Condensations in Expanding Cosmologic Models

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The consequence of a perturbation in the expanding universe is investigated. It is found that the density gradient decreases with time, while the ratio between the perturbed density and the background density diverges more and more from unity. These results are essentially in agreement with observation and explain why even in "point source" cosmologic models, condensations of extremely high densities are not to be expected at present.

INTRODUCTION

WHILE it is usual to regard the nebulae as condensations out of a homogeneous cosmic fluid, the presence of such condensations offers an interesting and rather intriguing problem. The earlier investigations of Tolman[1] and Sen[2] seemed to show that the Friedman model (i.e., pressureless homogeneous expanding universe) is fundamentally unstable to perturbations either in the distribution of material density or in the rate of expansion; and as such one could expect the formation of condensations as a result of infinitesimal perturbations. Gamow[3] has, however, more recently reported that an expanding universe is stable against small gravitational perturbations, and that in particular a rudimentary condensation is bound to expand and mix up with the rest of the universe. Gravitational influences, alone, cannot thus explain the formation of condensations, and Gamow and his collaborators considered the interaction between radiation and gas particles as a probable causative agent. Later calculations are, however, said to have belied this hope;[4] so that according to the Gamow school the occurrence of these condensations remain an unexplained phenomenon. Gamow has, however, partly used classical nonrelativistic concepts in his considerations, and this throws some doubt on the validity of his conclusions. Lifshitz,[5] too, in considering arbitrary small perturbations of the gravitational field and of the distribution of matter in the expanding universe, has found that these perturbations either decrease with time or increase so slowly that these cannot serve as centers of formation of nebulae.

In view of such conflicting conclusions, and the interest of the problem, it seems appropriate to investigate the problem anew and with a minimum of ad hoc assumptions. Our result seems to show that the difference in conclusion regarding the fate of a rudimentary condensation depends essentially on how one defines the degree of condensation.

Hoyle[6] has recently criticized the ever-expanding relativistic cosmologic models on the ground that "in the early stages of expansion, the material in the general background is unstable against formation of condensations. Thus condensations would be formed in the material with density much higher than the mean densities within the nebulae. No such condensations are observed. Detailed considerations show that this crucial objection cannot be overcome through the action of gas or radiation pressure." We shall see that our results will provide an answer to this criticism as well.

2. BASIC ASSUMPTIONS IN THE PRESENT DISCUSSION

We shall take the line element in the universe to be of the form

$$ds^2 = -e^{\mu}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) + e^{\nu} dt^2 \tag{1}$$

where $\mu = \mu(r, t)$ and $\nu = \nu(r, t)$. The line element of form (1) implies that we are assuming spherical symmetry as well as spatial isotropy. Spatial isotropy has not been assumed by Tolman[1] and Sen,[2] who have taken the line element in the form

$$ds^2 = -e^{\lambda} dr^2 - e^{\omega}(d\theta^2 + \sin^2\theta d\varphi^2) + e^{\nu} dt^2 \tag{2}$$

where λ, ω, ν are functions of r and t. It is easily seen that the necessary and sufficient condition that the line element (2) can be transformed to the form (1) without destroying the comoving nature of the coordinate system is that $\lambda - \omega$ must be a function of r alone.[7] In the perturbations considered by these authors, the time derivatives of λ and ω are not equal, so that although the unperturbed universe is isotropic, the isotropy is lost afterwards and the radial and cross radial dimensions increase at different rates. We do, however, think that the observable universe does not warrant such non-isotropic perturbations; and that no special loss of generality will result from the assumption of spatial isotropy.

We introduce also the usual assumption that the cosmic material and energy constitute a perfect fluid at

[1] R. C. Tolman, Proc. Nat. Acad. Sci. **20**, 169 (1934).
[2] N. R. Sen, Z. Astrophys. **9**, 215 (1935); **10**, 291 (1936).
[3] G. Gamow, Revs. Modern Phys. **21**, 367 (1949).
[4] R. A. Alpher and R. C. Herman, Revs. Modern Phys. **22**, 153 (1950).
[5] E. Lifshitz, J. Phys. U.S.S.R. **10**, No. 2, 116 (1946); Sci. Abstr. **A50**, No. 190 (1947).
[6] F. Hoyle, Nature **163**, 196 (1949).
[7] See, for example, R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, London, 1934), pages 364–367.

rest in the coordinate system of (1). This means that, besides isotropy, the possibility of a "*co-moving*"[8] coordinate system in which there is no net flux of energy or momentum is assumed. However such a coordinate system exists only under some restrictive conditions, e.g., we can use such a "*co-moving*" coordinate system only if all the different forms of energy existing at a particular point have the same proper macroscopic velocity. If, however, there be different velocities for different forms of energy at the same point, no such "*co-moving*" coordinate system is feasible. Thus, for example, when there is a flow of heat relative to the material contents, the coordinate system in which the matter is at rest will still show a flux of energy corresponding to the flow of heat.

It is clear, therefore, that in these perturbation problems introducing nonhomogeneity, this use of "*co-moving*" coordinate system is not quite justifiable. For instance, if the perturbation consists in a small adiabatic compression of the cosmic fluid in a certain region, there would then be a rise of temperature in this region and a consequent flow of heat away from this region. Similarly, any conversion between matter and radiation in a certain region would cause a change of pressure and a consequent flow of radiation relative to the cosmic mass.

However, in all these cases, if the perturbation be small, the flux of energy will be extremely small compared to the other (static) components of the energy tensor and can perhaps be neglected without introducing sensible error.

3. FATE OF NONHOMOGENEITIES IN EXPANDING UNIVERSE

With the foregoing assumptions, the energy momentum tensor components are $T_1{}^1=T_2{}^2=T_3{}^3=-p$, $T_4{}^4=\rho$, and $T_\nu{}^\mu=0$ when $\mu\neq\nu$; p and ρ denoting the pressure and energy density. The gravitational equations then assume the form[9]

$$8\pi p=e^{-\mu}[\tfrac{1}{4}\mu'^2+\tfrac{1}{2}\mu'\nu'+(\mu'+\nu')/r]-e^{-\nu}(\ddot{\mu}+\tfrac{3}{4}\dot{\mu}^2-\tfrac{1}{2}\dot{\mu}\dot{\nu})+\Lambda \quad (3)$$

$$8\pi p=e^{-\mu}[\tfrac{1}{2}(\mu''+\nu'')+\tfrac{1}{4}\nu'^2+(\mu'+\nu')/2r]-e^{-\nu}(\ddot{\mu}+\tfrac{3}{4}\dot{\mu}^2-\tfrac{1}{2}\dot{\mu}\dot{\nu})+\Lambda \quad (4)$$

$$8\pi\rho=-e^{-\mu}(\mu''+\tfrac{1}{4}\mu'^2+2\mu'/r)+\tfrac{3}{4}e^{-\nu}\dot{\mu}^2-\Lambda \quad (5)$$

$$0=2\dot{\mu}'-\dot{\mu}\nu'. \quad (6)$$

In the above expressions, dashes denote differentiation with respect to r and dots with respect to t. The energy momentum tensor satisfies the divergence identities $T_\mu{}^\nu{}_{,\nu}=0$ which gives two nontrivial relations

$$p'=-\tfrac{1}{2}(p+\rho)\nu' \quad (7)$$

$$\dot{\rho}=-\tfrac{3}{2}(p+\rho)\dot{\mu}. \quad (8)$$

These two equations are however not independent of Eqs. (3)–(6) and can be obtained directly from them.

Differentiating (8) with respect to r and substituting for p' from (7) and then eliminating ν' with the help of (6), we get,

$$\dot{\rho}'=-\tfrac{3}{2}\rho'\dot{\mu}. \quad (9)$$

Or,

$$(\partial/\partial t)(\log|\rho'|)=-\tfrac{3}{2}\dot{\mu}. \quad (10)$$

We assume, as seems reasonable, that a small perturbation does not reverse the expanding nature of the space at the point under consideration. This would obviously be true at least at those early stages when the rate of expansion was very large.[10] $\dot{\mu}$ is therefore positive, and Eq. (10) shows that the nonhomogeneity as measured by ρ' would decrease with time. Thus, if one takes the gradient of density (or the difference of density between two points) as a measure of the degree of condensation (or rarefaction), then the condensation (or rarefaction) is smoothed out in an expanding space.

One may, however, define the degree of condensation also by ρ'/ρ, i.e., the percentage rate of change of density.[11] We get using (8) and (9),

$$(\partial/\partial t)(\rho'/\rho)=3\rho'\dot{\mu}p/2\rho^2.$$

Or,

$$(\partial/\partial t)(\log|\rho'/\rho|)=3\dot{\mu}p/2\rho. \quad (11)$$

Equation (11) shows that if ρ'/ρ be taken as a measure of the degree of condensation (or rarefaction), then in expanding space ($\dot{\mu}$ positive) the condensation (or rarefaction) would go on increasing with time.

We can therefore picture the consequence of a perturbation in the following way. In view of (10), the difference of density between any two points will go on decreasing, while (11) shows that the ratio of density at any two points diverges more and more from unity. Thus, consequent to an original perturbation in the early career of the universe, we should have at the present epoch a much smaller difference and a much larger ratio between the densities of the condensations and the background. These conclusions are not inconsistent with observation. For suppose that, when the density was very large, a perturbation in the density actually took place. This perturbation, although very small compared to the then prevailing density, might still conceivably be much larger than the present differences of density between the nebulae and the background. (E.g., a perturbation in density, say, of the order 10^{-12} g/cc, when the background density was of the order 10^6 g/cc, constitutes a small perturbation but is much larger than the present average densities of the nebulae.) The ratio of the perturbed density to the

[8] See reference 7, pages 301–302.
[9] See reference 7, pp. 251–252.

[10] Gamow's semiclassical investigations also confirm this point. See reference 3.
[11] This, in effect, has been done by Tolman and Sen. See references 1 and 2.

background density, while originally very nearly unity, has now a far greater value, in agreement with the requirement of (11).

The limit to the excess of density in the condensations being thus set by the magnitude of the original perturbation (and not the original density), one sees that in spite of condensations occurring in the highly dense cosmic fluid early in the life of the universe, condensations of high densities, as envisaged by Hoyle,[6] cannot be met with at present.

In conclusion, we note that a particularly satisfactory feature of the present investigation is that we have not introduced any assumption regarding the pressure, the pressure gradient, or the mechanism and nature of the original perturbation. We have also not introduced any nongravitational interaction.

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Convergence of Intermolecular Force Series*

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The commonly used perturbation method of estimating the Van der Waals forces between atoms is shown to lead, when carried to the extreme, to divergent results. The method employs an expansion of the classical electrostatic interaction between the atoms in a series of inverse powers of the internuclear distance. The divergence arises because this expansion is utilized in regions of configuration space where it is not convergent. In this paper, the resulting divergent intermolecular force series are shown to be asymptotic to the true molecular interaction. The divergence is removed in an approximate way and the second-order attractive energy so obtained is added to the first-order exchange energy between the atoms. The method results in an electronic energy curve for H_2^+ in reasonable accord with the exact result of Hylleraas and in a new interaction between helium atoms in good agreement with recent low temperature experiment.

I. INTRODUCTION

THE Van der Waals force between two atoms, at an internuclear distance R, is usually estimated by the use of an expansion of the classical electrostatic interaction between the atoms in a series of inverse powers of R. This expansion is considered as a perturbation upon the combined system of the two atoms and the Schrödinger perturbation theory or the variational method is employed to evaluate the resultant shift in energy levels. The shift is identified with the potential energy of the interatomic force.[1]

The expansion of the electrostatic interaction between the atoms is convergent and meaningful only in a limited region of the configuration space of the combined system. It is, however, commonly employed throughout this configuration space. This paper reports an investigation of the uncertainties arising from this procedure and suggests a method of overcoming them.

Let H_A and H_B be the (unperturbed) Hamiltonians of atoms A and B. The corresponding state functions and energy levels will be designated by $\psi_A^{(s)}$, $E_A^{(s)}$ and $\psi_B^{(t)}$, $E_B^{(t)}$, where s and t are quantum numbers. We shall refer the Hamiltonian and wave functions of each atom to a coordinate system (rectangular or spherical) with origin at its nucleus. The coordinates of charges e_i belonging to atom A are (x_i, y_i, z_i) or (r_i, θ_i, ϕ_i). Those of charges ϵ_j belonging to atom B are (ξ_j, η_j, ζ_j) or $(\rho_j, \omega_j, \chi_j)$. The z and ζ-axes are directed along the internuclear line from A to B.

The intermolecular force results when the electrostatic interaction

$$V=\sum_{i,j} e_i\epsilon_j/r_{ij} \tag{1}$$

between the atoms is taken as a perturbation on the compound system having the Hamiltonian H_A+H_B, state functions $\psi_A^{(s)}\psi_B^{(t)}$ and energy levels $E_A^{(s)}+E_B^{(t)}$.

In carrying out a perturbation or variational calculation (1) is usually expanded in a series of inverse powers of the internuclear distance. Margenau[2,3] has given some of the lower terms in this series:‡

$$\begin{aligned}V'=\cdots-\frac{1}{R^3}\sum_{i,j} e_i\epsilon_j(2z_i\zeta_j-x_i\xi_j-y_i\eta_j)\\ +\frac{3}{2R^4}\sum_{i,j} e_i\epsilon_j[r_i^2\zeta_j-z_i\rho_j^2\\ +(2x_i\xi_j+2y_i\eta_j-3z_i\zeta_j)(z_i-\zeta_j)]\\ +\frac{3}{4R^5}\sum_{i,j} e_i\epsilon_j[r_i^2\rho_j^2-5z_i^2\rho_j^2-5r_i^2\zeta_j^2-15z_i^2\zeta_j^2\\ +2(4z_i\zeta_j-x_i\xi_j-y_i\eta_j)^2]+\cdots.\end{aligned} \tag{2}$$

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[1] F. London, Z. Physik 63, 245 (1930).

[2] H. Margenau, Phys. Rev. 38, 747 (1931).

[3] H. Margenau, Revs. Modern Phys. 11, 1 (1939).

‡ The symbol V' is used to denote an expanded form of V.