The most important effect of these new data is on the discussion in Sec. IV. The "possible" curve in Fig. 7 can now be constructed by extrapolating the $(dP/dT)_M$ curve in helium II from 1.74° to intersect the $(dP/dT)_M$ curve in helium I at 1.764°, so that a better agreement with the "predicted" slope is obtained (using a new $(dP/dT)_\lambda = 54.5$ atmos/deg). A difference of a factor of three or more still exists in the slopes, however, and this seems to be outside experimental error.

Details of the capillary blocking and λ-point-pressure experiments will be published elsewhere.

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Second-Order Radiative Corrections to Hyperfine Structure*

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In this paper the effect of second-order radiative displacements of the energy levels of a bound electron on hyperfine structure separation is examined, and a correction to the Fermi formula for s levels is obtained. The treatment is based on an approximate evaluation of a finite, completely renormalized, and exact expression for the second order energy. The correction, which is restricted to atoms of small αZ, is found to be $[\alpha/2\pi - \alpha^2 Z(5/2 - \ln 2)]E_H$, where E_H is the Fermi energy. The effect of this and other corrections to the Fermi formula on the presently accepted value of the fine structure constant is discussed.

INTRODUCTION

THE interaction of the quantized electromagnetic field with the electron field modifies the properties of the electron and thus causes displacement of the energy levels of electrons bound in atomic or prescribed external fields. The existence of electromagnetic displacements was first recognized through observation of energy level separations, and it is through precise measurement of these separations that the various predictions of quantum electrodynamics are accessible to detailed experimental investigation. Previous theoretical treatments of this problem have been limited by the fact that weak field approximations are made in the process of recognizing and removing charge and mass renormalizations. In the case of the hyperfine structure, the radiative corrections have never been calculated directly, previously obtained results having been inferred from the anomalous magnetic moment of the electron.[1] It has long been realized, however, that a direct evaluation of the second order radiative displacement might yield, in addition to the effect of the second order moment, corrections to the hyperfine structure frequency of order $\alpha^2 Z$. As such corrections arise from the high momentum parts of the external field and electron wave function, their evaluation and comparison with experiment may serve to extend the domain in which the theory has been investigated. Furthermore, in view of the important role played by the hyperfine structure formula in recent determinations of the fine structure constant, it is clear that such corrections may significantly affect the numerical value which one obtains.[2-4]

THE SELF-ENERGY FORMULA

In order to deal systematically with the energy level displacement problem it is desirable to begin with an exact expression for the second order correction. Such expressions in noncovariant form were in fact the starting point of the earliest calculations of the level shift[5,6] and have been given in covariant form by Feynman[7] and Schwinger.[8] We shall briefly derive these here following the methods of Dyson.[9]

As a starting point we consider our theory to be cast in a modified interaction representation in which the effects of the external potential appear in the equations of motion for the field variables, while that of the interaction between the electron and photon fields appears in the interaction Hamiltonian.[10] That is, the development of the wave functional in time is described by

$$\frac{\delta\Psi[\sigma]}{\delta\sigma(x)} = -\frac{i}{\hbar c} H_i(x)\Psi[\sigma], \tag{1}$$

where

$$H_i(x) = -j_\mu(x)A_\mu(x)/c, \tag{2}$$

and

$$j_\mu(x) = \tfrac{1}{2}iec[\bar{\psi}(x)\gamma_\mu\psi(x) - \psi(x)\gamma_\mu{}^T\bar{\psi}(x)]. \tag{3}$$

The operators $\psi(x)$, $\bar{\psi}(x)$, and $A_\mu(x)$, which describe the electron-positron and electromagnetic fields, respectively, satisfy the equations of motion

$$\gamma_\mu\frac{\partial\psi}{\partial x_\mu} + \left[-\frac{ie}{\hbar c}\gamma_\mu A_\mu{}^e(x) + \kappa_0\right]\psi(x) = 0, \tag{4}$$

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$$-\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\mu+\bar{\psi}(x)\left[-\frac{ie}{\hbar c}\gamma_\mu A_\mu{}^e(x)+\kappa_0\right]=0, \quad (5)$$

$$\Box^2 A_\mu(x)=0, \quad (6)$$

and the usual commutation relations. $A_\mu{}^e(x)$ is a specified external potential.

Mass renormalization can be formally inserted into these equations by means of a unitary transformation which takes (2), (4), and (5) into

$$H_i(x)=c^{-1}j_\mu(x)A_\mu(x)-[(\kappa-\kappa_0)/\kappa_0]mc^2\bar{\psi}(x)\psi(x) \quad (2')$$
$$=H_1(x)+H_2(x),$$

$$\gamma_\mu\frac{\partial\psi}{\partial x_\mu}+\left[-\frac{ie}{\hbar c}\gamma_\mu A_\mu{}^e(x)+\kappa\right]\psi(x)=0, \quad (4')$$

$$-\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\mu+\bar{\psi}(x)\left[-\frac{ie}{\hbar c}\gamma_\mu A_\mu{}^e(x)+\kappa\right]=0. \quad (5')$$

Equation (1) has the well-known solution

$$\Psi[t_+]=U(t_+,t_-)\Psi[t_-],$$

where

$$U(t_+,t_-)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar c}\right)^n\int_{t_-}^{t_+}$$
$$\times P[H_i(x_1),\cdots,H_i(x_n)]d_4x_1\cdots d_4x_n.$$

Second-order radiative corrections are contained in

$$U^{(2)}(t_+,t_-)=\frac{1}{2}\left(\frac{-i}{\hbar c}\right)^2\int_{t_-}^{t_+}d_4x_1d_4x_2$$
$$\times P[H_1(x_1),H_1(x_2)]+\frac{-i}{\hbar c}\int_{t_-}^{t_+}H_2(x_1)d_4x_1,$$

since H_2 is already of second order in the coupling constant. The second-order correction to the energy of an electron in a stationary state a of the external potential is given by the relation

$$-(i/\hbar)(t_+-t_-)\Delta E^{(2)}$$
$$=\text{imaginary part }(\Psi_a,U(t_+,t_-)\Psi_a), \quad (7)$$

where Ψ_a is the wave functional of the uncoupled electron photon system corresponding to the presence of a single electron in the specified state and no photons. The time difference (t_+-t_-) is to be taken large, and any oscillating dependence on the limits t_+ and t_- is to be ignored. Equation (7) simply states that a change in the energy of the state Ψ_a manifests itself by a change in phase. The real part of the expression on the right side of (7) corresponds to the decay of the state Ψ_a arising from a transition to a lower energy state accompanied by emission of a photon.

The expression (7) can be conveniently analyzed by means of Feynman diagrams in a manner entirely analogous to the corresponding problem for a free electron. One obtains

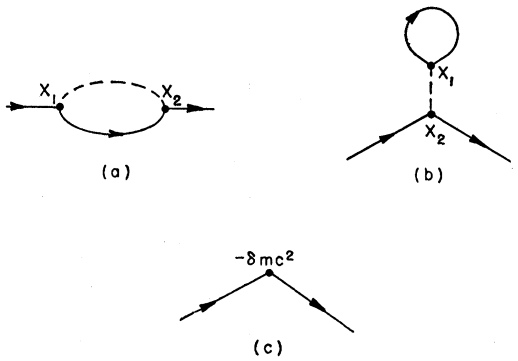

FIG. 1. Feynman diagrams for the self-energy of a bound electron.

$$-(i/\hbar)(t_+-t_-)\Delta E=$$
$$-\left(\frac{-i}{\hbar c}\right)^2(-ie)^2\frac{\hbar c}{4}\int_{t_-}^{t_+}\bar{\phi}_a(x_2)\gamma_\mu S_F{}^e(x_2,x_1)\gamma_\mu$$
$$\times\phi_a(x_1)D_F(x_2-x_1)d_4x_1d_4x_2$$
$$+\left(\frac{-i}{\hbar c}\right)^2(-ie)^2\frac{\hbar c}{4}\int_{t_-}^{t_+}\bar{\phi}_a(x_2)\gamma_\mu\phi(x_2)D_F(x_2-x_1)$$
$$\times\mathrm{Tr}[\gamma_\mu S_F{}^e(x_1,x_1)]d_4x_2d_4x_1$$
$$+\frac{i}{\hbar c}\frac{\kappa-\kappa_0}{\kappa_0}mc^2\int_{t_-}^{t_+}\bar{\phi}_a(x_1)\phi_a(x_1)d_4x_1, \quad (8)$$

corresponding to the diagrams in Fig. 1. $\phi_a(x)$ and $\bar{\phi}_a(x)$ are the normalized c-number solutions of the Dirac equation and its adjoint (4′ and 5′) corresponding to a designated stationary state. $D_F(x_2-x_1)$ is the photon propagation function defined by Feynman and Dyson (the notation and normalization are Dyson's), while $S_F{}^e(x_2,x_1)$ is defined in a manner analogous to the free electron propagation function by

$$\tfrac{1}{2}S_F{}^e(x_2,x_1)_{\alpha\beta}=\langle P[\psi_\alpha(x_2),\bar{\psi}_\beta(x_1)]\rangle_0\epsilon(x_1,x_2),$$

where $\langle\ \rangle_0$ denotes the vacuum expectation value.[11]

We shall refer to the energy represented by diagram (a) as the fluctuation energy, $\Delta E_F{}^{(a)}$. It contains a mass-like part which is to be just canceled by the energy corresponding to diagram (c).

The energy corresponding to diagram (b) will be referred to as the polarization energy. It can be written as[12]

$$\Delta E_P=\frac{-ie}{c(t_+-t_-)}\int_{t_-}^{t_+}\bar{\phi}_a(x_1)\gamma_\mu A_\mu{}^P(x_1)\phi_a(x_1)d_4x_1, \quad (9)$$

[11] Note that the vacuum state with respect to the matter field is defined by the condition $\psi^+(x)\Psi_0=0$ and $\bar{\psi}^-(x)\Psi_0=0$ and differs from the vacuum state used in the free particle interaction representation.

[12] $S_F{}^e(x_2,x_2)$ is to be understood as $\frac{1}{2}\lim_{x_2',x_2''\to x_2}[S_F{}^e(x_2',x_2'')+S_F{}^e(x_2'',x_2')]$ where x_2' is earlier, x_2'' is later than x_2. [See Schwinger, Phys. Rev. 82, 664 (1951).] This definition insures

where

$$A_\mu{}^P(x_1)=(i/2c)\int D_F(x_1-x_2)j_\mu{}^P(x_2)d_4x_2, \quad (10)$$

and

$$j_\mu{}^P(x_2)=\tfrac{1}{2}iec\ \mathrm{Tr}[\gamma_\mu S_F{}^e(x_2,x_2)]=\langle j_\mu(x_2)\rangle_0. \quad (11)$$

An explicit form for $S_F{}^e(x_2, x_1)$ in terms of the stationary states of the external potential is given by

$$\tfrac{1}{2}S_F{}^e(x_2,x_1)_{\alpha\beta}$$
$$=\sum_{\substack{r\\E_r>0}}(\phi_r(x_2))_\alpha(\bar\phi_r(x_1))_\beta;\quad (x_2-x_1)_0>0,$$
$$=-\sum_{\substack{\rho\\E_\rho<0}}(\phi_\rho(x_2))_\alpha(\bar\phi_\rho(x_1))_\beta;\quad (x_2-x_1)_0<0. \quad (12)$$

Use of this form in (8) followed by integration over the time variables in d_4x_1 and d_4x_2 reduces the above formula for ΔE to the possibly more familiar form used by Kroll and Lamb[5] and French and Weisskopf.[6]

We shall, in fact, find it convenient to carry out one time integration at this time. In view of (12), $S_F{}^e(x_2, x_1)$ depends upon the time-like components of x_1 and x_2 only through their difference $x_0=(x_2-x_1)_0$. This same remark applies, therefore, to the entire integrands of 8(a) and 8(b), while the integrand of 8(c) is independent of the time-like component. Thus in the limit $t_+\to\infty$, $t_-\to-\infty$ one time integration may be performed immediately, yielding

$$\Delta E=\text{real part}\Big\{-i\alpha\pi\hbar c\int\bar\phi_a(x_2)\gamma_\mu S_F{}^e(x_2,x_1)\gamma_\mu$$
$$\times\phi_a(x_1)D_F(x_2-x_1)d\mathbf{x}_1d\mathbf{x}_2dx_0$$
$$-ie\int\bar\phi_a(x_1)\gamma_\mu A_\mu{}^P(x_1)\phi_a(x_1)d\mathbf{x}_1$$
$$-\frac{\kappa-\kappa_0}{\kappa_0}mc^2\int\bar\phi_a(x_1)\phi_a(x_1)d\mathbf{x}_1\Big\}. \quad (13)$$

REDUCTION OF THE SELF-ENERGY FORMULA

Although the expression (13) is exact, $\Delta E^{(a)}$ contains an infinite mass term which must be isolated and canceled against $\Delta E^{(c)}$, while $\Delta E^{(b)}$ contains an infinite term corresponding to a charge renormalization which must be recognized and removed. The renormalization program will be carried through in such a way as to yield an expression which is still exact and whose individual terms are finite and well defined.

We first rewrite our expressions in momentum space by introducing the appropriate Fourier transforms defined by[13]

$$\phi_a(x)=\int\phi_a(\mathbf{p})\delta(p_0-p_a)e^{ip\cdot x}d_4p,$$
$$\bar\phi_a(x)=\int\bar\phi_a(\mathbf{p})\delta(p_0-p_a)e^{-ip\cdot x}d_4p,$$
$$S_F{}^e(x_2,x_1)=\int S_F{}^e(p_2,p_1)\delta[(p_2-p_1)_0]$$
$$\times e^{i(p_2\cdot x_2-p_1\cdot x_1)}d_4p_1d_4p_2, \quad (14)$$
$$D_F(x)=\int D_F(k)e^{ik\cdot x}d_4k=\frac{-2i}{(2\pi)^4}\int\frac{e^{ik\cdot x}}{k^2}d_4k,$$
$$A_\mu{}^e(x)=\int A_\mu{}^e(\mathbf{q})\delta(q_0)e^{iq\cdot x}d_4q,$$
$$A_\mu{}^P(x)=\int A_\mu{}^P(\mathbf{q})\delta(q_0)e^{iq\cdot x}d_4q.$$

Inserting these in (13), carrying out the coordinate space integrations, and the trivial momentum space integrations, we find

$$\Delta E=\mathrm{Re}\Big\{-\alpha\hbar c(2\pi)^4\int\bar\phi_a(\mathbf{p}_2)\gamma_\mu S_F{}^e(p_2-k,p_1-k)$$
$$\times\gamma_\mu\phi_a(\mathbf{p}_1)\frac{d_4k}{k^2}d\mathbf{p}_1d\mathbf{p}_2$$
$$-ie(2\pi)^3\int\bar\phi_a(\mathbf{p}_2)\gamma_\mu A_\mu{}^P(p_2-p_1)\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2$$
$$-\frac{\kappa-\kappa_0}{\kappa}mc^2(2\pi)^3\int\bar\phi_a(\mathbf{p})\phi_a(\mathbf{p})d\mathbf{p}\Big\}. \quad (15)$$

(a) Renormalization of the Polarization Energy

We first note that as a consequence of its definition and the equations of motion (4′) and (5′), $S_F{}^e(x_2, x_1)$ satisfies

$$\gamma_\mu\frac{\partial S_F{}^e}{\partial x_{2\mu}}+\left[-\frac{ie}{\hbar c}\gamma_\mu A_\mu{}^e(x_2)+\kappa\right]S_F{}^e(x_2,x_1)=2i\delta_4(x_2-x_1),$$

and (16)

$$-\frac{\partial S_F{}^e}{\partial x_{1\mu}}\gamma_\mu+S_F{}^e(x_2,x_1)\left[-\frac{ie}{\hbar c}\gamma_\mu A_\mu{}^e(x_1)+\kappa\right]$$
$$=2i\delta_4(x_2-x_1),$$

that $\langle j_\mu(x_2)\rangle_0=\frac{1}{2}iec\ \mathrm{Tr}(\gamma_\mu S_F(x_2,x_2)]=0$ for zero external field. It is for this reason that diagrams like (b) are omitted from the usual Feynman rules, which apply for a representation in which the matter field operators obey a free particle equation of motion.

[13] The following remarks concerning the notation may be useful. Four vectors will be denoted by ordinary type (p, k, etc.) while boldface type ($\mathbf{p}$, $\mathbf{k}$, etc.) will be used to denote their respective space parts. The notations $p_\mu k_\mu$ and $p\cdot k$ will be used interchangeably to denote the four dimensional scalar product, so that $p_\mu k_\mu=p\cdot k=\mathbf{p}\cdot\mathbf{k}-p_0k_0$. It is hoped that the use of the same symbol for a function and its Fourier transform will not be a source of confusion. They are, of course, not the same functions of their respective arguments.

which in momentum space take the form

$$(i\gamma\cdot p_2+\kappa)S_F^e(p_2,p_1)$$
$$=\frac{ie}{\hbar c}\int\gamma\cdot A^e(p_2-p_3)S_F^e(p_3,p_1)d\mathbf{p}_3+\frac{2i}{(2\pi)^4}\delta_3(\mathbf{p}_2-\mathbf{p}_1),$$

and (17)

$$S_F^e(p_2,p_1)(i\gamma\cdot p_1+\kappa)$$
$$=\frac{ie}{\hbar c}\int S_F^e(p_2,p_3)\gamma\cdot A^e(p_3-p_1)d\mathbf{p}_3+\frac{2i}{(2\pi)^4}\delta_3(\mathbf{p}_2-\mathbf{p}_1).$$

Iteration of these equations in combination then yields

$$\begin{aligned}S_F^e(p_2,p_1)&=\frac{-2i}{(2\pi)^4}\Big[\frac{i\gamma\cdot p_2-\kappa}{p_2^2+\kappa^2}\delta_3(\mathbf{p}_2-\mathbf{p}_1)-\Big(\frac{ie}{\hbar c}\Big)\frac{i\gamma\cdot p_2-\kappa}{p_2^2+\kappa^2}\\&\quad\times\gamma\cdot A^e(p_2-p_1)\frac{i\gamma\cdot p_1-\kappa}{p_1^2+\kappa^2}\Big]\\&\quad+\Big(\frac{ie}{\hbar c}\Big)^2\frac{i\gamma\cdot p_2-\kappa}{p_2^2+\kappa^2}\Big[\int\gamma\cdot A^e(\mathbf{q})\\&\quad\times S_F^e(p_2-q,p_1+q')\gamma\cdot A^e(q')d\mathbf{q}d\mathbf{q}'\Big]\frac{i\gamma\cdot p_1-\kappa}{p_1^2+\kappa^2}\\&=S_F(p_2)\delta_3(\mathbf{p}_2-\mathbf{p}_1)-\Big(\frac{(2\pi)^4}{-2i}\Big)\Big(\frac{ie}{\hbar c}\Big)S_F(p_2)\\&\quad\times\gamma\cdot A^e(p_2-p_1)S_F(p_1)\\&\quad+\Big(\frac{ie}{\hbar c}\Big)^2\Big(\frac{(2\pi)^4}{-2i}\Big)^2S_F(p_2)\Big[\int\gamma\cdot A^e(\mathbf{q})\\&\quad\times S_F^e(p_2-q,p_1+q')\gamma\cdot A^e(\mathbf{q}')d\mathbf{q}d\mathbf{q}'\Big]S_F(p_1).\end{aligned}\quad(18)$$

The definition of $A_\mu^P(x)$ given by (10) and (11) yields for $A_\mu^P(\mathbf{q})$,

$$A_\mu^P(\mathbf{q})=-e(2\pi)^4D_F(q)\int\mathrm{Tr}[\gamma_\mu S_F^e(p,p-q)]d_4p.\quad(19)$$

On inserting the expression (18) for S_F^e in (19) one finds that the contribution from the first term vanishes, while that of the second is identical to the weak field polarization potential which has been computed many times before. Using this result in the form quoted by Karplus and Kroll[14] we have

$$A_\mu^P(\mathbf{q})=\frac{\alpha}{2\pi}AA_\mu^e(\mathbf{q})+\frac{\alpha}{2\pi}q^2\int_0^1dv\frac{2v^2(1-\frac{1}{3}v^2)}{4\kappa^2+q^2(1-v^2)}\times A_\mu^e(\mathbf{q})+\delta A_\mu^P(\mathbf{q}).\quad(20)$$

[14] R. Karplus and N. M. Kroll, Phys. Rev. **77**, 540 (1950). Equation (14).

The first term of (20), which contains the infinite constant factor A, is the well-known charge renormalization term and is to be henceforth ignored. $\delta A_\mu^P(q)$ is the part of the polarization potential which arises from the third term of (18) and is of importance only in the case of a strong external field. It is, however, finite and entirely free of effects to be attributed to renormalization. Using (20) but omitting the renormalization term we find for the second-order polarization energy, the finite exact expression,

$$\begin{aligned}\Delta E_P&=-ie(2\pi)^2\alpha\int_0^1dv\int\bar\phi_a(\mathbf{p}_2)\\&\quad\times\frac{2v^2(1-\frac{1}{3}v^2)(p_2-p_1)^2\gamma\cdot A^e(p_2-p_1)}{4\kappa^2+(p_2-p_1)^2(1-v^2)}\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2\\&\quad-ie(2\pi)^3\int\bar\phi_a(p_2)\gamma\cdot\delta A^P(p_2-p_1)\phi_a(p_1)d\mathbf{p}_1d\mathbf{p}_2.\end{aligned}\quad(21)$$

(b) Rearrangement of the Fluctuation Energy

The fluctuation energy can be treated in a manner entirely analogous to our treatment of the polarization energy. That is, $S_F^e(p_2,p_1)$ appearing in $\Delta E^{(a)}$ can be replaced by the right side of (18). The first term then contains the mass term which one cancels with $\Delta E^{(c)}$, while the first and second terms each contain an infinite charge term which can be isolated and are found to just cancel. The remaining expression is then finite and exact. This particular separation of terms, while straightforward, turns out to be unsuited to the approximations which we shall subsequently make. We use instead a separation of terms which is due to Feynman[15] and is based on the pair of identities,[16]

$$\gamma_\mu=-[i\gamma\cdot(p_1-k)+\kappa]\frac{2ip_{1\mu}-i\gamma\cdot k\gamma_\mu}{k^2-2p_1\cdot k}+\frac{2ip_{1\mu}-i\gamma\cdot k\gamma_\mu}{k^2-2p_1\cdot k}[i\gamma\cdot p_1+\kappa],\quad(22a)$$

$$\gamma_\mu=[i\gamma\cdot p_2+\kappa]\frac{2ip_{2\mu}-i\gamma_\mu\gamma\cdot k}{k^2-2p_2\cdot k}-\frac{2ip_{2\mu}-i\gamma_\mu\gamma\cdot k}{k^2-2p_2\cdot k}[i\gamma\cdot(p_2-k)+\kappa].\quad(22b)$$

[15] R. P. Feynman, private communication. The authors are indebted to Professor Feynman for describing this method to them.

[16] It is to be noted that each of the two terms appearing in these identities are singular at the zeros of the denominator. It is convenient to displace this singularity from the real axis by replacing $k^2-2p\cdot k$ by $k^2-2p\cdot k-i\epsilon$, where ϵ is a small positive quantity. It is then possible to treat these denominators on the same footing as those arising from $D_F(k)$ and $S_F(p)$ and thus to apply Feynman's contour integral method to the evaluation of the k-space integrals. While ϵ will not be explicitly displayed in any of our formulas, it is to be tacitly understood that these quantities are to be set equal to zero only after the entire calculation is completed.

We insert these in expression (15) for $\Delta E_F^{(a)}$, the form (a) for the γ_μ on the right, the form (b) for the γ_μ on the left. Each free electron operator, $i\gamma\cdot p+\kappa$, then appears with the appropriate momentum and adjacent to a wave function or the propagation function $S_F{}^e$. Now in addition to Eq. (17) for $S_F{}^e$ we have

$$(i\gamma\cdot p_1+\kappa)\phi_a(\mathbf{p}_1)=\frac{ie}{\hbar c}\int\gamma\cdot A^e(\mathbf{p}_1-\mathbf{p}_1')\phi_a(\mathbf{p}_1')d\mathbf{p}_1',$$
$$\bar\phi_a(\mathbf{p}_2)(i\gamma\cdot p_2+\kappa)=\frac{ie}{\hbar c}\int\bar\phi_a(\mathbf{p}_2')\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_2')d\mathbf{p}_2'. \tag{23}$$

Applying these as well as (17) we obtain[17]

$$\Delta E^{(a)}=C+L+Q, \tag{24}$$

where

$$C=4i\hbar c\alpha\int\bar\phi_a(\mathbf{p})\frac{i\gamma\cdot(p+k)}{k^2-2p\cdot k}\phi_a(\mathbf{p})\frac{d_4k}{k^2}d\mathbf{p}, \tag{25}$$

$$L=-2i\hbar c\alpha\left(\frac{ie}{\hbar c}\right)\Big[\int\bar\phi_a(\mathbf{p}_2)\frac{2ip_{2\mu}-i\gamma_\mu\gamma\cdot k}{k^2-2p_2\cdot k}\times\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\frac{2ip_{1\mu}-i\gamma\cdot k\gamma_\mu}{k^2-2p_1\cdot k}\phi_a(p_1)\frac{d_4k}{k^2}d\mathbf{p}_1d\mathbf{p}_2$$
$$+4\int\bar\phi_a(\mathbf{p}_2)\frac{k^2-p_2\cdot k+p_2{}^2}{(k^2-2p_2\cdot k)^2}\times\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\phi_a(\mathbf{p}_1)\frac{d_4k}{k^2}d\mathbf{p}_1d\mathbf{p}_2\Big], \tag{26}$$

$$Q=-\hbar c\alpha(2\pi)^4\left(\frac{ie}{\hbar c}\right)^2\int\bar\phi_a(\mathbf{p}_2)\mathfrak{A}_\mu'(q_2,p_2,k)\times S_F{}^e(p_2-k-q_2,p_1-k+q_1)\mathfrak{A}_\mu(q_1,p_1,k)\times\phi_a(\mathbf{p}_1)\frac{d_4k}{k^2}d\mathbf{p}_1d\mathbf{p}_2d\mathbf{q}_1d\mathbf{q}_2, \tag{27}$$

where,

$$\mathfrak{A}_\mu(q_1,p_1,k)=-\gamma\cdot A^e(\mathbf{q}_1)\frac{2ip_{1\mu}-i\gamma\cdot k\gamma_\mu}{k^2-2p_1\cdot k}+\frac{2i(p_1+q_1)_\mu-i\gamma\cdot k\gamma_\mu}{k^2-2(p_1+q_1)\cdot k}\gamma\cdot A^e(\mathbf{q}_1), \tag{28}$$

$$\mathfrak{A}_\mu'(q_2,p_2,k)=-\gamma\cdot A^e(q_2)\frac{2ip_{2\mu}-i\gamma_\mu\gamma\cdot k}{k^2-2p_2\cdot k}+\frac{2i(p_2-q_2)_\mu-i\gamma_\mu\gamma\cdot k}{k^2-2(p_2-q_2)\cdot k}\gamma\cdot A^e(\mathbf{q}_2). \tag{29}$$

We now proceed to a rearrangement of (24) with the objective of isolating and removing renormalizations.

From (25),

$$C=4i\hbar c\alpha\int\bar\phi_a(\mathbf{p})I(p)\phi_a(\mathbf{p})d\mathbf{p},$$

with

$$I(p)=\int\frac{i\gamma\cdot(p+k)}{(k^2-2p\cdot k)}\frac{d_4k}{k^2}.$$

Following the procedure of Karplus and Kroll,[18] we find

$$I(p)=\int dxd_4k\left[\frac{i\gamma\cdot p(1+x)}{(k^2-p^2x^2)^2}+xi\gamma\cdot p\frac{x^2p^2}{(k^2-p^2x^2)^3}\right]$$
$$=-\kappa_0\left[\frac{9i\pi^2}{4}+\frac{3}{2}\int\frac{d_4k}{(k^2+\kappa^2)^2}\right]+(i\gamma\cdot p+\kappa)\left[-\frac{3}{4}i\pi^2+\frac{3}{2}\int\frac{d_4k}{(k^2+\kappa^2)^2}\right]$$
$$-\frac{3}{2}i\pi^2\int_0^1\frac{[i\gamma\cdot p+\kappa][i\gamma\cdot p(1-2z)-2\kappa(1-z)][i\gamma\cdot p+\kappa]}{\kappa^2-(p^2+\kappa^2)z}dz. \tag{30}$$

Inserting (30) into (25) and applying (23) we find

$$C=\frac{\kappa-\kappa_0}{k_0}mc^2\int\bar\phi_a(\mathbf{p})\phi_a(\mathbf{p})d\mathbf{p}+4\hbar c\alpha\left[\frac{3}{4}\pi^2+\frac{3}{2}i\int\frac{d_4k}{[k^2+\kappa^2]^2}\right]\left(\frac{ie}{\hbar c}\right)\int\bar\phi_a(\mathbf{p}_2)\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\phi_a(\mathbf{p}_1)d\mathbf{p}_2d\mathbf{p}_1$$
$$+6\pi^2\hbar c\alpha\left(\frac{ie}{\hbar c}\right)^2\int\bar\phi_a(\mathbf{p}_2)\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_3)\frac{[i\gamma\cdot p_3(1-2z)-2\kappa(1-z)]}{\kappa^2-(p_3{}^2+\kappa^2)z}\gamma\cdot A^e(\mathbf{p}_3-\mathbf{p}_1)\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}_3dz, \tag{31}$$

$$\frac{\kappa-\kappa_0}{\kappa_0}=\frac{\alpha}{(2\pi)^3}\left[\frac{9\pi^2}{4}-\frac{3i}{2}\int\frac{d_4k}{[k^2+\kappa^2]^2}\right].$$

[17] It is worth noting that the form of (24) is very similar to that of the expression which would have been obtained had (18) been substituted for $S_F{}^e(p_2,p_1)$ in $\Delta E^{(a)}$. The essential difference is the appearance of the factors $\mathfrak{A}$, $\mathfrak{A}'$ in Q in place of $i\gamma\cdot A^e(\mathbf{q}_1)\times S_F(p_1-k)\gamma_\mu$ and $\gamma_\mu S_F(p_2-k)i\gamma\cdot A^e(\mathbf{q}_2)$. The significance of this difference will be discussed later.

[18] See reference 14, Eqs. (16) to (23). The expression $I(p)$ above is very similar to $I(i\gamma\cdot p+\kappa)$ there evaluated.

The first term is the well-known divergent mass term and is just canceled by $\Delta E^{(c)}$. The second term is also divergent and corresponds to a modification of the electron's charge. It will be canceled by a similar term in L.

Applying similar methods to (26) we find

$$L=-4\hbar c\alpha\left[\frac{3}{4}\pi^2+\frac{3}{2}i\int\frac{d_4k}{[k^2+\kappa^2]^2}\right]\left(\frac{ie}{\hbar c}\right)\times\int\bar{\phi}_a(\mathbf{p}_2)\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2$$
$$-6\pi^2\hbar c\alpha\left(\frac{ie}{\hbar c}\right)^2\int\bar{\phi}_a(\mathbf{p}_2)\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_3)\times\frac{[i\gamma\cdot p_3-\kappa]}{\kappa^2-(p_3^2+\kappa^2)z}\gamma\cdot A^e(\mathbf{p}_3-\mathbf{p}_1)\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}_3dz$$
$$+2\pi^2\hbar c\alpha\left(\frac{ie}{\hbar c}\right)\int\bar{\phi}_a(\mathbf{p}_2)A_\mu{}^e(\mathbf{p}_2-\mathbf{p}_1)\times K_\mu(p_2,p_1)\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2+L_D, \quad (32)$$

where[19]

$$K_\mu(p_2,p_1)=\frac{(p_2-p_1)^2}{\kappa^2}\gamma_\mu\int_0^1\left(\frac{2-6y(1-y)}{\Lambda_1{}^2}-\frac{y(1-y)}{\Lambda_2{}^2}\right)dydz+\frac{i(p_2-p_1)_\nu\sigma_{\mu\nu}}{\kappa}\int_0^1\frac{dy}{\Lambda_1{}^2}$$
$$+\int_0^1\frac{dy}{\kappa^2\Lambda_1{}^2}\{(5/2)(1-2y)(p_2{}^2-p_1{}^2)\gamma_\mu+(i\gamma\cdot p_2+\kappa)\gamma_\mu(i\gamma\cdot p_1+\kappa)-\tfrac{1}{2}(i\gamma\cdot p_2+\kappa)^2\gamma_\mu-\tfrac{1}{2}\gamma_\mu(i\gamma\cdot p_1+\kappa)^2-iy(p_2-p_1)_\nu\sigma_{\mu\nu}(i\gamma\cdot p_1+\kappa)+i(1-y)(i\gamma\cdot p_2+\kappa)(p_2-p_1)_\nu\sigma_{\mu\nu}\}$$
$$+\int_0^1\frac{dydz}{\kappa^2\Lambda_2{}^2}y(p_2{}^2-p_1{}^2)\gamma_\mu, \quad (33)$$

with

$$\Lambda_1{}^2=1-\kappa^{-2}[(p_1{}^2+\kappa^2)y+(p_2{}^2+\kappa^2)(1-y)-(p_2-p_1)^2y(1-y)],$$
$$\Lambda_2{}^2=1-\kappa^{-2}[(p_1{}^2+\kappa^2)yz+(p_2{}^2+\kappa^2)(1-yz)-(p_2-p_1)^2yz(1-y)].$$

[19] In writing out (33) we have omitted terms of the form $q\cdot A^e(\mathbf{q})$. We are at liberty to choose our gauge so that this quantity vanishes and shall do so.

L_D is that part of L which diverges in the infrared. It is given by

$$L_D=-8i\hbar c\alpha\left(\frac{ie}{\hbar c}\right)\int\bar{\phi}_a(\mathbf{p}_2)\frac{p_{2\mu}}{k^2-2p_2\cdot k}\times\left(\frac{p_{2\mu}}{k^2-2p_2\cdot k}-\frac{p_{1\mu}}{k^2-2p_1\cdot k}\right)\gamma\cdot A^e(p_2-p_1)\times\phi_a(\mathbf{p}_1)\frac{d_4k}{k^2}d\mathbf{p}_2d\mathbf{p}_1. \quad (34)$$

There is, of course, a compensating divergence in Q. In order to make evident the fact that $Q+L_D$ is finite, we note that L_D may also be written

$$L_D=\hbar c\alpha(2\pi)^4\left(\frac{ie}{\hbar c}\right)^2\int\bar{\phi}_a(\mathbf{p}_2)\left[\frac{2i(p_2-q_2)_\mu}{k^2-2(p_2-q_2)\cdot k}-\frac{2ip_{2\mu}}{k^2-2p_2\cdot k}\right]\gamma\cdot A^e(\mathbf{q}_2)S_F{}^e(p_2-q_2,p_1+q_1)\times\left[\frac{2i(p_1+q_1)_\mu}{k^2-2(p_1+q_1)\cdot k}-\frac{2ip_{1\mu}}{k^2-2p_1\cdot k}\right]\times\gamma\cdot A^e(\mathbf{q}_1)\phi_a(\mathbf{p}_1)\frac{d_4k}{k}d\mathbf{p}_1d\mathbf{p}_2d\mathbf{q}_1d\mathbf{q}_2. \quad (35)$$

In order to verify the identity of (34) and (35) one need merely note that (17) and (22) imply

$$(ie/\hbar c)\int\bar{\phi}(\mathbf{p}_2)[f(p_2)-f(p_3)]\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_3)\times S_F{}^e(p_3,p)d\mathbf{p}_2d\mathbf{p}_3=-[2i/(2\pi)^4]\bar{\phi}(\mathbf{p})f(p),$$

where $f(p)$ is any function of momentum which is diagonal in the spinor indices. The coefficient of (35) is equal in magnitude and opposite in sign to that of Q, while it is apparent that in the limit of low k_μ, the two integrands become identical.

We obtain finally for the fluctuation energy

$$\Delta E_F=\Delta E^{(a)}+\Delta E^{(c)}=2\pi^2\hbar c\alpha\left(\frac{ie}{\hbar c}\right)\int\bar{\phi}_a(\mathbf{p}_2)\times A_\mu{}^e(\mathbf{p}_2-\mathbf{p}_1)K_\mu(p_2,p_1)\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2$$
$$-6\pi^2\hbar c\alpha\left(\frac{ie}{\hbar c}\right)^2\int\bar{\phi}_a(\mathbf{p}_2)\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_3)\times\frac{[2i\gamma\cdot p_3z+\kappa(1-2z)]}{\kappa^2-(p_3{}^2+\kappa^2)z}\gamma\cdot A^e(\mathbf{p}_3-\mathbf{p}_1)\times\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}_3dz+(L_D+Q). \quad (36)$$

THE FORM OF THE EXTERNAL POTENTIAL

In order to apply the general expressions derived above to the problem of hyperfine structure it is necessary to specify the form of the external potential. We shall confine our attention to hydrogenic atoms; for these the potential is given by

$$\frac{e}{\hbar c}A_\mu{}^e(r)=-\frac{i\alpha Z}{r}\delta_{\mu 4}-\frac{e}{\hbar c}\frac{\mathfrak{y}}{4\pi}\times\nabla\left(\frac{1}{r}\right),$$

where α is the fine structure constant, Z the atomic number, and $\mathfrak{y}$ the nuclear magnetic moment operator. The first term is simply the Coulomb potential produced by the nuclear charge, while the second is the magnetic dipole potential produced by the nuclear magnetic moment. $\mathfrak{y}$ can be conveniently expressed in terms of the gyromagnetic ratio, g, the nuclear magneton, $|e|\hbar/2Mc$, and the nuclear spin operator, $\mathbf{\Sigma}$, by $\mathfrak{y}=g(|e|\hbar/2Mc)\mathbf{\Sigma}$, yielding

$$\frac{e}{\hbar c}A_\mu{}^e(\mathbf{r})=-\frac{i\alpha Z}{r}\delta_{\mu 4}+\frac{\alpha}{2\kappa}\frac{m}{M}g\mathbf{\Sigma}\times\nabla\left(\frac{1}{r}\right). \quad (37)$$

From (14) one easily finds

$$\frac{e}{\hbar c}A_\mu{}^e(\mathbf{q})=-\frac{i\alpha Z}{2\pi^2q^2}\delta_{\mu 4}+\frac{\alpha}{2\kappa}\frac{m}{M}g\frac{i\mathbf{\Sigma}\times\mathbf{q}}{2\pi^2q^2}$$

$$=\frac{e}{\hbar c}A_\mu{}^{(E)}(\mathbf{q})+\frac{e}{\hbar c}A_\mu{}^{(H)}(\mathbf{q}), \quad (38)$$

where the superscripts (E) and (H) refer to the electric and magnetic parts of the potential, respectively. The Fermi formula for the hyperfine structure together with the Breit relativistic correction is obtained simply by treating the magnetic potential as a small perturbation, thus

$$E_H=-ie(2\pi)^3\int\bar{\phi}_a(\mathbf{p}_2)\gamma\cdot A^{(H)}(p_2-p_1)\phi_a(\mathbf{p}_1)d\mathbf{p}_2d\mathbf{p}_1.$$

For S states, which will be our principal concern here, this formula yields [20]

$$E_H=\tfrac{2}{3}\pi\alpha g(m/M)(\phi_0{}^2/\kappa^3)(1+b_n\alpha^2Z^2)\langle\boldsymbol{\sigma}\cdot\mathbf{\Sigma}\rangle mc^2, \quad (39)$$

where ϕ_0 is the value of the Schrödinger wave function at the origin. The energy for the higher angular momentum states is of the same order of magnitude.

In considering radiative corrections to (39) we shall confine ourselves to terms which are linear in the nuclear magnetic moment. These corrections will be expressed as multiples of E_H and will be confined to terms of order α and α^2Z.

[20] E. Fermi, Z. Physik **60**, 320 (1930); and G. Breit, Phys. Rev. **35**, 1447 (1930). The subscript n in b_n refers to the principal quantum number of the state in question. Breit's results imply $b_1=3/2$, $b_2=17/8$.

EVALUATION OF THE POLARIZATION ENERGY

The polarization energy appears as the expectation value of the polarization potential for the state in question. The polarization potential can be regarded as consisting of a Coulomb part, which is independent of the nuclear moment, and a magnetic part which involves the nuclear moment linearly. The magnetic part is simply a modification of the nuclear magnetic field and can accordingly be expected to produce a modification in the hyperfine structure. There is, however, also an effect from the Coulomb part which arises from the fact that the lower energy hyperfine structure states are the more tightly bound, and therefore spend more time in the region where the Coulomb polarization potential is large. These two effects are, in fact, exactly equal, and we shall arrange our calculation in a way which makes this apparent at an early stage.

We consider first the contribution of the first term of (21),

$$\Delta E_P{}'=-\hbar c\alpha(2\pi)^2\left(\frac{ie}{\hbar c}\right)\int_0^1 dv\int\bar{\phi}_a(\mathbf{p}_2)\times\frac{2v^2(1-\frac{1}{3}v^2)(p_2-p_1)^2\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)}{4\kappa+(p_2-p_1)^2(1-v^2)}\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2$$

$$=-\hbar c\alpha(2\pi)^2\left(\frac{ie}{\hbar c}\right)\int_0^1 dv2v^2(1-\tfrac{1}{3}v^2)\times\int\bar{\phi}_a(\mathbf{p}_2)\frac{\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\phi_a(\mathbf{p}_1)}{4\kappa^2+(p_2-p_1)^2(1-v^2)}[(p_2{}^2+\kappa^2)+(p_1{}^2+\kappa^2)-2(\kappa^2-p_a{}^2)-2\mathbf{p}_1\cdot\mathbf{p}_2]d\mathbf{p}_1d\mathbf{p}_2,$$

where p_a is the time like component of p_1 and p_2 and is determined by the state. For hyperfine structure the third term of the above expression is, apart from a numerical factor, essentially $\Delta E_H\times(\kappa^2-p_a{}^2)\alpha$. $(\kappa^2-p_a{}^2)$ is of course proportional to the binding energy or $(\alpha Z)^2$, so that this term is a correction of order $\alpha^3Z^2E_H$ and can be ignored. The contribution of the last term to hyperfine structure is also of order $\alpha^3Z^2E_H$. Now, noting that $p^2+\kappa^2=-(i\gamma\cdot p-\kappa)(i\gamma\cdot p+\kappa)$ and recalling (23), we find that the first two terms yield

$$\Delta E_P{}'=\hbar c\alpha(2\pi)^2\left(\frac{ie}{\hbar c}\right)^2\int_0^1 dv2v^2(1-\tfrac{1}{3}v^2)\times\Big[\int\bar{\phi}_a(\mathbf{p}_2)\frac{\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_3)[i\gamma\cdot p_3-\kappa]\gamma\cdot A^e(\mathbf{p}_3-\mathbf{p}_1)}{4\kappa^2+(p_3-p_1)^2(1-v^2)}\times\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}_3$$

$$+\int\bar{\phi}_a(\mathbf{p}_2)\frac{\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_3)[i\gamma\cdot p_3-\kappa]\gamma\cdot A^e(\mathbf{p}_3-\mathbf{p}_1)}{4\kappa^2+(p_3-p_2)^2(1-v^2)}\times\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}_3\Big]. \quad (40)$$

Equation (40) involves the external potential quadratically and makes no reference to which factor $A_\mu{}^e$ appeared originally in $\Delta E_P'$ and which factor arose from application of the wave equation. It is therefore clear that the two effects discussed in the first paragraph are equal. Relabeling variables and noting that the two terms of (40) are equal, we obtain

$$\Delta E_P' = 2\hbar c\alpha(2\pi)^2\left(\frac{ie}{\hbar c}\right)^2\int_0^1 dv 2v^2(1-\tfrac{1}{3}v^2)\int \bar\phi_a(\mathbf{p}_2) \times\frac{\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1-\mathbf{q})[i\gamma\cdot(p_1+q)-\kappa]\gamma\cdot A^e(\mathbf{q})}{4\kappa^2+q^2(1-v^2)} \times\phi_a(\mathbf{p}_1)d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{q}.$$

We now consider

$$\left(\frac{ie}{\hbar c}\right)^2\int dv 2v^2(1-\tfrac{1}{3}v^2)\int d\mathbf{q} \times\frac{\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1-\mathbf{q})[i\gamma\cdot(p_1+q)-\kappa]\gamma\cdot A^e(\mathbf{q})}{4\kappa^2+q^2(1-v^2)} = G(\mathbf{p}_1,\mathbf{p}_2).$$

The form of $\phi_a(\mathbf{p}_1)$, $\bar\phi_a(\mathbf{p}_2)$ is such as to confine $|\mathbf{p}_1|$ and $|\mathbf{p}_2|$ to values of the order of $\alpha Z\kappa$ while the q integration yields its main contribution for q's of the order of κ. Thus it is reasonable to suppose that replacing $G(\mathbf{p}_1,\mathbf{p}_2)$ by $G(0,0)$ gives rise to an error of order $\alpha Z\Delta E_P'$, which we shall soon see is of order $\alpha^3Z^2E_H$.[21] In evaluating $G(0,0)$ we may of course also replace p_a by κ. Thus,

$$G(0,0)=\left(\frac{ie}{\hbar c}\right)^2\int dv 2v^2(1-\tfrac{1}{3}v^2)\int d\mathbf{q} \times\frac{\gamma\cdot A^e(-\mathbf{q})[i\gamma\cdot\mathbf{q}-\kappa(1+\gamma_4)]\gamma\cdot A^e(\mathbf{q})}{4\kappa^2+q^2(1-v^2)}.$$

We now note that $\gamma\cdot A^e(-\mathbf{q})(1+\gamma_4)\gamma\cdot A^e(\mathbf{q})$ contains no terms linear in the nuclear moment, while

$$\left(\frac{ie}{\hbar c}\right)^2\gamma\cdot A^e(-\mathbf{q})i\gamma\cdot\mathbf{q}\gamma\cdot A^e(\mathbf{q})=\frac{\alpha^2Z}{4\pi^2\kappa}\frac{m}{M}g\gamma_4\frac{\boldsymbol{\sigma}\cdot\mathbf{q}\times(\boldsymbol{\Sigma}\times\mathbf{q})}{(q^2)^2} =\frac{\alpha^2Z}{6\pi^4\kappa}\frac{m}{M}g\gamma_4\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}}{q^2}, \tag{41}$$

on averaging over angles. Equation (41) plays an important part in the evaluation of all of the α^2Z corrections to be obtained later, as all operators which contribute to these corrections are easily reducible to the form on the left. We find therefore that

$$G(0,0)\to\frac{\alpha^2Z}{6\pi^4\kappa}\frac{m}{M}g\gamma_4\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}\int_0^1 dv 2v^2(1-\tfrac{1}{3}v^2) \times\int\frac{d\mathbf{q}}{q^2(4\kappa^2+q^2(1-v^2))} =\frac{\alpha^2Z}{6\pi^4\kappa}\frac{m}{M}g\gamma_4\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}\int_0^1 dv 2v^2(1-\tfrac{1}{3}v^2)\frac{\pi^2}{\kappa(1-v^2)^{\frac{1}{2}}} =\frac{\alpha^2Z}{16\pi\kappa^2}\frac{m}{M}g\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma},$$

since

$$\int_0^1\frac{v^2(1-\frac{1}{3}v^2)}{(1-v^2)^{\frac{1}{2}}}dv=\frac{3\pi}{16}.$$

Thus[22]

$$\Delta E_P'\to\frac{1}{2}\frac{\alpha^3Z^2\pi}{\kappa^3}\frac{m}{M}g mc^2\int\bar\phi_a(\mathbf{p}_2)\gamma_4\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2 =\frac{1}{2}\frac{\alpha^3Z^2\pi}{\kappa^3}\frac{m}{M}g\phi_0{}^2\langle\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}\rangle mc^2 =\tfrac{3}{4}\alpha^2ZE_H. \tag{42}$$

It is clear that there are no α^2Z corrections for states with nonzero angular momentum.

We now consider briefly the second term (21). The explicit form of $\delta A_\mu{}^P(q)$ is

$$\delta A_\mu{}^P(\mathbf{q})=\frac{2ie}{q^2}\left(\frac{ie}{\hbar c}\right)^2\int\mathrm{Tr}\left[\gamma_\mu\frac{i\gamma\cdot p-\kappa}{p^2+\kappa^2}\gamma\cdot A^e(\mathbf{q}_2) \times S_F{}^e(p-q_2,p-q+q_1)\gamma\cdot A^e(\mathbf{q}_1) \times\frac{i\gamma\cdot(p-q)-\kappa}{(p-q)^2+\kappa^2}\right]d\mathbf{q}_1d\mathbf{q}_2(d_4p).$$

To estimate its magnitude we again insert (18) for $S_F{}^e(p-q_2,p-q+q_1)$ and drop the last term obtaining

$$\delta A_\mu{}^P(q)\approx\frac{-2ie}{q^2}\left(\frac{ie}{\hbar c}\right)^3\int\mathrm{Tr}\left[\gamma_\mu\frac{i\gamma\cdot p-\kappa}{p^2+\kappa^2}\gamma\cdot A^e(\mathbf{q}_2) \times\frac{i\gamma\cdot(p-q_2)-\kappa}{(p-q_2)^2+\kappa^2}\gamma\cdot A^e(\mathbf{q}-\mathbf{q}_1+\mathbf{q}_2) \times\frac{i\gamma\cdot(p-q+q_1)-\kappa}{(p-q+q_1)^2+\kappa^2}\gamma\cdot A^e(\mathbf{q}_1) \times\frac{i\gamma\cdot(p-q)-\kappa}{(p-q)^2+\kappa^2}\right]a_{\lambda_1}d\mathbf{q}_2(d_4p),$$

[21] The authors are indebted to H. A. Bethe for pointing out to them the fact that integrals like (40) might be approximated in this way.

[22] One must use Pauli-Schrödinger wave functions with spin rather than Dirac wave functions here. For Dirac wave functions the momentum space integral diverges unless $G(\mathbf{p}_1,\mathbf{p}_2)$ is used. The error is of order $\alpha^2Z^2\ln(\alpha Z)\Delta E_P'$.

the leading term having vanished as a consequence of the Furry Theorem.[23] It is clear from the occurrence of two extra factors $(ie/\hbar c)\gamma\cdot A^e$ that $\delta A_\mu{}^P$ would be at least a factor α^2Z^2 smaller than the leading term of $A_\mu{}^P$, so that the contribution of this term to the hyperfine structure is of order $\alpha^4Z^3E_H$.[24]

EVALUATION OF THE FLUCTUATION ENERGY

(a) First-Order Part

It is convenient to consider separately those parts of ΔE_F which involve $S_F{}^e(p_2, p_1)$ and those parts which do not. By the first-order part of ΔE_F we shall mean that part which comes from C and L. However, in view of the fact that L_D has been cast in a form similar to Q, we shall postpone discussing it and consider it in connection with Q.

The simplest of the first-order terms is the second term of Eq. (36), which, for convenience, we denote by $\Delta E_F{}^{(1)}$:

$$\Delta E_F{}^{(1)}=-6\pi^2\hbar c\alpha\left(\frac{ie}{\hbar c}\right)^2\int\bar{\phi}_a(\mathbf{p}_2)\Big\langle\bar{\ }\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_3)\times\frac{[2zi\gamma\cdot p_3+\kappa(1-2z)]}{\kappa^2-(p_3{}^2+\kappa^2)z}\gamma\cdot A^e(\mathbf{p}_3-\mathbf{p}_1)\bar{\ }\Big\rangle\times\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}_3dz. \quad (43)$$

This expression is very similar in form to (40). We therefore treat it in the same manner, that is, after writing $p_3=q+p_1$, we set $p_1=(0,0,0,i\kappa)=p_2$ in the portion of (43) bracketed thus $\langle\bar{\ }\bar{\ }\rangle$. Making use of (41) we find

$$\Delta E_F{}^{(1)}\longrightarrow-\frac{2}{\pi^2}\frac{\hbar c\alpha^3Z}{\kappa^2}\frac{m}{M}g\langle\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}\rangle\phi_0{}^2\int zdz\frac{d\mathbf{q}}{q^2}\frac{\kappa}{\kappa^2-q^2z}=-\frac{3}{\pi^3}E_H\alpha^2Z\int zdz\frac{\kappa}{\kappa^2-q^2z}\frac{d\mathbf{q}}{q^2}.$$

We find, therefore, thus $\Delta E_F{}^{(1)}$ is a term of order α^2ZE_H multiplied by a definite integral. We now note that the real part of

$$\int\frac{\kappa dq}{\kappa^2-q^2z}d\Omega=4\pi\mathcal{P}\int_0^\infty\frac{\kappa dq}{\kappa^2-q^2z}=0,$$

where $\mathcal{P}$ denotes the principal part, so that $\Delta E_F{}^{(1)}$ does not contribute in the order of interest. The integral does have an imaginary part which we could evaluate had we paid due attention to the small imaginary quantities which should be retained in the denominators throughout the calculation.[16] As noted before, however, we are interested only in the real part of the energy. It is to be noted that the essential difference between this result and that obtained for the vacuum polarization arises from the fact that here the coefficient of q^2 in the denominator is negative for all values of the auxiliary variables. We shall see that this is a characteristic feature of the denominators of K_μ as well.

We turn now to a consideration of the first term of (36). We consider first the contribution of the first term of $K_\mu(p_1, p_2)$ which we may write

$$K_\mu{}^{(A)}(p_2, p_1)=(p_2-p_1)^2\gamma_\mu\int\left[\frac{2-6y(1-y)}{\Lambda_1{}^2}-\frac{y(1-y)}{\Lambda_2{}^2}\right]dydz,$$

yielding

$$\Delta E_F{}^{(2)}=2\pi^2\hbar c\alpha\left(\frac{ie}{\hbar c}\right)\int dydz\int\bar{\phi}_a(\mathbf{p}_2)\times(p_2-p_1)^2\gamma\cdot A^e(p_2-p_1)\phi_a(\mathbf{p}_1)\times\left[\frac{2-6y(1-y)}{\kappa^2\Lambda_1{}^2}-\frac{y(1-y)}{\kappa^2\Lambda_2{}^2}\right]d\mathbf{p}_1d\mathbf{p}_2.$$

This expression is identical in form with $\Delta E_P{}'$ and may be dealt with in a precisely similar way. One finds that

$$\Delta E_F{}^{(2)}\longrightarrow-\frac{\alpha^2Z}{\pi^3}E_H\int dydz\frac{d\mathbf{q}}{q^2}\left[\frac{(2-6y(1-y))\kappa}{\kappa^2-y^2q^2}-\frac{y(1-y)\kappa}{\kappa^2-y^2zq^2}\right].$$

Again the integration over q gives rise to a vanishing real part so that like $\Delta E_P{}^{(1)}$, $\Delta E_P{}^{(2)}$ yields no α or α^2Z contributions. The third and fourth terms of K_μ are similar in structure to $K_\mu{}^A$ and can be similarly shown to give no α or α^2Z contributions.

There remains

$$K_\mu{}^{(B)}=i(p_2-p_1)_\nu\sigma_{\mu\nu}\int_0^1\frac{dy}{\kappa\Lambda_1{}^2}$$

to be considered. We note first that $K_\mu{}^{(B)}$ may be split into two terms $K_\mu{}^{(B)'}+K_\mu{}^{(B)''}$ with

$$K_\mu{}^{(B)'}=[i(p_2-p_1)_\nu/\kappa]\sigma_{\mu\nu},$$

and

$$K_\mu{}^{(B)''}=i(p_2-p_1)_\nu\sigma_{\mu\nu}\times\int dy\frac{y(p_1{}^2+\kappa^2)+(1-y)(p_2{}^2+\kappa^2)-(p_2-p_1)^2y(1-y)}{\kappa^3\Lambda_1{}^2}.$$

The contributions from $K_\mu{}^{(B)''}$ are of the same general character as those from the previously considered parts of K_μ and can similarly be shown to give no α or α^2Z contributions.

$K_\mu{}^{(B)'}$ corresponds to the anomalous moment of the electron, which appears here as a point dipole. It is clear, therefore, that it will contribute a correction to

[23] Wendell H. Furry, Phys. Rev. **51**, 125 (1937).

[24] This argument might be insufficient if $\delta A_\mu{}^P(q)$ were singular at small q. However, the gauge invariance of $j_\mu{}^P$ and the absence of charge renormalization effects in $\delta A_\mu{}^P$ imply that the momentum dependence of $\delta A_\mu{}^P(q)$ is $q^2A_\mu{}^e(q)f(q^2)$, where $f(q^2)$ is finite at $q=0$.

the hyperfine structure $\delta_\mu E_H$ where δ_μ is the fractional change in the electron moment. Thus

$$\Delta E_F^{(3)}=2\pi^2\hbar c\alpha\left(\frac{ie}{\hbar c}\right)\int\bar{\phi}_a(\mathbf{p}_2)\frac{i(p_2-p_1)_\nu}{\kappa}\sigma_{\mu\nu}\times A_\mu^{(H)}(\mathbf{p}_2-\mathbf{p}_1)\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2. \quad (44)$$

Now,

$$i(p_2-p_1)_\nu\sigma_{\mu\nu}A_\mu^{(H)}(\mathbf{p}_2-\mathbf{p}_1)\left(\frac{ie}{\hbar c}\right)=\frac{\alpha}{2\kappa}\frac{m}{M}g\frac{\boldsymbol{\sigma}\cdot(\mathbf{p}_2-\mathbf{p}_1)\times(\boldsymbol{\Sigma}\times(\mathbf{p}_2-\mathbf{p}_1))}{2\pi^2(p_2-p_1)^2},$$

so that

$$\Delta E_F^{(3)}=\frac{\alpha^2}{2}\frac{m}{M}g\frac{mc^2}{\kappa^3}\int\bar{\phi}_a(\mathbf{p}_2)\frac{\boldsymbol{\sigma}\cdot(\mathbf{p}_2-\mathbf{p}_1)\times(\boldsymbol{\Sigma}\times(\mathbf{p}_2-\mathbf{p}_1))}{(p_2-p_1)^2}\times\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2=\frac{\alpha}{2\pi}E_H.$$

We note that the anomalous moment is $\alpha/2\pi$ Bohr magnetons, in agreement with previous calculations.

The anomalous moment also gives rise to a spin-orbit interaction obtained by using $A_\mu^{(E)}$ instead of $A_\mu^{(H)}$ in (44). Now one might inquire as to whether this interaction is different for different hyperfine structure states in a manner analogous to the interaction with the Coulomb polarization potential. One might anticipate that this will not be the case as the spin-orbit interaction is, roughly speaking, distributed throughout the atom instead of being concentrated at the center as is the case for the polarization potential. Therefore the small differences in the wave functions of different hyperfine structure states cannot have a large effect on the spin orbit energy. A detailed examination shows this effect to be of order $\alpha^3Z^2E_H$.

We summarize the results of this section with the remark that apart from L_D, the total significant contribution of the first-order part to the hyperfine structure is simply $(\alpha/2\pi)E_H$ arising from the anomalous electron moment. A more detailed investigation than has been given here indicates that the omitted terms are all of order $\alpha^3Z^2E_H$.

(b) Second-Order Part and L_D[25]

The remaining part of the fluctuation energy consists of the second-order part, which has been denoted as Q, and the infrared divergent part of L, namely L_D. These expressions both involve $S_F^e(p_2, p_1)$, for which one has either an expansion in powers of the external potential or a sum over the stationary states of the potential. On the other hand, one notes that $Q+L_D$ involves the external potential quadratically and has an over-all coefficient of order α, so that in view of the results obtained for the polarization energy and first-order part, one would expect it to yield at most an α^2Z correction. Thus one might hope to evaluate these expressions in lowest approximation by replacing $S_F^e(p_2, p_1)$ by $S_F(p_2)\delta_3(\mathbf{p}_2-\mathbf{p}_1)$. In order to see that this is in fact the case, it is worth inquiring as to the circumstances under which $S_F(p)\delta_3(\mathbf{p}-\mathbf{p}')$ is a good approximation to $S_F^e(p, p')$. Some answer to this question can be obtained by considering the first two terms in an expansion of $S_F^e(p, p')$ for the Coulomb field. Referring to (18) and (38) one finds

$$S_F^e(p, p+q)\approx S_F(p)\delta(q)-S_F(p)\gamma_4\frac{\alpha Z}{2\pi^2q^2}\frac{i\gamma\cdot(p-q)-\kappa}{q^2-2\mathbf{p}\cdot\mathbf{q}+p^2+\kappa^2}.$$

A comparison of the two terms is simplified by integrating over q, thus

$$\int S_F^e(p, p+q)d\mathbf{q}\approx S_F(p)-S_F(p)\gamma_4\frac{\alpha Z}{2}\int_0^1 dx\frac{i\gamma\cdot p(1+x)-\kappa}{[x(p^2+\kappa^2)-x^2\mathbf{p}^2]^{\frac{1}{2}}}.$$

It is clear that when $p^2+\kappa^2$ and $\mathbf{p}^2$ are of order $\alpha^2Z^2\kappa^2$, the two terms are of the same order of magnitude, while for "large" p (i.e., $(p^2+\kappa)$ large) the second term is a factor αZ smaller than the first. Thus the validity of replacing $S_F^e(p_2, p_1)$ by $S_F(p_2)\delta_3(\mathbf{p}_2-\mathbf{p}_1)$ depends upon the weight given low momenta in the integral in which S_F^e appears. In the case of Q the arguments of S_F^e are of the form $p+k+q$. Now $p^2+\kappa^2$ and $\mathbf{p}^2$ are kept small by the form of the wave function; low k is favored by the factors $k^2(k^2-2p\cdot k)(k^2-2p'\cdot k)$ appearing in the denominator, while low q is favored by the factors $1/q^2$ in the potentials. On the other hand, one notes that, if $\mathfrak{A}_\mu(q, p, k)$ is written

$$\mathfrak{A}_\mu(q, p, k)=\left(\frac{\gamma\cdot A^e i\gamma\cdot k\gamma_\mu}{k^2-2p\cdot k}-\frac{i\gamma\cdot k\gamma_\mu\gamma\cdot A^e}{k^2-2(p+q)\cdot k}\right)+2i\left(\frac{(p+q)_\mu}{k^2-2(p+q)\cdot k}-\frac{p_\mu}{k^2-2p\cdot k}\right)\gamma\cdot A^e,$$

the numerator of the first term vanishes with k while the coefficient of $\gamma\cdot A^e$ in the second term vanishes with q. In addition, the fact that one of the external potentials refers to the dipole field introduces an additional factor q. It turns out that these last two factors are just sufficient, and one can accordingly show that contributions from the next term in the expansion of S_F^e are no larger than $\alpha^3Z^2\ln(\alpha Z)E_H$.[26] It should be observed that had

[25] The treatment given Q is essentially the same as that used by M. Baranger [Phys. Rev. 84, 866 (1951)] in the evaluation of its contribution to the αZ corrections for the Lamb effect. We are indebted to him for providing us with a copy of his thesis.

[26] A more complete discussion of the validity both of this approximation and the small momentum approximation first introduced in evaluating the expression in Eq. (40) will be given by H. A. Bethe and M. Baranger in a forthcoming publication.

we based our treatment on an expansion of S_F^e as it appears in Eq. (18) rather than on the Feynman identities (22), then the terms corresponding to $\mathfrak{A}_\mu$ would not have had the afore-mentioned important property.[17] An attempt to demonstrate the validity of the expansion based on an inspection of individual portions of the higher order terms fails.[27]

The evaluation of $Q+L_D$, after replacing S_F^e by the propagation function for a free electron, is a straightforward matter and proceeds in a manner entirely similar to the previously considered terms. Considering L_D first, and recalling that

$$S_F(p)=\frac{-2i}{(2\pi)^4}\frac{i\gamma\cdot p-\kappa}{p^2+\kappa^2},$$

one obtains

$$L_D=-2i\hbar c\alpha\left(\frac{ie}{\hbar c}\right)^2\int\bar{\phi}_a(\mathbf{p}_2)\times\left[\left(\frac{2i(p_1+q)_\mu}{k^2-2(p_1+q)\cdot k}-\frac{2ip_{2\mu}}{k^2-2p_2\cdot k}\right)\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1-\mathbf{q})\times\frac{i\gamma\cdot(p_1+q)-\kappa}{q^2+2\mathbf{p}_1\cdot\mathbf{q}+p_1^2+\kappa^2}\gamma\cdot A^e(\mathbf{q})\times\left(\frac{2i(p_1+q)_\mu}{k^2-2(p_1+q)\cdot k}-\frac{2ip_{1\mu}}{k^2-2p_1\cdot k}\right)\frac{d_4k}{k^2}d\mathbf{q}\right]\phi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2.$$

Using again our previous approximation, we neglect the small momenta and binding energy of the initial and final state when these appear in the bracketed [ ] portion of the expression. That is, we set $p_1=p_2=p=(0,0,0,i\kappa)$. Making use of this fact, integrating over p_1 and p_2 and using Eq. (41) we find for hyperfine structure

$$L_D\rightarrow\frac{4i\alpha^2Z\kappa}{\pi^5}\left[\frac{2}{3}\alpha\pi\frac{\phi_0^2}{\kappa^2}\frac{m}{M}g\langle\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}\rangle mc^2\right]\int d\mathbf{q}dk\left(\frac{1}{q^2}\right)^2\frac{1}{k^2}\times\left[\frac{q^2}{(k^2-2(p+q)\cdot k)^2}-\frac{4\kappa^2(k\cdot q)^2}{(k^2-2p\cdot k)(k^2-2(p+q)\cdot k)}\right].$$

It has been remarked previously that L_D diverges in the infrared. It might therefore be preferable to combine it with Q before integrating over k. We shall see immediately, however, that in the above approximation L_D vanishes, so that the suggested combination is unnecessary. Considering the first term only and replacing $1/k^2$ by $1/(k^2+\lambda^2)$ to control the infrared divergence, we obtain

$$i\int d\mathbf{q}d_4k\frac{1}{q^2(k^2+\lambda^2)(k^2-2(p+q)\cdot k)^2}=-\pi^2\int_0^1dx\frac{xd\mathbf{q}}{[\lambda^2(1-x)+(\kappa^2-q^2)x^2]q^2}=-4\pi^3\int xdx\frac{dq}{\kappa^2x^2+\lambda^2(1-x)-q^2x^2},$$

whose real part vanishes.

The other part of L_D vanishes similarly. The fact that the contribution from L_D vanishes in this order is not surprising in view of its similarity to the rest of the first-order part of ΔE_F.

For Q we have, on replacing $S_F^e(p,p')$ by $S_F(p)\times\delta(p-p')$,

$$Q=2i\hbar c\alpha\left(\frac{ie}{\hbar c}\right)^2\int\bar{\phi}_a(\mathbf{p}_2)\left[\mathfrak{A}_\mu'(p_2-p_1-q,p_2,k)\times\frac{i\gamma\cdot(p_1+q-k)-\kappa}{k^2-2(p_1+q)\cdot k+q^2+2\mathbf{p}_1\cdot\mathbf{q}+p_1^2+\kappa^2}\times\mathfrak{A}_\mu(q,p_1,k)\frac{d_4k}{k^2+\lambda^2}d\mathbf{q}\right]\phi_a(\mathbf{p}_1)d\mathbf{p}_2d\mathbf{p}_1.$$

Again we set $p_1=p_2=p$ in the bracketed expression so that using (28) and (29),

$$Q=2i\hbar c\alpha\phi_0^2\left(\frac{ie}{\hbar c}\right)^2\left\langle\int\frac{d_4k}{k^2+\lambda^2}d\mathbf{q}\,\mathfrak{A}_\mu'(-\mathbf{q},p,k)\times\frac{i\gamma\cdot(p+q-k)-\kappa}{k^2-2(p+q)\cdot k+q^2}\mathfrak{A}_\mu(\mathbf{q},p,k)\right\rangle=2i\hbar c\alpha\phi_0^2\left(\frac{ie}{\hbar c}\right)^2\langle\mathrm{I}+\mathrm{II}+\mathrm{III}\rangle,$$

with

$$\mathrm{I}=\int d\mathbf{q}d_4k\frac{(2ip_\mu-\gamma_\mu i\gamma\cdot k)\gamma\cdot A^e(-\mathbf{q})(i\gamma\cdot(p+q-k)-\kappa)\gamma\cdot A^e(\mathbf{q})(2ip_\mu-i\gamma\cdot k\gamma_\mu)}{(k^2+\lambda^2)(k^2-2p\cdot k)^2(k^2-2(p+q)\cdot k+q^2)},$$

$$\mathrm{II}=\int d\mathbf{q}d_4k\frac{\gamma\cdot A^e(-\mathbf{q})[2i(p+q)_\mu-\gamma_\mu i\gamma\cdot k][i\gamma\cdot(p+q-k)-\kappa][2i(p+q)_\mu-i\gamma\cdot k\gamma_\mu]\gamma\cdot A^e(\mathbf{q})}{(k^2+\lambda^2)(k^2-2(p+q)\cdot k)^2(k^2-2(p+q)\cdot k+q^2)},$$

$$\mathrm{III}=-\int\frac{d\mathbf{q}d_4k}{D}\{\gamma\cdot A^e(-\mathbf{q})[2i(p+q)_\mu-\gamma_\mu i\gamma\cdot k][i\gamma\cdot(p+q-k)-\kappa]\gamma\cdot A^e(\mathbf{q})[2ip_\mu-i\gamma\cdot k\gamma_\mu]+[2ip_\mu-\gamma_\mu i\gamma\cdot k]\gamma\cdot A^e(-\mathbf{q})[i\gamma\cdot p+q-k-\kappa][2i(p+q)_\mu-i\gamma\cdot k\gamma_\mu]\gamma\cdot A^e(\mathbf{q})\},$$

[27] Notwithstanding the above remark, an evaluation based on such an expansion broken off at the quadratic turn does in fact yield the same result as will be obtained here. This suggests that if one groups the higher order terms properly, a proof of their smallness might be possible.

and

$$D=(k^2+\lambda^2)(k^2-2p\cdot k)\times[k^2-2(p+q)\cdot k][k^2-2(p+q)\cdot k+q^2].$$

For brevity we confine our discussion for the moment to II. The treatments of I and III are identical. Using the well-known denominator combining formula

$$\frac{1}{ab^2c}=6\int_0^1\frac{x^2ydxdy}{[a(1-x)+bxy+cx(1-y)]^4},$$

and letting $k\rightarrow k+(p+q)x$, one finds

$$\mathrm{II}=6\int d\mathbf{q}d_4k\frac{x_2ydxdyN_2}{[k^2+\kappa^2x^2+q^2x(1-x-y)+\lambda^2(1-x)]^4}.$$

The numerator N_2 involves various products of spinor matrices and momenta from which one is to pick out contributions to the hyperfine structure. This operation is facilitated by reducing all terms to the form $\gamma\cdot A^e(-\mathbf{q})i\gamma\cdot\mathbf{q}\gamma\cdot A^e(\mathbf{q})$. For this purpose the following facts and equivalence relations are useful.

$$\gamma\cdot\mathbf{q}\gamma\cdot A^e(\mathbf{q})=-\gamma\cdot A^e(\mathbf{q})\gamma\cdot\mathbf{q},$$
$$\gamma\cdot A^e(-\mathbf{q})\gamma\cdot A^e(\mathbf{q})\rightarrow 0,$$

$(\ )\gamma\cdot A^e(-\mathbf{q})\gamma_4\gamma\cdot A^e(\mathbf{q})(\ )\rightarrow 0$ regardless of the content of the brackets. A factor γ_4 appearing on the extreme right or left of an expression may be dropped.

Using the above one finds

$$N_2\rightarrow[2(\kappa^2-q^2)(2-4x+x^2+x^3)-k^2(5+3x)]\times\gamma\cdot A^e(-\mathbf{q})i\gamma\cdot\mathbf{q}\gamma\cdot A^e(\mathbf{q}).$$

Carrying out the k space integration and inserting (41) yields

$$Q_{\mathrm{II}}=-\frac{2\hbar c\alpha}{3\pi^2\kappa}\phi_0{}^2\alpha^2Zg\frac{m}{M}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}\rangle\int x^2ydxdy\frac{d\mathbf{q}}{q^2}\times\left\{\frac{(2-4x+x^2+x^3)(\kappa^2-q^2)}{[\kappa^2x^2+\lambda^2(1-x)+q^2(1-x-y)]^2}-\frac{5+3x}{[x^2+\lambda^2(1-x)+q^2x(1-x-y)]}\right\}.$$

We now recall (39), carry out the angular q integration and transform $q\rightarrow q\kappa$, $\lambda\rightarrow\lambda\kappa$ so as to make the integral dimensionless, yielding

$$Q_{\mathrm{II}}=-\frac{4\alpha^2Z}{\pi^2}E_H\int dqx^2ydxdy\times\left\{\frac{(2-4x+x^2+x^3)(1-q^2)}{[x^2+\lambda^2(1-x)+q^2x(1-x-y)]^2}-\frac{5+3x}{x^2+\lambda^2(1-x)+q^2x(1-x-y)}\right\}.$$

Identical methods yield for the rest of Q,

$$Q_{\mathrm{I}}=-\frac{4\alpha^2Z}{\pi^2}E_H\int dqx^2ydxdy\left\{\frac{2-2x+3x^2-x^3-2x^2y+x^3y-q^2x^2(1-y)^2(1-x+xy)}{[x+\lambda^2(1-x)+q^2x(1-y)(1-x+xy)]^2}-\frac{1-3x(1-y)}{x^2+\lambda^2(1-x)+q^2x(1-y)(1-x+xy)}\right\},$$

$$Q_{\mathrm{III}}=\frac{8\alpha^2Z}{\pi^2}E_H\int dqx^2ydxdydz\left\{\frac{2-3x+q^2x(1-yz)[1-x(1-yz)]}{[x^2+\lambda^2(1-x)+q^2x[1-y-x(1-yz)^2]]^2}-\frac{1}{x^2+\lambda^2(1-x)+q^2x[1-y-x(1-yz)^2]}\right\}.$$

There remains only the purely numerical problem of evaluating the above integrals. The main points can be illustrated by considering the leading terms of Q_{II}, namely,

$$A=\int\frac{dqx^2ydxdy}{[x^2+\lambda^2(1-x)+q^2x(1-x-y)]^2},$$

$$B=\int\frac{q^2dqx^2ydxdy}{[x^2+\lambda^2(1-x)+q^2x(1-x-y)]^2}.$$

In the first integral it is simplest to do the q integration first, thus,

$$A=\frac{\pi}{4}\int_0^1dx\int_{1-x}^1dy\frac{x^2y}{[x^2+\lambda^2(1-x)]^{\frac{3}{2}}x^{\frac{1}{2}}(1-x-y)^{\frac{1}{2}}}.$$

It is to be observed that the coefficient of q^2 appearing in the denominator may be positive or negative, depending upon the values of the auxiliary variables. As the integral is to be evaluated as a principal part when the denominator has a pole on the real axis, it has a nonvanishing value only when this coefficient is positive. This is the origin of the lower limit $1-x$ in the y integration. The remainder of the integration is straightforward and will not be discussed further.[28]

For the integral B one must integrate over y first,[29]

[28] It might be mentioned that for Q_{III}, the fixing of the region for which the coefficient of q^2 is positive is facilitated by the substitution $z=(1-u)/y$, $1>u>1-y$.

[29] If one integrates over q first, the y integration diverges at $y=1-x$. The procedure used can be justified by remarking that the auxiliary variables have no physical significance and are in fact part of the k integration. Thus in any case in which it makes a difference one should complete the auxiliary variable integration before performing the q integration. One can also deal with the problem by rounding off the singularity of the Coulomb field. That is, one modifies the Fourier transform of the Coulomb potential by a factor $\Lambda^2/(\Lambda^2+q^2)$, where Λ is assumed to be large. With this factor present the integral is finite and independent of the order of integration. The result obtained in the limit $\Lambda\rightarrow\infty$ is identical with that given above.

obtaining

$$B=\int\left\{\frac{\ln\left(1+q^2\frac{1-x}{x}+\lambda^2\frac{1-x}{x^2}\right)}{q^2}-\frac{\ln\left|1-q^2\frac{1-x}{x}+\lambda^2\frac{1-x}{x^2}\right|}{q^2}+\frac{x}{x^2+\lambda(1-x)-x^2q^2}\right\}dqdx$$
$$=\pi\int_0^1\left(\frac{1-x}{x}\right)^{\frac{1}{2}}dx=\frac{\pi^2}{2}.$$

Only the first of the three terms contributes.

Proceeding in the manner outlined above all terms reduce to simple integrals over one, two or three auxiliary variables, a typical form being

$$\int_0^1 dx\int_0^1 dy\frac{x^m y^n}{[xy(1-xy)]^{\frac{1}{2}}}.$$

Certain of these, like A above, diverge in the limit $\lambda\to 0$, but yield a finite limit when taken together. The final result obtained is simply

$$Q=-\alpha^2 ZE_H(13/4-\ln 2). \tag{45}$$

SUMMARY AND CONCLUSIONS

The second-order radiative corrections to the hyperfine structure of S states are given by

$$\Delta E_P=\tfrac{3}{4}\alpha^2 ZE_H,$$
$$\Delta E_F=\left[\frac{\alpha}{2\pi}-\alpha^2 Z\left(\frac{13}{4}-\ln 2\right)\right]E_H,$$

yielding[30]

$$\Delta E=\left[\frac{\alpha}{2\pi}-\alpha^2 Z\left(\frac{5}{2}-\ln 2\right)\right]E_H. \tag{46}$$

For higher angular momentum states there is an order α-correction arising from the anomalous electron moment but no $\alpha^2 Z$ correction. While our derivation has been confined to hydrogenic atoms, the effect of screening on the radiative corrections is, aside from its effect on ϕ_0^2, only of order $\alpha^3 Z^2$. This result is a consequence of the fact that the $\alpha^2 Z$ correction comes chiefly from distances within a Compton wavelength of the nucleus.

A principal application of this result is to the deduction of the fine structure constant from the hyperfine structure formula. The hyperfine structure frequency of the ground state of hydrogen is related to other accurately known experimental quantities by

$$\Delta\nu=\frac{16}{3}\alpha^2 cR_\infty\left(\frac{\mu_P}{\mu_S}\right)\left(\frac{m_r}{m}\right)^3(1+\tfrac{3}{2}\alpha^2)\times\left(1+\frac{\alpha}{2\pi}-\frac{2.973\alpha^2}{\pi^2}\right)\left(1+\frac{\alpha}{2\pi}-\frac{2.973\alpha^2}{\pi^2}+\alpha^2\left(\frac{5}{2}-\ln 2\right)\right)P_r P_s. \tag{47}$$

In applying the above expression we use

$$c=2.997902\times 10^{10}\text{ cm/sec},^4 \quad (3.0\text{ ppm})$$
$$R_\infty=109{,}737.324\text{ cm}^{-1},^4 \quad (0.1\text{ ppm})$$
$$\Delta\nu=1420.4051\times 10^6\text{ sec}^{-1},^{31} \quad (0.3\text{ ppm})$$
$$(\mu_S/\mu_P)=658.2087,^{32} \quad (2.0\text{ ppm})$$
$$(m_r/m)^3=0.99836790.^4$$

The small α-dependent corrections are evaluated using $1/\alpha=137.036$. The factors P_r and P_s are inserted to take into account proton recoil effects and possible departure of the proton magnetic field from that of a point dipole. From (47) one finds

$$1/\alpha=(137.03651\pm 0.00028)P_r^{\frac{1}{2}}P_s^{\frac{1}{2}}. \tag{48}$$

The contribution of the $\alpha^2 Z$ term in Eq. (46) to this result is -0.00658.

An independent determination of $1/\alpha$ by Lamb and Retherford[33] from the fine structure separation of the $2P_{\frac{3}{2}}$, $2P_{\frac{1}{2}}$ levels in deuterium yields

$$1/\alpha=137.033\pm 0.006, \tag{49}$$

a result which tends to confirm the presence of the new electrodynamic correction obtained above. It is worth noting that the comparison of (49) and (48) is independent of the values of c and R_∞, and substantially independent of the anomalous electron moment.

The factors P_r and P_s in (48) might possibly affect the value of $1/\alpha$ by a few parts per 10^5. A more accurate measurement of the fine structure of deuterium would make possible an experimental estimate of their magnitude.

It is a pleasure to acknowledge several stimulating and informative conversations with Professor H. A. Bethe.

[30] This result has been reported previously by N. M. Kroll and F. Pollock, Phys. Rev. **84**, 594 (1951). The same result has been obtained by a different method by Karplus, Klein, and Schwinger, Phys. Rev. **84**, 597 (1951).

[31] P. Kusch and A. G. Prodell, Phys. Rev. **79**, 1009 (1950).

[32] Koenig, Prodell, and Kusch, Phys. Rev. **83**, 687 (1951). The value given above is based on 658.2289 ± 0.0006 (the value quoted by P. Kusch at the 1952 Annual Meeting of the American Physical Society) for μ_S/μ_P with the proton resonance measured in oil. The diamagnetic correction used is $(1-30.5\times 10^{-6})$ and is based on the results of Ramsey [Phys. Rev. **78**, 699 (1950)] and H. S. Gutowsky and J. W. McClure [Phys. Rev. **81**, 276 (1951)]. The uncertainty given by Kusch has been doubled to allow for possible uncertainties in the diamagnetic corrections.

[33] W. E. Lamb, Jr. and R. C. Retherford (to be published).