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Scattering of Plane Waves by Soft Obstacles. III. Scattering by Obstacles with Spherical and Circular Cylindrical Symmetry*

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A new variational method is devised for obtaining the "best" parameters for trial wave functions of a given type for insertion into the integral equation [see Eq. (1.4a)] for the scattering of scalar plane waves by obstacles with spherical and cylindrical symmetry.

The variational procedure is applied to the determination, in the least square sense, of the square well potential which most closely approximates the potential under consideration for a given propagation constant ($k_0=2\pi/\lambda_0$) of the incident wave. We then use as a trial function in the scattering integral equation the sphere (or cylinder) wave function (with the "best" parameters) which was derived in the first two papers of this series.

The differential and total scattering cross sections for scattering by Gaussian, exponential, and screened Coulomb potentials are obtained in simple closed forms. The Born approximation predicts the ratio of the total scattering cross section to πb^2 (where b is qualitatively the effective range of the scatterer) to be a monotone increasing function of $2\pi b/\lambda_0$, while in our theory it is a bounded function.

The main advantage of the procedure developed in this paper is that, while it is superior to the Born approximation, especially for large b/λ_0, it is equally simple to apply.

INTRODUCTION

THE tremendous volume of recent experimental data on the scattering of plane waves by obstacles (scattering of light by macromolecules, of high energy particles by nuclei, of sound waves by various types of obstacles, etc.) has led to a renewed interest in the development of approximation methods for the solution of scattering problems.

Variational principles have been derived and applied by Schwinger, Hulthén, and their collaborators and followers. In most applications to two- and three-dimensional problems the practice has been to express the scattered wave function as a series in Legendre polynomials and to find the "phases" and hence the coefficients by variational principles.

In an investigation of scattering by obstacles with sharp boundaries (spheres, circular cylinders, spheroids, etc.) Hart and one of the authors[1] have been led to approximation procedures which avoid the calculation of phases and lead to scattering cross sections directly. This paper is concerned with a generalization of these procedures to scattering by spherically and cylindrically symmetrical scatterers which have no sharp boundaries but which diminish in strength monotonically with the distance from the scattering center.

One may develop the theory of scattering by considering each infinitesimal volume element of the scatterer as a point Rayleigh scatterer and characterizing the total scattered wave at a point of observation as the sum of all the waves scattered by the point scatterers. To apply this procedure one must know the exciting field at each infinitesimal point scatterer. Mathematically the method is expressed through an integral equation (1.4a) in which the wave function at a given point is an integral over that at all other points.

The classical approximation technique is that of Rayleigh, Gans, and Born[2] (henceforth referred to as the R.G.B. or the Born approximation), in which one assumes that each infinitesimal Rayleigh scatterer is excited by the incident plane wave in the same manner that it would be in the absence of the remainder of the obstacle. In terms of the integral equation (1.4a) one substitutes the wave function of the incident wave into the integrand of the integral and carries out the required integration to compute the scattered wave function. The limitations of the R.G.B. approximation are very striking if one plots the ratio of the total scattering cross section to the geometrical cross section as a function of $2\pi a/\lambda_0$, where a is a characteristic length of the scatterer and λ_0 is the wavelength of the incident plane wave in the absence of the scatterer. This quantity is a monotone increasing function in the R.G.B. approximation, while in the exact theory[1,3] it is usually a bounded function of $2\pi a/\lambda_0$ (see Fig. 1).

In all iterative procedures for solving integral equations the quick achievement of accurate results is facilitated by a good first approximation to the solution. The problem of improving on the R.G.B. approximation then resolves itself into finding reasonably accurate and convenient forms to represent the trial wave functions.

In S-I and S-II this has been done for the scattering by spherical and elongated obstacles having sharp

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[1] R. Hart and E. Montroll, J. Appl. Phys. **22**, 376 (1951) and **22**, 1278 (1951); R. Hart, J. Acoust. Soc. Am. **23**, 373 (1951). See also A. L. Latter, Phys. Rev. **83**, 1056 (1951), where some of the same ideas have been discussed independently. The first two papers listed above will be referred to as S-I and S-II, respectively.

[2] Lord Rayleigh, Proc. Roy. Soc. (London) **A84**, 25 (1910); **A90**, 219 (1914); R. Gans, Ann. Physik **76**, 29 (1925); M. Born, Z. Physik **37**, 863 (1926); **38**, 803 (1926).

[3] H. C. van de Hulst, Recherches Astronomiques de l'observatoire d'Utrecht **11** (1946).

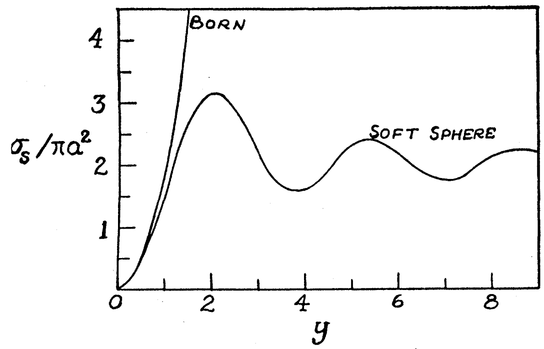

FIG. 1. Ratios of the total scattering cross section of a spherical potential well $[k^2(r)-k_0^2=k_1^2-k_0^2,\ r<a;\ =0,\ r>a]$ with geometrical cross section in the limit as $k_1/k_0\rightarrow1$. The upper curve is the Born approximation [Eq. (3.21)]. The lower curve is the soft sphere approximation [Eq. (3.20)]. The parameter y is defined by $y=k_0a(k_1/k_0-1)$.

boundaries (square well potential). If $k_0=2\pi/\lambda_0$ and $k_1=2\pi/\lambda_1$ are the propagation constants for the exterior and interior of the scatterer, it was found that the essential features of the correct theory are retained by choosing the interior trial wave function to be of the form $\exp(ik_1z)$ rather than $\exp(ik_0z)$ as in the R.G.B. approximation.

In this paper we shall generalize the previous results by approximating the wave functions of scatterers which do not have sharp boundaries by those of spheres and cylinders with width and depth of potential well chosen so as to give the best comparison in a certain variational sense. This procedure consists essentially in determining, first, in the least square sense, the square well potential which most closely approximates the potential under consideration for a given incident propagation constant. We then use as a trial function in the original integral equation the sphere (or cylinder) wave function with the appropriate parameters.

We shall consider mainly soft scatterers ($|k_1-k_0|/k_0\ll1$), but one should be able to extend the range of our results without too much difficulty. In particular we shall examine the scattering by Gaussian, exponential, and screened Coulomb fields. The primary merit of the new method is that while it is significantly superior to the R.G.B. theory, it is nevertheless equally simple in application, and, in fact, gives all results for the differential and total scattering cross sections in rather compact closed forms for the scatterers considered.

1. WAVE EQUATION FOR SPHERICALLY SYMMETRICAL SCATTERERS

Let us consider the scattering of a scalar plane wave of propagation constant $k_0=2\pi/\lambda_0$ (λ_0 being its wavelength in the absence of an obstacle) by an obstacle in which the wavelength of a transmitted wave and hence $k=2\pi/\lambda$ varies from point to point. The character of the scattered wave can be discussed in terms of the solution of the wave equation.

$$\nabla^2\psi+k^2(r)\psi=0. \tag{1.1}$$

We restrict ourselves in this paper to situations in which the required solutions of (1.1) are continuous and have continuous first derivatives, even though $k(r)$ might be a discontinuous function. These correspond to the usual quantum-mechanical boundary conditions, to the scattering of sound waves by obstacles whose density is the same as that of the medium in which they are embedded, and to the scalar analog of the scattering of electromagnetic waves by nonconducting dielectric particles. Equation (1.1) is equivalent to Schrödinger's equation if k_0^2 is identified with $2mE/\hbar^2$ and $k^2(r)$ with $2m(E-V)/\hbar^2$.

We locate the origin of our coordinate system at the center of a spherically symmetrical obstacle and postulate $k(r)\rightarrow k_0$ as $r\rightarrow\infty$. Actually our analysis will be limited to forms of $k(r)$ with the properties: $[k^2(r)-k_0^2]\rightarrow0$ more rapidly than $1/r$ as $r\rightarrow\infty$ and increases as $r\rightarrow0$ at a rate no faster than $1/r$ (if it increases at all for small r). More precisely we shall require that

$$\int_0^\infty[k^2(r)-k_0^2]^2r^2dr<\infty \quad\text{and}\quad \int_0^\infty[k^2(r)-k_0^2]rdr<\infty.$$

The wave function of an incident plane wave propagated in the direction of the unit vector $\mathbf{s}_0$ is

$$\psi_i=\exp[ik_0(\mathbf{r}\cdot\mathbf{s}_0)],\quad k_0=2\pi/\lambda_0. \tag{1.2}$$

If $\psi_s(\mathbf{r})$ is the wave function of the scattered field, the solution of (1.1) is of the form

$$\psi(\mathbf{r})=\psi_s(\mathbf{r})+\exp ik_0(\mathbf{r}\cdot\mathbf{s}_0), \tag{1.3}$$

where $\psi_s(\mathbf{r})\rightarrow0$ as $1/r$. Hence $\psi_s(\mathbf{r})$ satisfies the integral equation[4]

$$\psi_s(\mathbf{R})=-\frac{1}{4\pi}\int\frac{e^{ik_0|\mathbf{R}-\mathbf{r}|}}{|\mathbf{R}-\mathbf{r}|}[k_0^2-k^2(r)]\psi(\mathbf{r})d\tau, \tag{1.4a}$$

where the integration extends over all space.

When $|R|$ is much larger than the range of the scatterer, the scattered wave function $\psi_s(\mathbf{R})$ can be approximated by

$$\psi_s(\mathbf{R})\sim\frac{e^{iRk_0}}{4\pi R}\int\psi(\mathbf{r})[k^2(r)-k_0^2]\times\exp[-ik_0(\mathbf{r}\cdot\mathbf{s}_1)]d\tau. \tag{1.4b}$$

The Born (Rayleigh-Gans in the case of vector waves) approximation for the scattered wave function is obtained by replacing $\psi(\mathbf{r})$ in (1.4) by the wave function of the incident wave, $\exp ik_0(\mathbf{r}\cdot\mathbf{s}_0)$, and integrating over all space.

[4] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1949), p. 114.

Equation (1.4) has, in S-II, been applied to the scattering of plane waves by uniform isotropic soft scatterers with sharp boundaries. For example, in a spherical scatterer of radius a and propagation constant k_1 (i.e., the case of scattering by a square well potential $V(r)$ of width a and height proportional to k_1^2) Eq. (1.4) becomes (since $k(r)=k_1$ for $r<a$ and $=k_0$ for $r>a$):

$$\psi_s=\frac{(k_1^2-k_0^2)}{4\pi}\int\frac{e^{ik_0|\mathbf{R}-\mathbf{r}|}\psi_t(\mathbf{r})}{|\mathbf{R}-\mathbf{r}|}d\tau$$

$$\sim\frac{k_1^2-k_0^2}{4\pi R}e^{iRk_0}\int\psi_t(\mathbf{r})\exp[-ik_0(\mathbf{r}\cdot\mathbf{s}_1)]d\tau, \quad (1.5)$$

where the integration extends over the volume of the sphere and $\psi_t(\mathbf{r})$ is the wave function of the interior of the sphere.

The following approximate wave function has been derived for a sphere in S-I:

$$\psi_t=\frac{2k_1e^{ia(k_1-k_0)}[\exp\{ik_1(\mathbf{r}\cdot\mathbf{s}_0)\}-\kappa e^{2iak_1}\exp\{-ik_1(\mathbf{r}\cdot\mathbf{s}_0)\}]}{(k_1+k_0)[1-\kappa^2\exp 4iak_1]}. \quad (1.6)$$

The first term in the numerator corresponds to the transmitted wave through the sphere and the second to the internal reflected wave. The approximate scattered wave function that is derived by substituting (1.6) into (1.5) is, for $r\gg a$ ($J_{\frac{3}{2}}(z)$ is the Bessel function of order $\frac{3}{2}$),

$$\psi_s(r)=A_sr^{-1}\{J_{\frac{3}{2}}(\omega a)/(\omega a)^{\frac{3}{2}}-\kappa e^{2iak_1}J_{\frac{3}{2}}(va)/(va)^{\frac{3}{2}}\}, \quad (1.7)$$

with ω and v defined by (3.5), and

$$A_s=a^3(2\pi)^{\frac{1}{2}}k_1(k_1-k_0)e^{i(k_1-k_0)a+irk_0}/(1-\kappa^2e^{4iak_1}), \quad (1.8)$$
$$\kappa=(m-1)/(m+1), \quad m=k_1/k_0.$$

This result is quite accurate when $m=k_1/k_0<1.3$ and is useful for $m<1.5$. The ratio of the total scattering cross section to the geometrical cross section approaches the exact value for all size spheres as $k_1\rightarrow k_0$. This is not the case for the Born approximation.

Expressions similar to (1.6) and (1.7) have been obtained in S-II for the internal and scattered wave functions of infinite circular cylinders. Approximate scattered wave functions were derived for finite cylinders and oblate spheroids in S-II by approximating their internal wave functions by those of infinite cylinders, substituting these wave functions in (1.7) and integrating over the entire volume of the obstacles.

We shall now apply this technique to the determination of approximate scattered wave functions of spherically and cylindrically symmetrical scatterers which do not have sharp boundaries. We shall approximate the internal wave functions of these scatterers by those of spheres and cylinders whose radii a and propagation constants k_1 for a given propagation constant of the incident wave k_0 will be determined by the variational scheme discussed in the next section.

2. A VARIATIONAL SCHEME[5]

Let the propagation function of the scatterer of interest be $k(r)$. Then the wave equation for an incident plane wave is (1.1). Now suppose we cannot easily solve (1.1), but that we can solve the wave equation that corresponds to a similar scatterer with propagation function $k_0(r)$ (which might depend on several parameters):

$$\nabla^2\psi_0+k_0^2(r)\psi_0=0. \quad (2.1)$$

The identity,

$$\nabla^2\psi_0+k^2(r)\psi_0=[k^2(r)-k_0^2(r)]\psi_0, \quad (2.2)$$

suggests a variational scheme. If $k_0(r)$ were equal to $k(r)$, the quantities on both sides of (2.2) would vanish for all r. Hence, if we could choose the parameters in the approximating function $k_0(r)$ [and hence in $\psi_0(\mathbf{r})$] in such a way as to minimize the integral

$$I=\int|k^2(r)-k_0^2(r)|^2|\psi_0(\mathbf{r})|^2d\tau, \quad (2.3)$$

we would, in the least square sense, have the best approximating function of the initially chosen form of $k_0(r)$. Equation (2.2) implies that minimizing I is equivalent to minimizing $\nabla^2\psi_0+k^2(r)\psi_0$, or finding the parameters of ψ_0 which make ψ_0 the best solution of (1.1) of the given form.

It is to be noted that $\psi_0(\mathbf{r})$ and hence the "best" parameters of $k_0(r)$ depend on the wavelength, or energy, of the incident plane wave.

The physical meaning of the integral (2.3) is immediately apparent if the incident waves correspond to a steady stream of particles of energy E emitted in the direction of the scattering center of potential $V_0(r)$ and we interpret $|\psi_0|^2d\tau$ as a quantity proportional to the particle density in the volume element $d\tau$. Let the parameters of $k_0(r)$ be fixed, while those of $k(r)$ are varied to minimize I. Then the solution of the variational problem $\delta I=0$ yields the best least square fit of $k(r)$ to $k_0(r)$ with each volume element in space weighted by its proper quantum-mechanical particle density function. Actually it will be easier to solve the variational problem with $k_0(r)$ fixed and $k(r)$ varied, and through successive guesses of the parameters of $k_0(r)$ to find those which will give the best $k(r)$ its required parameters. In terms of the scattering potential $V(r)$, $\delta I=0$ corresponds to

$$\delta I=\delta\int|V-V_0|^2|\psi_0|^2d\tau=0. \quad (2.4)$$

Equation (2.4) is a general variational principle that can be applied either to an initially chosen form for the

[5] Variational methods for solving scattering problems have been used quite extensively by Harold Levine and Julian Schwinger, Phys. Rev. **74**, 958 (1948); **75**, 1423 (1949); W. Kohn, Phys. Rev. **74**, 1763 (1948); and others. To the best of our knowledge the simple particular variational scheme of this section has not been used explicitly before.

wave function $\psi_0(\mathbf{r})$ or of the scattering potential V_0. Let $\psi_0(\mathbf{r})$ (with several parameters) be chosen to have roughly the form which should correspond to the potential $V(r)$. Then define the approximating potential function $V_0(r)$ by

$$V_0=(1/\psi_0)\{2\hbar^{-2}\nabla^2\psi_0+E\psi_0\}, \quad (2.5)$$

where E is the energy of the incident particles. The best values of the parameters in ψ_0 are then derived from (2.4).

The approximate wave function $\psi_0(\mathbf{r})$ with the best parameters can be improved further by inserting it into the integral equation (1.4) and integrating to obtain a better scattered wave function.

The integral (2.3) is useful for comparing two different forms of approximating wave functions. If the best parameters are found in two different forms of $\psi_0(\mathbf{r})$, that form which gives the smaller value to (2.3) is the better approximation of $\psi(\mathbf{r})$.

3. APPROXIMATE SCATTERED WAVE FUNCTIONS FOR SPHERICALLY SYMMETRICAL SCATTERERS

We shall now consider scatterers for which $|k^2(r)-k_0^2|$ is a monotone decreasing function of r. Some important examples are $k^2(r)-k_0^2=Ae^{-Br}$, $A\exp(-Br^2)$, and $Ar^{-1}e^{-Br}$. The Gaussian form occurs in scattering by randomly coiled polymer chains, and all forms have been used for the scattering of fundamental particles by nuclei.

A natural approximating form for such $k^2(r)-k_0^2$ is that of a uniform, isotropic spherical scatterer of radius a. Then

$$k_0^2(r)-k_0^2=\begin{cases}k_1^2-k_0^2 & \text{if } r<a,\\ 0 & \text{if } r>a.\end{cases} \quad (3.1)$$

Since the wave function ψ_0 is well known for this type of scatterer, one can obtain best values of k_1 and a for a given spherically symmetrical scatterer and k_0 by applying the variational principle discussed in the last section. Suppose this has been done. Then an improved scattered wave function ψ_s is derived by substituting ψ_0 for ψ in the integrand of the integral equation (1.4b).

Let ψ_t be the transmitted or internal wave function of the sphere and $\psi_s^{(0)}$ the scattered wave function. Then

$$\psi_0(\mathbf{r})=\begin{cases}\psi_t(\mathbf{r}) & r<a\\ \psi_s^{(0)}(\mathbf{r})+\exp\{ik_0(\mathbf{r}\cdot\mathbf{s}_0)\} & r>a\end{cases}$$

and

$$\psi_s(\mathbf{R})\sim\frac{e^{iRk_0}}{4\pi R}\Big\{2\pi\int_0^a\int_0^\pi \exp\{-ik_0(\mathbf{r}\cdot\mathbf{s}_1)\}$$
$$\times[k^2(r)-k_0^2]\psi_t r^2\sin\vartheta' dr d\vartheta'$$
$$+2\pi\int_a^\infty\int_0^\pi \exp\{-ik_0(\mathbf{r}\cdot\mathbf{s}_1)\}[k^2(r)-k_0^2]$$
$$\times[\psi_s^{(0)}(\mathbf{r})+\exp\{ik_0(\mathbf{r}\cdot\mathbf{s}_0)\}]r^2\sin\vartheta' dr d\vartheta'\Big\}. \quad (3.2)$$

Now we could substitute the exact sphere wave functions into (3.2). This procedure gives us new scattered wave functions which are complicated infinite series of Legendre polynomials and which must be summed numerically. Since we are interested here in soft scatterers (high energies of incident waves) we can better use the closed form approximate sphere wave functions that were derived in S-I. This allows us to obtain simple closed form wave functions for many scatterers of interest (as we shall soon see, these new wave functions will be no more difficult to use than those of the Born approximation even though they are significantly more accurate). The main contribution to $\psi_s(\mathbf{R})$ in (3.2) comes from the integral from 0 to a because $k^2(r)-k_0^2$ is small for $r>a$. Hence we can use approximate wave functions in the second integral of (3.2) which need only be accurate in the neighborhood of $r=a$ but which might be rather poor (provided they are bounded) for very large values of r.

We shall use the internal sphere wave function for all values of r (including those outside the sphere) as an approximate wave function. Since $\psi_t(\mathbf{r})$ and its first derivative are equal, respectively, to $[\psi_s^{(0)}(\mathbf{r})+\exp\{ik_0(\mathbf{r}\cdot\mathbf{s}_0)\}]$ and its first derivative at $r=a$, $\psi_t(\mathbf{r})$ is a good approximation to the exterior wave function for a short distance beyond a sphere boundary (especially if $k^2(r)-k_0^2$ decreases exponentially with r). Hence we can write

$$\psi_s(R)\sim\frac{e^{ik_0R}}{4\pi R}\int \exp\{-ik_0(\mathbf{r}\cdot\mathbf{s}_1)\}$$
$$\times[k^2(r)-k_0^2]\psi_t(\mathbf{r})d\tau, \quad (3.3)$$

where the integral extends over all space.

Let us substitute the soft sphere internal wave function (1.6) into (3.3) to obtain

$$\psi_s(\mathbf{R})\sim\frac{k_1e^{ia(k_1-k_0)+iRk_0}[I_1-\kappa e^{2iak_1}I_2]}{R(k_1+k_0)[1-\kappa^2\exp(4iak_1)]}, \quad (3.4a)$$

where κ is defined by Eq. (1.8),

$$I_1=\int_0^\infty\int_0^\pi [k^2(r)-k_0^2]e^{ir\omega\cos\gamma}r^2\sin\gamma d\gamma, \quad (3.4b)$$

$$I_2=\int_0^\infty\int_0^\pi [k^2(r)-k_0^2]e^{irv\cos\beta}r^2\sin\beta d\beta, \quad (3.4c)$$

γ is the polar angle between the variable vector $\mathbf{r}$ which spans space and the fixed vector $(k_1\mathbf{s}_0-k_0\mathbf{s}_1)$, and β is the polar angle between $\mathbf{r}$ and the fixed vector $(k_1\mathbf{s}_0+k_0\mathbf{s}_1)$. Also

$$\omega^2=|k_1\mathbf{s}_0-k_0\mathbf{s}_1|^2=k_1^2+k_0^2-2k_1k_0\cos\vartheta, \quad (3.5a)$$

$$v^2=k_1^2+k_0^2+2k_1k_0\cos\vartheta, \quad (3.5b)$$

θ being the angle of scattering (the angle between $\mathbf{s}_1$ and $\mathbf{s}_0$).

The integrations over γ and β in (3.4) are elementary and yield

$$\psi_s \sim A_s R^{-1} \int_0^\infty [k^2(r)-k_0^2] r^2 \left\{ \frac{\sin \omega r}{\omega r} - \kappa e^{2iak_1} \frac{\sin vr}{vr} \right\} dr, \quad (3.6)$$

where

$$A_s = \frac{2k_1}{(k_1+k_0)} \frac{\exp i[a(k_1-k_0)+k_0R]}{[1-\kappa^2 \exp(4iak_1)]}. \quad (3.6a)$$

The differential scattering cross section is then

$$\sigma(\theta) = R^2 |\psi_s(R)|^2 = |A_s|^2 \left| \int_0^\infty [k^2(r)-k_0^2] r^2 \left\{ \frac{\sin \omega r}{\omega r} - \kappa e^{2iak_1} \frac{\sin vr}{vr} \right\} dr \right|^2. \quad (3.7)$$

We must now find relations between the parameters of the scattering potentials and those of the approximating sphere. This is done by applying the variational scheme of Sec. 2. At this point we derive the relations for very soft scatterers. The method of improving the accuracy of these is straightforward but leads to lengthy equations which we shall not exhibit here.

We wish to minimize the integral

$$F = \int [(k^2(r)-k_0^2)-(k_0^2(r)-k_0^2)]^2 |\psi_t|^2 d\tau, \quad (3.8)$$

for the special case

$$k^2(r)-k_0^2 = A^2 f(r/b),$$

$$k_0^2(r)-k_0^2 = \begin{cases} k_1^2-k_0^2, & r<a, \\ 0, & r>a, \end{cases}$$

with ψ_t given by (1.6). For very soft spheres we can neglect the term of order κ, which is the coefficient of the reflected wave in the sphere. Then $|\psi_t|^2 = \text{constant}$ and the variational equations,

$$\partial F/\partial(A^2) = 0 \quad \text{and} \quad \partial F/\partial b = 0, \quad (3.9)$$

imply

$$A^2 = (k_1^2-k_0^2) \int_0^{a/b} x^2 f(x) dx \bigg/ \int_0^\infty x^2 f^2(x) dx, \quad (3.9a)$$

$$A^2 = (k_1^2-k_0^2) \int_0^{a/b} x^3 f'(x) dx \bigg/ \int_0^\infty x^3 f(x) f'(x) dx. \quad (3.9b)$$

By equating (3.9a) to (3.9b) and integrating by parts, we find that if

$$\alpha = a/b, \quad (3.10a)$$

then we obtain the equation

$$\tfrac{2}{3}\alpha^3 f(\alpha) = \int_0^\alpha x^2 f(x) dx. \quad (3.10b)$$

Hence (3.10a) and (3.10b) give the relation between a and b. Once α has been determined, A^2 is expressed in terms of α through (3.9a) and (3.10b):

$$A^2 = \tfrac{2}{3}\alpha^3 (k_1^2-k_0^2) f(\alpha) \bigg/ \int_0^\infty x^2 f^2(x) dx = \tfrac{1}{2}\beta^{-1}(k_1^2-k_0^2). \quad (3.11)$$

As $A \to 0$ this is the correct limiting relation.

Since the terms of order κ and κ^2 in all of the equations derived below (3.18–3.37) are of the same order of magnitude as the corrections required in the relations (3.10) and (3.11) between A's, b's and k_1's and a's, they should not be used without these corrections.

The total scattering cross section σ_s can be obtained in two ways: by integrating $\sigma(\theta)$ over all angles or by using the imaginary part of the scattered amplitude in the forward direction. These two results are identical only when $\sigma(\theta)$ is exact. Since it is possible to integrate our approximate expressions for $\sigma(\theta)$, we prefer to do the former to obtain a more unified presentation. It should also be noted that in the integration over θ one uses the weight function $\sin\theta$. The largest errors in our approximate formulas occur in the range of small θ. Hence the integration procedure should give the better results. We shall now summarize the formulas for $\sigma(\theta)$ and σ_s for very soft scatterers.

After neglecting the terms of order κ and κ^2 in (3.7), we have

$$\sigma(\theta) = \frac{4k_1^2 A^4 b^4}{\omega^2 (k_1+k_0)^2} \left| \int_0^\infty x f(x) \sin(bx\omega) dx \right|^2. \quad (3.12)$$

The total scattering cross section is given by

$$\sigma_s = \frac{8k_1^2 A^4 b^4 \pi}{(k_1+k_0)^2} \int_0^\pi \left| \int_0^\infty x f(x) \sin \omega bx dx \right|^2 \sin\theta \omega^{-2} d\theta. \quad (3.13)$$

It is desirable to introduce $u = b\omega$ as the new angle variable to replace θ. Then

$$\frac{\sigma}{\pi b^2} = \frac{8k_1 A^4 b^2}{k_0(k_1+k_0)^2} \times \int_{b(k_1-k_0)}^{b(k_1+k_0)} u^{-1} \left\{ \int_0^\infty x f(x) \sin ux dx \right\}^2 du. \quad (3.14)$$

In the limit of super soft scatterers or high energies ($k_1 \to k_0$), the natural parameter in which to express the results is

$$y = (A/k_0)^2(bk_0). \quad (3.15)$$

Now

$$b(k_1+k_0) = 2\beta A^2 b/(k_1-k_0) = y\beta/[(k_1/k_0)-1],$$
$$b(k_1-k_0) = 2\beta A^2 b/(k_1+k_0) = y\beta/[(k_1/k_0)+1].$$

Hence as $k_1 \to k_0$ for fixed y, $b(k_1+k_0) \to \infty$ and $b(k_1-k_0) = y\beta$. The coefficient of the integral in (3.14) becomes $2y^2$.

With these asymptotic results, the total scattering cross section of a super soft spherically symmetrical scatterer is

$$\frac{\sigma_s}{\pi b^2} = 2y^2 \int_{y\beta}^{\infty} u^{-1} \left\{ \int_0^{\infty} x f(x) \sin uxdx \right\}^2 du. \quad (3.16)$$

The first special case of this formula was derived by van de Hulst[3] for a uniform spherical scatterer. There $f(x)=0$ for $r>b$, 1 for $r<b$, $\beta=1$, and $y=bk_0[(k_1/k_0)^2-1]$ and Eq. (3.20) results.

We shall now give the results of the application of the above equations to four types of scattering potentials.

(a) Sphere of Radius a

Let

$$k^2(r)-k_0^2 = \begin{cases} k_1^2-k_0^2 & \text{if } r<a, \\ 0 & \text{if } r>a. \end{cases} \quad (3.17)$$

Then

$$\sigma(\theta) = \tfrac{1}{2}|A_s|^2(k_0^2-k_1^2)^2 a^6 \left\{ \frac{J_{\frac{3}{2}}^2(a\omega)}{(a\omega)^3} - 2\kappa \frac{\cos(2ak_1) J_{\frac{3}{2}}(a\omega) J_{\frac{3}{2}}(av)}{(a\omega)^{\frac{3}{2}}(av)^{\frac{3}{2}}} + \frac{\kappa^2 J_{\frac{3}{2}}^2(av)}{(av)^3} \right\}. \quad (3.18)$$

This result agrees with that derived in S-I and S-II for soft spheres.

It is shown in S-I and S-II that

$$\sigma_s \simeq 2\pi^{\frac{3}{2}} a^2 x^2 m^{-1}(m-1)^2 \left\{ \frac{\mathbf{H}_{\frac{3}{2}}(2x[m-1])}{(x[m-1])^{5/2}} - \frac{\mathbf{H}_{\frac{3}{2}}(2x[m+1])}{(x[m+1])^{5/2}} \right\}, \quad (3.19)$$

where $x = ak_0 = 2\pi a/\lambda_0$, $m = k_1/k_0$, and $\mathbf{H}_{\frac{3}{2}}(z)$ is the $\frac{3}{2}$th-order Struve function,

$$\mathbf{H}_{\frac{3}{2}}(z) = (z/2\pi)^{\frac{1}{2}}(1+2z^{-2}) - (2/\pi z)^{\frac{1}{2}}(\sin z + z^{-1}\cos z).$$

If we define $y = x(m-1)$ and let $m \to 1$, we obtain the exact total scattering cross section of super soft spheres,[2,3]

$$\sigma_s/\pi a^2 = 2(\pi/y)^{\frac{1}{2}} \mathbf{H}_{\frac{3}{2}}(2y), \quad (3.20)$$

as $y \to \infty$, $\sigma_s/\pi a^2 \to 2$.

The Born approximation expression which corresponds to (3.19) is

$$\sigma_s/\pi a^2 = 2a^2(k_1-k_0)^2\{1-\pi^{\frac{1}{2}}\mathbf{H}_{\frac{3}{2}}(4ak_0)/(2ak_0)^{5/2}\}.$$

In the limit as $m \to 1$, the range in which the Born approximation is best (provided that $2\pi a/\lambda_0$ is small), we get

$$\sigma_s/\pi a^2 = 2y^2. \quad (3.21)$$

This equation represents $\sigma_s/\pi a^2$ as a monotonically increasing function of y, which becomes infinite as the sphere radius becomes infinite. The exact equation (3.20) represents a bounded oscillating function of y. The exact total scattering cross section is compared with the Born approximation in Fig. 1.

(b) Gaussian Scatterer

Let

$$k^2(r)-k_0^2 = \pm A^2 \exp[-(r/b)^2]. \quad (3.22)$$

Then

$$\sigma(\theta) = |A_s|^2(A^4 b^6 \pi/16)\{\exp(-\tfrac{1}{2}b^2\omega^2) - 2\kappa\cos(2k_1 a) \times \exp[-\tfrac{1}{4}b^2(\omega^2+v^2)] + \kappa^2 \exp(-\tfrac{1}{2}b^2 v^2)\}. \quad (3.23)$$

Since $f(x) = \pm\exp(-x^2)$, α satisfies

$$(4/3)\alpha^3 + \alpha = \exp(\alpha^2)\int_0^{\alpha} \exp(-x^2)dx$$

so that

$$\alpha = a/b = 1.235. \quad (3.24a)$$

It follows that

$$A^2 = \pm\frac{16}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(k_1^2-k_0^2)\alpha^3\exp(-\alpha^2)$$
$$\simeq \pm 1.744(k_1^2-k_0^2). \quad (3.24b)$$

The total scattering cross section of a Gaussian scatterer is obtained by integrating (3.23) over all angles. Then

$$\sigma_s = \frac{\tfrac{1}{2}\pi^2 A^4 b^6 k_1^2}{(k_1+k_0)^2(1+\kappa^4-2\kappa^2\cos 4ak_1)} \left[\frac{(1+\kappa^2)}{k_1 k_0 b^2} \times \{\exp[-\tfrac{1}{2}b^2(k_1-k_0)^2] - \exp[-\tfrac{1}{2}b^2(k_1+k_0)^2]\} - 4\kappa\cos(2ak_1)\exp[-\tfrac{1}{2}b^2(k_1^2+k_0^2)] \right]. \quad (3.25)$$

The limiting expression for super soft scatterers (high energies) is derived by substituting (3.24) into (3.25) and neglecting terms of order κ and κ^2. Then, for small A,

$$\sigma_s/\pi b^2 = \tfrac{1}{8}\pi y^2 \exp(-\tfrac{1}{2}y^2\beta^2), \quad y = (A/k_0)^2(bk_0), \quad \beta = 0.2867. \quad (3.26)$$

This total scattering cross section is compared with the Born approximation,

$$\sigma_s/\pi b^2 = \tfrac{1}{8}\pi y^2, \qquad (3.27)$$

in Fig. 2. Notice that as $y \to \infty$, $\sigma_s/\pi b^2 \to \infty$ in the Born approximation and zero in our approximation.

(c) Exponential Scatterer

Let

$$k^2(r) - k_0^2 = \pm A^2 \exp(-r/b). \qquad (3.28)$$

Then

$$\sigma(\theta) = 4b^6A^4|A_s|^2\{(1+\omega^2b^2)^{-4} + \kappa^2(1+v^2b^2)^{-4} - 2\kappa\cos(2ak_1)(1+v^2b^2)^{-2}(1+\omega^2b^2)^{-2}\}. \qquad (3.29)$$

Here $f(x) = \pm\exp(-x)$. Hence $1+\alpha+\frac{1}{2}\alpha^2+\frac{1}{3}\alpha^3 = e^\alpha$, $\alpha = 2.318$, and we have

$$A^2 = \pm(8/3)(k_1^2 - k_0^2)\alpha^3e^{-\alpha} \simeq \pm 3.270(k_1^2 - k_0^2). \qquad (3.30)$$

Since

$$\int_0^\pi \frac{\sin\theta d\theta}{(1+\omega^2b^2)^4} = \frac{1}{6b^2k_1k_0}\left\{\frac{1}{[1+b^2(k_1-k_0)^2]^3} - \frac{1}{[1+b^2(k_1+k_0)^2]^3}\right\} = L_1,$$

and

$$\int_0^\pi \frac{\sin\theta d\theta}{(1+\omega^2b^2)^2(1+v^2b^2)^2} = \frac{1}{4k_1k_0b^2[1+b^2(k_1^2+k_0^2)]^2} \times\left\{\frac{4k_1k_0b^2}{[1+b^2(k_1+k_0)^2][1+b^2(k_1-k_0)^2]} + \frac{1}{[1+b^2(k_1^2+k_0^2)]}\log\left(\frac{1+b^2(k_1+k_0)^2}{1+b^2(k_1-k_0)^2}\right)\right\} = L_2,$$

we have

$$\sigma_s = 8b^6A^4\pi\{(1+\kappa^2)L_1 - 2\kappa L_2\cos(2ak_1)\}|A_s|^2. \qquad (3.31)$$

The limiting expression for super soft scatterers is

$$\sigma_s/\pi b^2 = (4/3)y^2(1+\beta^2y^2)^{-3}, \quad \beta = 0.1529. \qquad (3.32)$$

The corresponding Born approximation is

$$\sigma_s/\pi b^2 = (4/3)y^2. \qquad (3.33)$$

Equations (3.33) and (3.32) are compared in Fig. 3.

(d) Screened Coulomb Scatterer

Let

$$k^2(r) - k_0^2 = \pm[A^2/(r/b)]\exp(-r/b). \qquad (3.34)$$

Then

$$\sigma(\theta) = A^4b^6|A_s|^2\{(1+\omega^2b^2)^{-2} + \kappa^2(1+v^2b^2)^{-2} - 2\kappa\cos(2ak_1)(1+\omega^2b^2)^{-1}(1+v^2b^2)^{-1}\}. \qquad (3.35)$$

Here $f(x) = \pm x^{-1}e^{-x}$. Then α satisfies $e^\alpha = 1+\alpha+\frac{2}{3}\alpha^2$ and $\alpha = 0.807$; hence,

$$A^2 = \pm(4/3)\alpha^2(k_1^2 - k_0^2)e^{-\alpha} \simeq \pm 0.379(k_1^2 - k_0^2). \qquad (3.36)$$

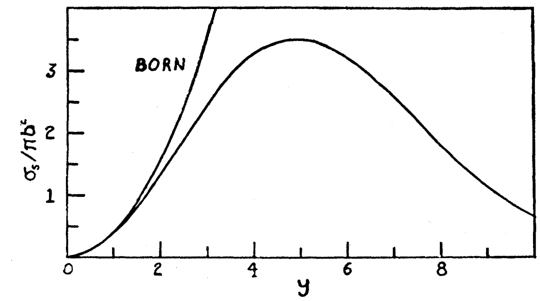

FIG. 2. Ratios of the total scattering cross section of a Gaussian. sphere $[k^2(r) - k_0^2 = \pm A^2\exp\{-(r/b)^2\}]$ with πb^2 in the limit as $A/k_0 \to 0$. The upper curve is the Born approximation [Eq. (3.27)]. The lower curve is the present approximation [Eq. (3.26)]. The parameter y is defined by $y = (A/k_0)^2(bk_0)$.

We obtain the total scattering cross section by integrating (3.35):

$$\sigma_s = \pi A^4b^4|A_s|^2(k_1k_0)^{-1} \times\left\{(1+\kappa^2)\left[\frac{1}{1+b^2(k_1-k_0)^2} - \frac{1}{1+b^2(k_1+k_0)^2}\right] - \frac{2\kappa\cos 2ak_1}{1+b^2(k_1^2+k_0^2)}\log\left[\frac{1+b^2(k_1+k_0)^2}{1+b^2(k_1-k_0)^2}\right]\right\}. \qquad (3.37)$$

The limiting expression for super soft scatterers is

$$\sigma_s/\pi b^2 = y^2/(1+y^2\beta^2), \quad \beta = 1.319, \qquad (3.38)$$

while the Born approximation yields

$$\sigma_s/\pi b^2 = y^2. \qquad (3.39)$$

Equations (3.39) and (3.38) are compared in Fig. 4.

It is of interest at this point to see if (3.35) reduces as $b \to \infty$ to the exact differential cross section of a Coulomb scatterer. Let us consider the scattering of an electron by an ion of charge Ze. Then

$$k^2(r) - k_0^2 = -(2mZe^2/rh^2)e^{-r/b}, \quad k_0^2 = 2mE/h^2 = mv^2/h^2,$$

where e = charge on the electron, m = electron mass, and v = velocity of incident electron.

From (3.34), $A^2b = 2mZe^2/h^2$. Then, from (3.36),

$$2Zme^2/bh^2 = 0.379(k_1^2 - k_0^2).$$

Hence as $b \to \infty$, $k_1 \to k_0$ so that

$$\omega^2 = k_1^2 + k_0^2 - 2k_1k_0\cos\vartheta = 4m^2v^2h^{-2}\sin^2\tfrac{1}{2}\vartheta,$$

and (3.35) approaches

$$\sigma(\theta) = (2mZe^2/h^2)^2\omega^{-4} \to [(Ze^2\csc^2\tfrac{1}{2}\theta)/2mv^2]^2,$$

the Rutherford scattering formula. Our approximation is equivalent to that of the Born approximation for this problem. Equations (3.18), (3.23), and (3.29) all reduce to the corresponding Born approximations when $k_1 = k_0$, for then $\omega^2 \to 4k_0^2\sin^2\frac{1}{2}\theta$ and $\kappa = 0$. The resulting expressions are independent of the parameters of the sphere whose approximate wave functions were used.

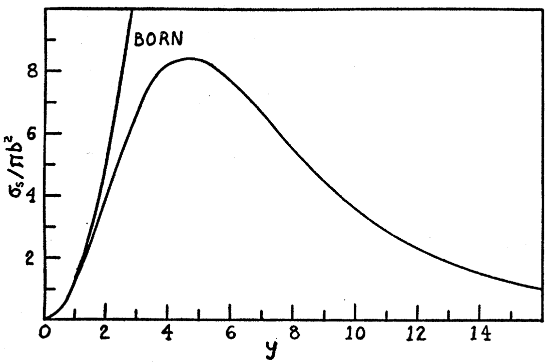

FIG. 3. Ratios of the total scattering cross section of an exponential sphere $[k^2(r)-k_0^2=\pm A^2\exp(-r/b)]$ with πb^2 in the limit as $(A/k_0)\to 0$. The upper curve is the Born approximation [Eq. (3.33)]. The lower curve is the present approximation [Eq. (3.32)]. The parameter y is defined by $y=(A/k_0)^2(bk_0)$.

As A, and therefore as $(k_1^2-k_0^2)$, increases one must use a better approximation to ψ_t in (3.8) in order to obtain more accurate relations between the sphere parameters and those of the scatterer of interest.

The above results are generalized in the next section to scatterers with the symmetry of an infinite cylinder.

4. SCATTERING BY SOFT OBSTACLES WITH SYMMETRY OF INFINITE CIRCULAR CYLINDER

Since the theory of scattering by an infinite circular cylinder is essentially the same as that of a sphere, we shall not include as much detail in this section as we did in the last. We shall merely summarize the main formulas and give a brief indication of their derivation.

We let the z axis of our coordinate system be the axis of our scatterer and let our incident plane wave be propagated in the direction of a unit vector s_0 which lies in the $x-z$ plane and makes an angle α_0 with the z axis. We shall use the approximate internal wave function of an isotropic circular cylinder of radius a and propagation constant k_1. An angle α_1 defined by

$$k_1\cos\alpha_1=k_0\cos\alpha_0, \tag{4.1}$$

and a set of "starred" parameters

$$k_j^*=k_j\sin\alpha_j,\quad m^*=k_1^*/k_0^* \tag{4.2}$$

enter naturally into the theory.

The starting point of our analysis is the integral equation (1.4a). We approximate $\psi(r)$ in the integrand by the wave function ψ_t of a soft cylinder. As was shown in S-II, the value of ψ_t at a point r, z, θ (in cylindrical coordinates) is approximately

$$\psi_t(r)=\frac{2(m^*)^{\frac{1}{2}}\exp[ia(k_1^*-k_0^*)]\exp(izk_0\cos\alpha_0)[\exp(ik_1^*r\cos\theta)-i\kappa\exp(2iak_1^*)\exp(-irk_1^*\cos\theta)]}{(1+m^*)[1+\kappa^2\exp(4iak_1^*)]}. \tag{4.3}$$

If (4.3) is substituted into (1.4a), the integration over z and θ can be carried out in the manner discussed in S–II. Then

$$\psi_s(R)=\frac{i(m^*)^{\frac{1}{2}}(2\pi/Rk_0^*)^{\frac{1}{2}}\exp[ia(k_1^*-k_0^*)+i(Rk_0^*-\tfrac{1}{4}\pi+k_0z\cos\alpha_0)]}{(1+m^*)[1+\kappa^2\exp(4iak_1^*)]}\times\int_0^\infty r[k^2(r)-k_0^2][J_0(r\omega_1)-i\kappa\exp(2iak_1^*)J_0(rv_1)]dr, \tag{4.4}$$

where

$$\omega_1^2=k_1^{*2}+k_0^{*2}-2k_1^*k_0^*\cos\Theta \tag{4.5a}$$

$$v_1^2=k_1^{*2}+k_0^{*2}+2k_1^*k_0^*\cos\Theta. \tag{4.5b}$$

The differential scattering cross section is then given by

$$\sigma(\Theta)=R|\psi_s(R)|^2=\frac{2\pi m^*}{k_0^*(1+m^*)^2(1+\kappa^4+2\kappa^2\cos4a_1k_1^*)}\times\left|\int_0^\infty r[k^2(r)-k_0^2][J_0(r\omega_1)-i\kappa\exp(2iak_1^*)J_0(rv_1)]dr\right|^2. \tag{4.6}$$

The integrations are easily carried out in three of the four special cases considered in the last section. The screened Coulomb potential leads to a divergence in the integrals needed to relate the parameters of the cylindrical scatterer with those of the screened Coulomb potential.

As in the case of scattering by spherically symmetrical scatterers, we use the variational scheme of Sec. 2 to relate a and k_1 to A and b. We again limit ourselves to soft scatterers so that we can assume $|\psi_0|^2=$constant. The integral to be minimized is

$$I=2\pi\int_0^\infty([k^2(r)-k_0^2]-[k_0^2(r)-k_0^2])^2rdr.$$

In our cases of interest

$$k^2(r)-k_0^2=A^2f(r/b),$$

while

$$k_0^2(r)-k_0^2=\begin{cases}k_1^2-k_0^2 & \text{if}\quad r<a\\ 0 & \text{if}\quad r>a.\end{cases}$$

The variational equations $\partial I/\partial A^2=0$ and $\partial I/\partial b=0$ imply

$$A^2=(k_1^2-k_0^2)\int_0^\alpha xf(x)dx\Big/\int_0^\infty xf^2(x)dx \quad (4.7)$$

$$A^2=(k_1^2-k_0^2)\int_0^\alpha x^2f'(x)dx\Big/\int_0^\infty x^2f(x)f'(x)dx \quad (4.8)$$

$$\alpha=a/b. \quad (4.9)$$

By equating these expressions for A^2, we find that α is the real positive root of

$$\alpha^2f(\alpha)=\int_0^\alpha xf(x)dx, \quad (4.10)$$

and A^2 is related to k_1 by

$$A^2=(k_1^2-k_0^2)\alpha^2f(\alpha)\Big/\int_0^\infty xf^2(x)dx. \quad (4.11)$$

General expressions can be derived for the total scattering cross sections of very soft scatterers in the following way. Equation (4.6) implies in this case

$$\sigma_s=\frac{4\pi mA^2}{k_0(1+m)^2}\int_0^\pi\left[\int_0^\infty rf(r/b)J_0(r\omega)dr\right]^2d\Theta.$$

Let us introduce the new integration variable $u=b\omega$. Then

$$d\Theta=udu/b^2k_1k_0\sin\Theta.$$

$\sin\Theta$ is obtained as a function of u from (4.5). Then

$$\sigma_s/2b=\frac{4\pi mA^4b^3}{k_1(1+m)^2}\times\int_{b(k_1-k_0)}^{b(k_1+k_0)}\frac{udu\left\{\int_0^\infty xf(x)J_0(xu)dx\right\}^2}{[b^2(k_0+k_1)^2-u^2]^{\frac{1}{2}}[u^2-b^2(k_1-k_0)^2]^{\frac{1}{2}}}.$$

In the limit as $A\rightarrow 0$ (i.e., $m\rightarrow 1$) it is convenient to introduce a new variable

$$y=(A/k_0)^2(bk_0)=\tfrac{1}{2}\beta^{-1}(k_1^2-k_0^2)/k_0.$$

Then for fixed y

$$(k_1+k_0)b=y\beta/[k_1/k_0-1]\rightarrow\infty$$
$$(k_1-k_0)b=y\beta/[k_1/k_0+1]\rightarrow\tfrac{1}{2}\beta y$$

and

$$\sigma_s/2b\sim\tfrac{1}{2}\pi y^2\int_{\beta y}^\infty\frac{udu\left\{\int_0^\infty xf(x)J_0(xu)dx\right\}^2}{(u^2-\beta^2y^2)^{\frac{1}{2}}}\Big/. \quad (4.12)$$

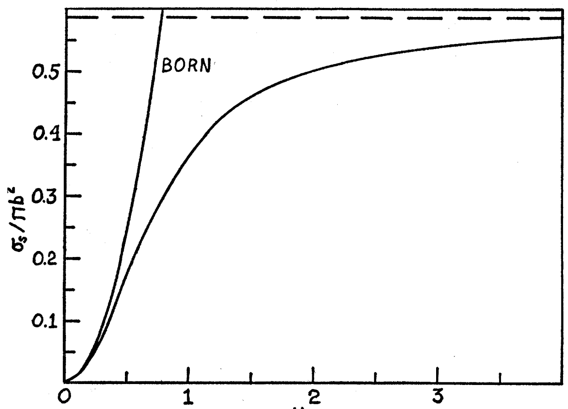

FIG. 4. Ratios of the total scattering cross section of a screened Coulomb scatterer $[k^2(r)-k_0^2=\pm\{A^2/(r/b)\}\exp(-r/b)]$ with πb^2 in the limit as $(A/k_0)\rightarrow 0$. The upper curve is the Born approximation [Eq. (3.39)]. The lower curve is the present approximation [Eq. (3.38)]. The parameter y is defined by $y=(A/k_0)^2(bk_0)$.

As an example, we shall present here only the results for the Gaussian scatterer. In that case

$$k^2(r)-k_0^2=\pm A^2\exp-(r/b)^2.$$

Hence

$$\sigma(\Theta)=\frac{\pi m^*b^4A^4}{2k_0^*(1+m^*)^2[1+\kappa^4+2\kappa^2\cos(4ak_1^*)]}\times\{\exp(-\tfrac{1}{2}b^2\omega_1^2)+\kappa^2\exp(-\tfrac{1}{2}b^2v_1^2)+2\kappa\sin(2ak_1^*)\exp[-\tfrac{1}{2}b^2(k_1^{*2}+k_0^{*2})]\}. \quad (4.13)$$

The defining equation for α for a Gaussian scatterer is $1+2\alpha^2=\exp\alpha^2$. From this we find $\alpha=1.121$ and

$$A^2=\pm 4\alpha^2\exp(-\alpha^2)(k_1^2-k_0^2)=\pm 1.431(k_1^2-k_0^2). \quad (4.14)$$

The total scattering cross section of a very soft Gaussian cylinder to plane waves where direction of propagation is normal to the cylinder axis is (we have dropped the stars on our m's and k_1's and let $m=k_1/k_0$)

$$\sigma_s=\frac{\pi mb^4A^4}{k_0(1+m)^2}\int_0^\pi\exp\{-\tfrac{1}{2}b^2(k_1^2+k_0^2-2k_1k_0\cos\Theta)\}d\Theta$$
$$=\frac{m\pi^2b^4A^4}{k_0(1+m)^2}\exp[-\tfrac{1}{2}b^2k_0^2(1+m^2)]I_0(b^2k_0^2m);$$
$$m=1+(A^2/1.431k_0^2)$$

where $I_0(x)$ is the zeroth-order Bessel function of purely imaginary argument. In the limit as $A\rightarrow 0$ we can use the asymptotic formula (4.12) to obtain

$$\sigma_s/2b\sim\tfrac{1}{8}\pi y^2\int_{\beta y}^\infty\frac{u^2\exp(-\tfrac{1}{2}u^2)du}{[u^2-\beta^2y^2]^{\frac{1}{2}}}$$
$$=\frac{\pi}{8}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}y^2\exp(-\tfrac{1}{2}\beta^2y^2).$$

One might attempt to generalize the results of this paper to scatterers in which the scattering force cannot be expressed in terms of a potential which depends only on the distance from the scattering center; to the scattering of vector waves; to the scattering by shell structures; and to the scattering by periodically distributed scatterers. One might also attempt to make improvements by using step function potentials as the trial forms. Some of these problems are now under investigation. It would be desirable to investigate some slightly more complicated improved forms of the approximating wave functions then those used here.

The general program on scattering, of which this work is one phase, was initiated through a grant of the Research Corporation while one of the authors (E.W.M.) was at the University of Pittsburgh.

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Magnetostrictive Vibration of Prolate Spheroids. Ni-Fe and Ni-Cu Alloys*

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The study of the different magnetic properties determined from the magnetostrictive vibrational behavior of small ferromagnetic prolate spheroids was extended to a large number of Ni-Fe and Ni-Cu alloys. The computed values for the saturation magnetostriction, the total change of Young's modulus with magnetization, and the initial permeability of plain-annealed (i.e., annealed in zero magnetic field) specimens of these materials, were inserted in two independent expressions, derived by Kersten, for the internal stress. The agreement between the two sets of values was quite good. The Ni-Fe alloys were also studied after having been magnetic-annealed; moreover, the 68-percent Ni Permalloy, having exceedingly strain-sensitive properties, was investigated in cold-worked, baked, and annealed conditions. In general, the changes of the calculated internal stress with heat treatment appeared consistent with the corresponding variations of properties such as the initial permeability. Even more revealing in this way were the values for the domain size determined from the computed dissipation factors and internal stresses and found to agree very well with previous measurements of Barkhausen discontinuities. When, for all the plain-annealed specimens, the relative change of Young's modulus was plotted against the relative magnetization, it was discovered that all the curves were almost identical. With the support of previous work, we were able to conclude that these curves should be similar for all plain-annealed face-centered cubic structures at room temperature.

A SMALL centrally-clamped prolate spheroid of a ferromagnetic material was forced magnetostrictively into longitudinal vibration in its fundamental mode. The perturbing agent was a small axial high frequency magnetic field superimposed on a uniform and axial static field. It was shown in a previous paper[1] how the incremental permeability, magnetostriction constant, modulus of elasticity, and dissipation factor of the ferromagnetic spheroid could be accurately computed from the resonance changes of the impedance of the high frequency magnetizing coil. Moreover, it was found that the values obtained for these parameters for a wide composition range of Ni-Fe alloys were in qualitative agreement with previous theory and experiment.

Here, the results are presented for a larger number of compositions and for a variety of heat treatments, and the extent of corroboration with previous work is indicated as quantitatively as possible. More important, on the basis of our results, it is now possible to extend present information on certain properties of ferromagnetics at low magnetizing fields.

All the measurements were made at room temperature and with a high frequency field of 0.35-oersted amplitude.

PLAIN-ANNEAL AND MAGNETIC-ANNEAL

The compositions chosen for study were nickel, the Ni-Fe alloys of atomic percent nickel: 88, 84, 79, 68, 56, 45, and the Ni-Cu alloys of atomic percent nickel: 90, 84, 81, 71. They were all commercially pure, the impurities amounting to less than 0.5 percent.

Structurally, all the above compositions are face-centered cubic. The two alloy systems represent, however, examples of addition to one ferromagnetic metal, of various amounts of either another ferromagnetic or of a nonferromagnetic. It is therefore likely that any statement found to apply to both systems will be applicable with equal validity to most other f.c.c. binary nickel alloys.

A spheroidal specimen of each composition was studied in what hereafter will be called the plain-annealed condition, which is achieved by a three-hour anneal at 700°C in a hydrogen atmosphere followed by a slow cool. The nickel and the Ni-Fe specimens were subsequently heated again to 700°C in hydrogen but

* Assisted by the ONR.
† Now in the Physics Department of the University of Leeds, England.
[1] Beck, Kouvelites, and McKeehan, Phys. Rev. 84, 957 (1951). The following corrections for errors missed in proof should be made in this paper. On p. 958 the last part of Eq. (9) should contain the additional factor ω, the last two parts of Eq. (10) should have the first plus sign changed to a minus sign, the last part of Eq. (11) should contain the additional factor $X_a e^{2i\omega t}$.