-Fermi model is that all states are filled equally as long as a certain maxima momentum is not exceeded, after which they are all empty. Thus, with $\hbar=1$,

$$dN(\mathbf{p},\mathbf{r})=2d^3\mathbf{r}d^3\mathbf{p}/(2\pi)^3 \quad \text{if} \quad p\leq P(r),$$
$$dN=0 \quad \text{if} \quad p>P(r).$$

The factor 2 comes from the fact that the number of states is doubled by the presence of the spin. $P(r)$ is the maximum momentum allowable at a given point. By integrating this expression over $\mathbf{p}$ one obtains

$$\rho(r)=2\cdot(4\pi/3)P^3(r)/(2\pi)^3. \tag{2}$$

This is the fundamental relationship between the density of particles $\rho(r)$ and the maximum momentum. To find the angular momentum distribution function it is convenient to consider not $n(L)$ itself but its integral,

$$N(L)\equiv\int_L^\infty n(L)dL.$$

$N(L)$ is the number of particles present with angular momentum greater than L. Now $\mathbf{L}=\mathbf{r}\times\mathbf{p}$, and therefore $L^2=r^2p^2\sin^2\vartheta$, where ϑ is the angle between $\mathbf{r}$ and $\mathbf{p}$. Hence,

$$N(L)=\int dN(\mathbf{p},\mathbf{r}),$$

where the limits of the r and p integrations are given by the conditions $pr\sin\vartheta\geqq L$ and $p\leq P(r)$. By integration over the angles and p we obtain immediately

$$N(L)=(4/3\pi)\int[P^2r^2-L^2]^{\frac{3}{2}}dr/r. \tag{3}$$

The integration over r is to be extended over that region where the radical is real. From $N(L)$ we obtain $n(L)$ by differentiation,

$$n(L)=-dN(L)/dL=(4L/\pi)\int[P^2r^2-L^2]^{\frac{1}{2}}dr/r, \tag{4}$$

which is a well-known result.[5] Using this, we obtain from Eq. (1):

$$\langle L^2\rangle_{\text{Av}}=\frac{8}{15\pi Z}\int_0^\infty\frac{dr}{r}(rP)^5. \tag{5}$$

By means of (2) we know $\langle L^2\rangle_{\text{Av}}$ as soon as we know the density ρ.

In the case of the Thomas-Fermi atom this density is well-known and can be expressed in terms of a universal function φ for all atoms. If we put $r=\mu x$, where $\mu=3^{\frac{2}{3}}2^{-7/3}\pi^{\frac{2}{3}}Z^{-\frac{1}{3}}/me^2$, and make use of the definition $\varphi(x)=(r/Ze^2)(P^2(r)/2m)$, we obtain

$$\langle L^2\rangle_{\text{Av}}=Z^{\frac{7}{3}}\frac{2}{5}\left(\frac{3\pi}{4}\right)^{\frac{2}{3}}\left(\int_0^\infty\frac{dx}{x}(x\varphi)^{5/3}\right).$$

The integral in the brackets is a pure number and has been obtained by a numerical integration; its value is 0.370. This gives us finally

$$\langle L^2\rangle_{\text{Av}}=0.262Z^{\frac{7}{3}}. \tag{6}$$

Figure 1 gives a plot of this function along with a plot of the empirical value of $\langle L^2\rangle_{\text{Av}}$ as computed from the known ground state configurations of the elements.

[4] We have chosen units such that $\hbar=1$.

[5] E. Fermi, Z. Physik **49**, 550 (1928).

One sees that there is a general tendency to give too high angular momenta for the lighter elements, but as Z increases the Thomas-Fermi model gives a very good average variation of $\langle L^2\rangle_{Av}$ with Z. This is just in line with the usual idea that the Thomas-Fermi model improves with increasing atomic number. (Needless to say, we could not possibly hope for more than such a correctness in the average, since in the Thomas-Fermi model all properties of atoms vary smoothly with their atomic number, and therefore, all effects due to closing of shells are averaged out.)

In the nuclear case we are faced with the difficulty of not knowing ρ. The equations determining the density in the Thomas-Fermi model are not even known exactly, since the nuclear forces are unknown, and even if they were it would seem to be a prohibitively difficult task to solve them. None the less one can ask whether or not it is possible by a reasonable assumption concerning the density of protons and of neutrons in the nucleus to obtain a fit of the observed angular momenta distribution. In treating the nucleus we shall treat the Z protons and N neutrons as separate systems, each obeying its own Pauli principle, and each with its own density function. Such a treatment is certainly implicit in the usual conception of the Thomas-Fermi model. We shall therefore deal with the proton system, though of course the neutron system may be treated in an identical fashion. Let us write

$$r=Rx, \quad P(r)=P_0 f(x), \tag{7}$$

where R is a quantity of the dimension of a length and of the order of magnitude of the nuclear radius, while P_0 is a constant with the dimensions of a momentum. The function $f(x)$ is defined such that $f(0)=1$, so that P_0 is the maximum proton momentum at the center of the nucleus. Since the relationship of density to maximum momentum is given by (2), we can eliminate R and P_0 from the expression for $\langle L^2\rangle_{Av}$, by making use of $\int\rho d^3\mathbf{r}=Z$, and obtain

$$\langle L^2\rangle_{Av}=\pi^{\frac{2}{3}}2^{-\frac{1}{3}}3^{7/3}5^{-2}Z^{\frac{2}{3}}\frac{5\int\frac{dx}{x}\{xf(x)\}^5}{\left[3\int\frac{dx}{x}\{xf(x)\}^3\right]^{5/3}}. \tag{8}$$

The numerical factor in front of $Z^{\frac{2}{3}}$ is 0.885. Exactly the same equation holds for the neutron $\langle L^2\rangle_{Av}$ if we replace Z by N and f by another function suitable for describing the neutron density in a nucleus. Since the empirical data on neutron shells and proton shells indicates that the angular momentum distribution is roughly the same for both, we shall restrict ourselves to the same function f for both neutrons and protons.

The simplest choice of $f(x)$ would be

$$f=1 \quad \text{for} \quad x<1, \quad \text{and} \quad f=0 \quad \text{for} \quad x>1.$$

This assumes that the nucleus is a sphere of radius R and constant density. The ratio of the integrals in the expression for $\langle L^2\rangle_{Av}$ is then unity, and we obtain

$$\langle L^2\rangle_{Av}=0.885Z^{\frac{2}{3}}.$$

A plot of this function along with the plot of $\langle L^2\rangle_{Av}$ obtained from the shell model is given by the dashed curve in Fig. 2. One sees that this density distribution gives on the whole too large angular momenta. This result is not very surprising since the assumption of constant density right up to the edge of the nucleus over-emphasizes very much the contribution to $\langle L^2\rangle_{Av}$ from larger values of x. These however—as can be seen from the expression for $\langle L^2\rangle_{Av}$ which involves an integral over the fourth power of x—make the largest contribution to $\langle L^2\rangle_{Av}$. Therefore, our result is very sensitive to the density near the edge of the nucleus. Certainly a more realistic density distribution is given if we assume that the density is constant up to a certain point and then drops off to zero in a Gaussian fashion. Other distributions are also possible, but we choose this for ease of calculation. By a very reasonable choice of the thickness of this surface layer we shall see that it is possible to obtain as good an agreement as in the

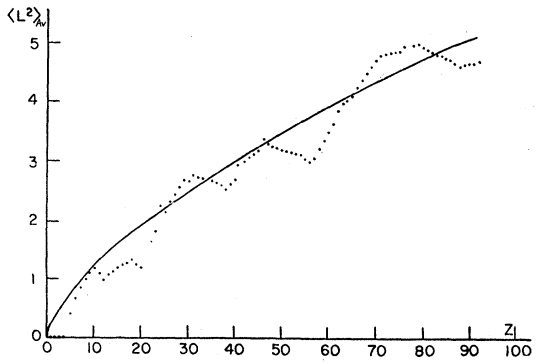

FIG. 1. Comparison of the mean squared angular momentum of an atom as calculated in the Thomas-Fermi model with the empirical values. ——— Thomas-Fermi model; ···· empirical values.

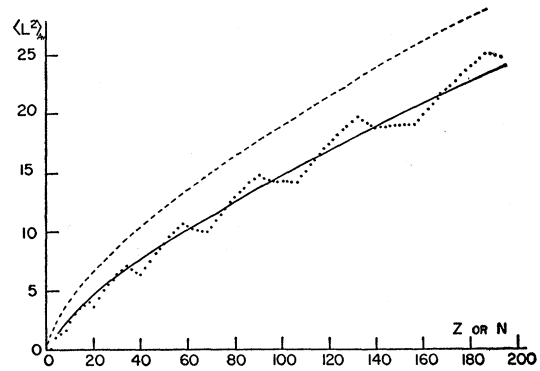

FIG. 2. Comparison of the mean squared angular momentum of the protons or neutrons in a nucleus as calculated in the Thomas-Fermi model with the values obtained from the "shell model." – – – – Constant nuclear density; ——— Gaussian drop in nuclear density at the surface; ···· values from shell model.

TABLE I. Values of Z for the first appearance of various values of l.

	$l=1$	$l=2$	$l=3$
Fermi (approximate integration)	5	21	55
Fermi (exact integration)	4.0	16.2	44.0
Definition in text	2.4	14.8	43.0
Empirical	5	21	58

atomic case. Let us choose then

$$f(x)=1 \quad \text{for} \quad x<1$$
$$=\exp(-(x-1)^2/3\beta) \quad \text{for} \quad x>1.$$

Choosing for β the value

$$\beta=\left(\frac{1.8\times10^{-13}\text{ cm}}{R}\right)^2\Big/\log 2,$$

where R is determined by the condition that inside the nucleus the proton density has the value $(Z/A)(1.4\times10^{-13}\text{ cm})^{-3}$, we obtain the solid curve given in Fig. 2. This choice means that we have chosen a surface layer of "thickness" 1.8×10^{-13} cm independent of R. We notice that the agreement with the shell model predictions is excellent, and therefore it is not necessary—as proposed by Yang[3]—to assume a surface layer with thickness proportional to $A^{\frac{1}{3}}$. Since it is not easy to decide exactly which curve is to be considered the "best fit" for the empirical facts, the β-value given is necessarily more an order of magnitude than a precise conclusion from the data.

With this choice of the constant β, we can obtain a value of the nuclear radius. Let us define the nuclear radius R_0 as the distance from the origin at which the density falls to half its value. Thus we obtain

$$R_0=R\{1+[(\log 2)\beta]^{\frac{1}{2}}\}=R+1.8\times10^{-13}\text{ cm}.$$

The resulting R_0 does not exactly satisfy the requirement that it is proportional to $A^{\frac{1}{3}}$, though it does to a very good approximation. We obtain for the ratios of the values of $R_0A^{-\frac{1}{3}}$ the values 0.95, 0.97, 0.98, 1.0 for $Z=12$, 30, 90, ∞, respectively. In the limiting case of large A we have, of course, $R_0=1.4\times10^{-13}A^{\frac{1}{3}}$. Thus although we cannot satisfy simultaneously both the requirements of constant internal density and nuclear radius proportional to $A^{\frac{1}{3}}$, we can do it well within the experimental accuracy of these facts. We could, of course, have just as well determined R by the requirement of $R_0A^{-\frac{1}{3}}=$ constant, in which case we would have found that the density at the origin decreases slightly for lighter atoms; a result which in itself is perhaps not too implausible.

III. THE FIRST APPEARANCE OF A GIVEN ORBITAL ANGULAR MOMENTUM

Since in the Thomas-Fermi model the angular momentum is not quantized, but has a continuous distribution of possible values, the question, at which Z a particle with a certain $L=[l(l+1)]^{\frac{1}{2}}\cdot(l=1, 2, 3, \text{etc.})$ appears for the first time, is less unique than would appear from the literature.

We would like to discuss here a definition which we believe to be the best possible within the framework of the Thomas-Fermi model: All particles with a given angular momentum L will be said to have the quantum value l which makes $[l(l+1)]^{\frac{1}{2}}$ closest to L. In this case the "number of particles with an angular momentum l," say $\nu(l)$, will be given by

$$\nu(l)=N\left\{\frac{[l(l+1)]^{\frac{1}{2}}+[l(l-1)]^{\frac{1}{2}}}{2}\right\}$$
$$-N\left\{\frac{[l(l+1)]^{\frac{1}{2}}+[(l+1)(l+2)]^{\frac{1}{2}}}{2}\right\}.$$

This differs from the definition of Fermi in two ways: he replaces $[l(l+1)]^{\frac{1}{2}}$ by $l+\frac{1}{2}$, and replaces the integral involved in $N(L)$ by its approximate value obtained by multiplying the value of the integrand at the center of the interval by the length of the interval. The latter approximation turns out to be quite poor because the integrand is really a very rapidly varying function of L, and therefore, we have undertaken to repeat the work carrying out the integral in question numerically. As soon as one has $\nu(l)$ one may define the "first appearance" of l by the condition that for that Z, we must have $\nu(l)=1$.

The results of these calculations for the Fermi atom are given in Table I. By comparison of the first and second entries one can see the effect of replacing the exact integration by the approximate one. The exact integration leads to much poorer agreement with the empirical data. Further, the effect of replacing $[l(l+1)]^{\frac{1}{2}}$ by $l+\frac{1}{2}$ can be seen on comparing the second and third entries. With the most straightforward definition the agreement with the empirical data is poorest. If we consider the percent error we find for the Fermi case (exact integration) 20 percent, 23 percent, 24 percent, and for our case 52 percent, 32 percent, 26 percent for $l=1$, 2, 3, respectively. Thus the error for Fermi's definition increases with Z, while that for the definition above decreases. Since all properties of the Fermi model improve with increasing Z, it seems to us that definition given above would be the preferable one. However, in any case, we feel that the quantity $\nu(l)$ is so poorly defined in a model with continuous L, that it is not worth considering in the nuclear case.[3] For a comparison with experiment the natural quantities are averages such as $\langle L^2\rangle_{\text{Av}}$.

In conclusion we would like to thank the University of Wisconsin's Computing Service for having carried out the various numerical integrations involved in this paper. One of us (J.H.D.J.) participated in this work while on a Carl Schurz visiting professorship at the University of Wisconsin. He would like to indicate his thanks to the Carl Schurz committee and to the University in general for having made his stay so pleasant and profitable.