# Possibility of Frequency Multiplication and Wave Amplification by Means of Some Relativistic Effects

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A method is proposed of converting the frequency of an electromagnetic wave to a higher frequency by reflection from an electron cloud moving with relativistic velocity. Such an electron cloud can be reaized by compressing all or part of the electron beam of an electron accelerator into one or more groups. It is shown that there is a gain of wave energy arising from the relativistic law of reflection of a wave which is reflected from such a cloud. It is also shown on the basis of Bailey's relativistic electro-magneto-ionic theory that under certain circumstances the reflection from a moving slab of electrons may be increased considerably if the slab moves through a longitudinal or transverse magnetic field. It is estimated that a wave of length 1 mm and of a power at least one milliwatt can be generated by reflecting a wave of length 3 cm from the beam of a small betatron. Equipment is being designed to test this prediction experimentally.

THE purpose of this note is to point out that through the development of electron accelerators (such as the van de Graaff machine and the betatron) the generation of very short electromagnetic waves by means of some relativistic effects has come within the range of possibility.[1]

## I. RELATIVISTIC DOPPLER EFFECT

Consider first the expression for the relativistic Doppler effect in the form

$$\nu'=\nu[1+(u/c)\cos\phi](1-u^2/c^2)^{-\frac{1}{2}}, \qquad (1)$$

where $u$=velocity of an electron in the beam of an accelerator, $c$=velocity of light in vacuum, $\nu'$=frequency of an electromagnetic wave in a coordinate system $S'$ at rest with respect to the observer, $\nu$=frequency of an electromagnetic wave in a coordinate system $S$ moving with an electron in the beam, and $\phi$ is the angle between the wave normal in the system $S$ of a spherical wave originating at the electron and the direction of the relative velocity $u$.

Thus,

$$\nu'=\nu F \quad \text{for} \quad \phi=0, \qquad (2)$$

$$\nu'=\nu f \quad \text{for} \quad \phi=\pi, \qquad (3)$$

where

$$F=\{(1+u/c)/(1-u/c)\}^{\frac{1}{2}}=f^{-1}. \qquad (4)$$

TABLE I. Conversion factor $F$ for typical accelerating voltages.

| $V$(Mev) | 0.1 | 1 | 2 | 5 | 10 | 25 | 100 |
|---|---|---|---|---|---|---|---|
| $F$ | 1.85 | 5.75 | 9.7 | 21.5 | 41.1 | 97.7 | 394 |
| $K=F^2$ | 3.44 | 33 | 94 | 463 | 1690 | 9570 | 154800 |

[1] Since this paper was submitted for publication an article by H. Motz entitled "Applications of the radiation from fast electron beams" [J. Appl. Phys. 22, 527 (1951)], which deals with a similar problem, has become available in this country. We agree with this author on the importance of coherence of the radiation from the electron beam. However, a large increase of intensity above incoherent radiation will be obtained even if grouping or bunching of the electrons is not perfect (see Secs. IV and V of this paper).

$F$ and $f$ in (2) and (3) may be termed "conversion factors" because the frequency of an electromagnetic wave radiated by the moving electron appears to a stationary observer $F$ times greater [in the case of Eq. (2)] and $F$ times less [in the case of Eq. (3)], i.e., the observed frequency depends on the "direction of observation" as specified by $\phi$. Also $F$ and $f$ interchange when $u$ changes its sign.

It may easily be seen that as $(u/c)\rightarrow 1$, $F\rightarrow\infty$ in (2) and $f\rightarrow 0$ in (3). The first case is of course the one of interest for the present purpose.

In Table I the values of $F$ are given for a selection of accelerating voltages typical of present day machines. As may be seen from this table, a large frequency multiplication is theoretically possible even for accelerating voltages of 5 Mev, e.g., electrons radiating a wavelength of 3 cm in the system $S$ would appear to radiate a wavelength of about 1.5 mm in the laboratory frame.

The question naturally arises how the electrons in the beam of an accelerator, linear or circular, can be made to oscillate at a desired frequency. Several schemes seem feasible, but that which appears to be most practicable is better discussed in terms of the motion of a mirror moving with relativistic velocity.

## II. MIRROR MOVING WITH RELATIVISTIC VELOCITY

Consider an electromagnetic wave originating in the laboratory frame $S'$ and traveling towards the oncoming electron stream of an accelerator and also a slab, cut out of this stream by planes perpendicular to the direction of the beam, in which the wave is absorbed by imparting energy of oscillation to the electrons and then re-radiated.

It will be recognized that the situation is then analogous to reflection by a mirror moving with relativistic velocity.

An electromagnetic wave of frequency $\nu$ incident at an angle $\theta$ on such a mirror is reflected back with the increased frequency $\nu'$ if the mirror moves towards the observer with the velocity $u$. This frequency is known to be given by

$$\nu'=\nu[1+2(u/c)\cos\theta+u^2/c^2]/(1-u^2/c^2). \quad (5)$$

In the present case where the directions of incidence and reflection are perpendicular to the surface of the slab (the "mirror") $\theta=0$ and the expression reduces to

$$\nu'=\nu K, \quad (6)$$

where

$$K=(1+u/c)/(1-u/c)=F^2. \quad (7)$$

Clearly, when $u$ approaches $c$, then

$$K\sim 2/(1-u/c).$$

The values of $K$ are also given in Table I. It will be observed that if this mechanism can be used, a notable multiplication of frequency ($K\sim 30$) will occur even at velocities corresponding to 1 Mev.

### III. RELATIVISTIC WAVE AMPLIFICATION

Before attempting to obtain at least an approximate estimate of the efficiency of such a scheme, it may be pointed out that another relativistic theorem greatly increases the chances of observing the present mode of frequency conversion. As is well known, the theory of relativity shows[2] that the field amplitudes $A$ and $A'$ measured by two observers in two different systems of coordinates $S$ and $S'$ are in the ratios of the frequencies measured by these observers, i.e., the amplitudes transform as follows:

$$A/\nu=A'/\nu'. \quad (8)$$

Applied to the present case this means that if the conversion factors $F$ or $K$ are large, the electrons in the slab will experience a field amplitude much larger than that originally measured in the laboratory frame, and an observer in the laboratory frame will observe a field amplitude much larger than the amplitude of the wave re-radiated from the electrons and, *a fortiori*, much larger (by the factor $K$) than the amplitude of the wave originally sent out from the laboratory frame. In fact, we should expect a total increase of wave energy by the factor $K^2$.

### IV. A NUMERICAL EXAMPLE

For the purpose of a first estimate of the conversion efficiency of such an arrangement [corresponding to Eq. (6)] we take as a concrete example a wave in the laboratory ($S'$) frame of 3 cm wavelength, i.e., 10,000 Mc/sec frequency and an amplitude of 100 v/cm at the position of the slab. This can be provided by a standard magnetron radar transmitter. We disregard initially the effect described by Eq. (8) and also assume that in the electron beam of the accelerator there is no restoring force on the electrons at right angles to the beam. Although in most machines there exists a small radial magnetic field for focusing purposes, this will be allowable because under these assumptions the amplitude $a$ of the electron vibrations created by the wave is given by

$$a=(e/m)(E/\omega^2)\text{ cm}, \quad (9)$$

where $E$= field strength in esu and $\omega/2\pi$= frequency of the incident wave. For the present numerical example this yields $a\sim 4.5\times 10^{-5}$ cm. We may therefore restrict our considerations, if necessary, to regions where the transverse magnetic field is small or even zero, e.g., with a betatron to the immediate neighborhood of the median plane. We shall return to this aspect of the problem in Sec. V, where it will be shown that this restriction on the magnetic field may be removed by using the results of Bailey's relativistic electro-magneto-ionic theory ("E.M.I." theory).[3]

We will now consider the electron density in the beam of the machine. Data published for the original betatron built at the University of Illinois[4] yield $\sim 10^9$ as the total number of electrons in the orbit of this machine.

The final answer to our problem depends on how many of these electrons can be made to oscillate coherently as a group. If by suitable means it is possible for the wave to excite all the $10^9$ electrons as a group then, by the classical radiation theory, the energy $S$ radiated per second is given by

$$S=\tfrac{1}{3}(q^2/c^3)(e^2/m^2)E^2$$

or

$$S=\tfrac{1}{3}[(Ne)^2/c^3](e/m)^2E^2\text{ erg/sec}, \quad (10)$$

where $N$ is the electron density and $q=Ne$ is the total charge in the group of electrons. For our numerical example this expression yields $S\sim 10^2$ erg/sec $=10^{-2}$ mw. In practice it may not be possible to group all the electrons in the beam together in the manner indicated. In the betatron being designed in this laboratory to test the generation of short waves by the present method (see Sec. VI) the electrons are compressed into regions about 30 wavelengths apart.

### V. THE PROCESS OF WAVE AMPLIFICATION AND FREQUENCY CONVERSION ON THE BASIS OF BAILEY'S RELATIVISTIC E.M.I. THEORY

Professor V. A. Bailey has kindly drawn the writer's attention to the fact that a more detailed analysis of the wave conversion process considered may be carried out by using the results of his relativistic E.M.I. theory.[3] In fact, this theory makes it possible to take into account the effect of any transverse or longitudinal magnetic field pervading the electron beam which introduces an electronic gyrofrequency and which we were forced to exclude in our above approximation. Professor Bailey has carried out an analysis of the

[2] Abraham-Becker, *Theorie der Elektrizitaet* (Leipzig-Berlin 1933), p. 314, Vol. 2.

[3] V. A. Bailey, Phys. Rev. **78**, 428 (1950) and earlier communications.

[4] D. W. Kerst, Phys. Rev. **60**, 47 (1941).

reflection of a circularly polarized wave by a uniform semi-infinite slab of electrons moving with the velocity $U$ along or transverse to a uniform magnetic field $H$. Clearly this is the configuration that exists in most electron accelerators. The calculations are here reproduced with Professor Bailey's permission.

Let $S$ be a frame of reference in which the slab is at rest and $S'$ a parallel frame moving relatively to $K$ with a velocity $-U$ along the axis $0x$.

In $S$ let $\mathbf{E}_i$, $\mathbf{E}_r$, and $\mathbf{E}_n$ be the electric vectors, near the face of the slab, of the incident, reflected, and transmitted waves, respectively. The corresponding magnetic vectors are proportional to $-\mathbf{E}_i$, $\mathbf{E}_r$, and $-M\mathbf{E}_n$, where the refractive index $M$ is given by the relevant dispersion equation.

The boundary conditions at the face of the slab yield the following relations between parallel tangential components:

$$E_r+E_i=E_n, \quad E_r-E_i=-ME_n,$$

and so

$$E_r/E_i=(1-M)/(1+M). \tag{11}$$

If $P_i$ and $P_r$ are the incident and reflected Poynting fluxes, then the coefficient of reflection $\rho$ is given by

$$\rho=P_r/P_i=|1-M|^2/|1+M|^2. \tag{12}$$

The phases of the three waves are, respectively, $\phi_i$, $\phi_r$, $\phi_n$, where

$$\begin{gathered}\phi_i=\omega t+(\omega/c)x, \quad \phi_r=\omega t-(\omega/c)x,\\ \phi_n=\omega t+(M\omega/c)x.\end{gathered} \tag{13}$$

Since the phase $\phi=\omega t-lx$ of any wave is invariant to a Lorentz transformation, it follows that $(l, 0, 0, i\omega/c)$ is a four-vector and so transforms like $(x, 0, 0, ict)$. Hence, in the frame $S'$ we have

$$\left.\begin{aligned} l'&=\beta(l+\omega U/c^2)\\ \omega'&=\beta(\omega+Ul)\end{aligned}\right\}, \tag{14}$$

where

$$\beta=(1-U^2/c^2)^{-\frac{1}{2}}.$$

Also the field components of this wave are given by

$$\left.\begin{aligned} E_2'&=\beta(E_2+UH_3/c), & H_2'&=\beta(H_2-UE_3/c)\\ E_3'&=\beta(E_3-UH_2/c), & H_3'&=\beta(H_3+UE_2/c)\end{aligned}\right\}. \tag{15}$$

The Poynting flux in $S$ is

$$P=(E_2H_3-E_3H_2)/4\pi,$$

and so, by (15), the flux in $S'$ is

$$P'=\beta^2(1+U/c)^2P/4\pi.$$

i.e.,

$$P'=KP, \tag{16}$$

where

$$K=(1+U/c)/(1-U/c). \tag{17}$$

For the three waves considered we have from (13)

$$\begin{aligned} \omega_i&=\omega, & l_i&=-\omega/c,\\ \omega_r&=\omega, & l_r&=\omega/c,\\ \omega_n&=\omega, & l_n&=-M\omega/c,\end{aligned}$$

and so by (14) we have in $S'$

$$\left.\begin{aligned} \omega_i'&=\beta(1-U/c)\omega, & l_i'&=\beta(1-U/c)\omega/c\\ \omega_r'&=\beta(1+U/c)\omega, & l_r'&=\beta(1+U/c)\omega/c\\ \omega_n'&=\beta(1-MU/c)\omega, & l_n'&=\beta(-M+U/c)\omega/c.\end{aligned}\right\} \tag{18}$$

Hence,

$$\omega_r'/\omega_i'=K, \tag{19}$$

$$\omega_n'/\omega_i'=(1-MU/c)/(1-U/c). \tag{20}$$

Also by (16) and (17) we have

$$P_i'=K^{-1}P_i, \quad P_r'=KP_r,$$

and so, on using (12) we find

$$\rho'=\frac{P_r'}{P_i'}=\left(\frac{1+U/c}{1-U/c}\right)^2\left|\frac{1-M}{1+M}\right|^2=K^2\rho. \tag{21}$$

We now consider in turn the two important cases in which the magnetic field is, respectively, longitudinal and transverse to the electron beam.

In the first case, where the magnetic field is parallel to the electron beam, the refractive index $M$ in the frame $S$ is given by

$$M=\{1-p_0^2/\omega(\omega-k_n\Omega)\}^{\frac{1}{2}}, \tag{22}$$

where $p_0^2=4\pi Ne^2/m_0$, $\Omega=-He/m_0c$, $k_n=\pm1$, and $\omega$ is the wave angular frequency in $S$.

Since by (21) $\rho'$ is the coefficient of reflection in $S'$ of a wave of frequency $\omega_i'$, $M$ is now given by (22) with $\omega=\omega_i'\sqrt{K}$, i.e., by

$$M=\{1-p_0^2/\omega_i'K^{\frac{1}{2}}(\omega_i'K^{\frac{1}{2}}-k_n\Omega)\}^{\frac{1}{2}}. \tag{23}$$

To an observer at rest in $S'$, (19) and (21) give the frequency multiplication $\mu=\omega_r'/\omega_i'$ and intensity amplification $\alpha=P_r'/P_i'$, which are caused by reflection from the moving slab of electrons. Since $\mu=K$ and $\alpha=K^2$, it follows that only $\alpha$ depends on $\rho$.

When $M$ is real and $\ll1$ or $\gg1$, or when $M$ is purely imaginary, then $\alpha=K^2$ and the slab behaves like a perfect mirror in motion.

It is interesting to note that when $M=U/c$ then $\rho'=1$, i.e., the moving slab reflects like a perfect mirror at rest (i.e., without amplification) but multiplies the frequency by $K$. So when $U\sim c$ we have $M\sim1$ and may use a beam of low density for frequency multiplication. However, since there is no amplification this case will probably not be important in practice.

A low density can also be used with amplification $(\alpha=K^2)$ when $|M|\gg1$, i.e., by (23) when $k_n\Omega>0$ and

$$\omega_i'\sim|\Omega|/\sqrt{K}=1.77\times10^7H/\sqrt{K}. \tag{24}$$

This requires the proper choice of the direction of the magnetic field $H$, and a value of $H$ which is $\sqrt{K}$ times larger than that corresponding to a gyrofrequency equal to the frequency of the incident wave. For example, for an incident 3-cm wave the magnetic field would need to be about $3300\sqrt{K}$ gauss; hence, $K$ could easily be as high as 25. If $H_m$ and $K_m$ are the practically maximum possible values of $H$ and $K$, respectively, then

$$\left.\begin{aligned} \omega_i' &= 1.77\times 10^7 H_m/\sqrt{K_m}, \\ \omega_r' &= 1.77\times 10^7 H_m\sqrt{K_m}. \end{aligned}\right\} \qquad (25)$$

Thus, if $H_m=20{,}000$ gauss and $K_m=94$ (corresponding to 2-Mev electrons), we have $\lambda_i'=5.2$ cm, $\lambda_r'=0.55$ mm and an amplification $\alpha=8900$. The resonance required here is the less critical the higher the electron density.

Equation (20) suggests also that the wave inside the medium will have a multiplied frequency when $|M|U \ll c$ or $\gg c$. But the observable wave intensity is then very much diminished.

For the other case of interest, where the magnetic field is transverse to the electron beam, the equation of dispersion (22) is replaced by the equations

$$M=(1-p_0^2/\omega^2)^{\frac{1}{2}} \qquad (26)$$

and

$$M=\{1-p_0^2(\omega^2-p_0^2)/\omega^2(\omega^2-p_0^2-\Omega^2)\}^{\frac{1}{2}}, \qquad (27)$$

which relate to linearly polarized waves in the frame $S$ with their electric vectors, respectively, parallel and perpendicular to the magnetic field.

Transforming to the frame $S'$ by replacing $\omega$ in (26) and (27) by $\omega_i'\sqrt{K}$, we find, as before, that the frequency multiplication $\mu=K$ is independent of the coefficient of reflection $\rho$, but the intensity amplification $\alpha=K^2\rho$ depends on $\rho$. Also, as before, when $M$ is real and $\ll 1$ or $\gg 1$, or when $M$ is pure imaginary, then $\alpha=K^2$. When $M=U/c$ and $U\sim c$ we may again use a beam of low intensity for frequency multiplication, but again we lose the advantage of amplification.

A low density can also be used with amplification of intensity ($\alpha=K^2$) when the wave's electric vector is perpendicular to the magnetic field and $|M|\gg 1$, i.e., by (27), when

$$\omega_i'\sim\{(\Omega^2+p_0^2)/K\}^{\frac{1}{2}}.$$

If $p_0^2\ll\Omega^2$, i.e., $(4\pi Nec)^2\ll H^2$, this last condition reduces to (24), and we are led to the same formulas (25) and conclusions as in the case in which the magnetic field is parallel to the electron beam.

It thus appears that simultaneous frequency multiplication and wave amplification is possible both with and without a magnetic field; but the presence of a longitudinal or transverse magnetic field of suitable strength allows this to be done with beams of lower electron density.

## VI. CONCLUSION. EXPERIMENTS PLANNED

Irrespective of how one may visualize the reflection process, it is not possible to state in simple terms how much of the energy is reflected in such a direction as to be detected at a higher frequency in the laboratory frame, since this also depends amongst other factors on the band width of the detecting device. However, certain simple considerations lead one to believe that the useful part of the radiation differs from the total radiation by a factor not very different from $\frac{1}{2}$.[5] This factor has not been taken into account in the following discussion.

Referring to the numerical example of Sec. IV, we may now introduce the relativistic increase of intensity and select a suitable value of $K$ from Table I. If we aim at the very moderate value $K\sim 30$, which requires only about 1 Mev, the total radiation from the beam at the frequency $3\times 10^5$ Mc/sec ($\lambda=1$ mm) should be about 10 mw.

From this estimate we conclude that, by using the frequency conversion and wave amplification indicated by the foregoing theory, it should be possible to generate waves of wavelength equal to a small fraction of a millimeter and of useful intensity. It is hoped to verify the theoretical conclusions and produce such short waves in this laboratory with the help of a small betatron at present under construction. It is clear that a van de Graaff machine or linear accelerator of corresponding energy would make the experimental work easier because the electron density in the beams of these machines is much higher than in a betatron. But such instruments are not available in this laboratory. On the other hand, a betatron has the advantage that the disturbing $\gamma$-radiation may be largely eliminated by omitting the target. It is also hoped that by a slight elaboration of the method of observation planned for these experiments it may be possible to investigate the transverse Doppler effect which, as far as we are aware, has not hitherto been observed.

This problem arose out of work on wave amplification in ionized media carried out by the writer in association with Professor Bailey.

[5] From more accurate calculations carried out by Motz (reference 1) it appears that this factor is 1/20 for a 10 percent spread of frequency. However, the value of the field strength assumed in our numerical example for the "original" wave (100 volts/cm) was purposely chosen very small. There is no difficulty in increasing this value at least one hundredfold.