The Solution of the Fluctuation Problem in Electron-Photon Shower Theory

H. MESSEL AND R. B. POTTS
University of Adelaide, Adelaide, South Australia
(Received February 25, 1952)

The analytical solution is given for the fluctuation problem arising in electron-photon shower theory under approximation A. The diffusion equations for two fundamental distribution functions are derived, and by transforming them into matrix recurrence relations their solution is obtained directly. From one of these distribution functions follows the analytical solution for the (n, m)th moments. The method of solution is similar to that previously employed by Messel and Potts to solve the fluctuation problems in nucleon shower theory. The G-equations used by Janossy and Scott play no role in the solution of the problem.

I. INTRODUCTION

ALTHOUGH the fluctuation problem in electron-photon shower theory has received much attention in the last fifteen years, little progress has been made towards its solution. Furry,[1] Arley,[2] Euler,[3] Nordsieck *et al.*,[4] and Scott and Uhlenbeck,[5] who were among the first to discuss the problem, were mainly concerned with the simplified "Furry" model of the shower. More recently the problem has been investigated by Arley,[6] Bhabha,[7] Bhabha and Ramakrishnan,[8] Janossy,[9] Janossy and Messel,[10] Messel[11] and Scott.[12] Even when ionization loss has been neglected (approximation A) explicit analytical solutions have been obtained for only the first and second moments of the distribution function;[5,8,10] for these, Janossy and Messel[10] have given extensive numerical results, and using them Messel[11] has calculated the probability function, assuming it to be a Polya distribution.

It is the purpose of the present paper to give the analytical solution of the fluctuation problem in approximation A. We have previously[13–16] given the solutions of similar problems in nucleon cascade theory and the present work is a further application of the methods developed there. In the present problem, the method consists essentially in transforming the "last-collision" diffusion equation for the distribution function into a matrix recurrence relation, the solution of which follows immediately. A similar procedure is used for obtaining the nth moments.

II. DEFINITIONS

Let $p_{n,m}^{(j)}(\eta_1, \cdots, \eta_n; \eta_{n+1}, \cdots, \eta_{n+m}; x)d\eta_1\cdots d\eta_{n+m}$ be the differential probability that after a depth x cascade units a primary (j) of unit energy has given rise to n electrons with energies in the ranges η_k, $\eta_k+d\eta_k$, $k=1, \cdots, n$ in any order, and m photons with energies in the ranges η_{n+l}, $\eta_{n+l}+d\eta_{n+l}$, $l=1, \cdots, m$ in any order. When $j=1$, the primary is an electron and when $j=2$, a photon. Further, let $q_{n,m}^{(j)}(\eta_1, \cdots, \eta_n; \eta_{n+1}, \cdots, \eta_{n+m}; x)d\eta_1\cdots d\eta_{n+m}$ be the differential probability that that after a depth x a primary (j) of unit energy has given rise to n electrons with energies in the intervals $d\eta_k$, $k=1, \cdots, n$ and to m photons with energies in the intervals $d\eta_{n+l}$, $l=1, \cdots, m$ and to any numbers of electrons and photons with arbitrary energies.

The relation between $q_{n,m}^{(j)}$ and $p_{n,m}^{(j)}$ is expressed by

$$q_{n,m}^{(j)}=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{1}{a!b!}\int_0^1 d\eta_{n+1}\cdots\int_0^1 d\eta_{n+a} \times\int_0^1 d\eta_{n+a+m+1}\cdots\int_0^1 d\eta_{n+a+m+b}p_{n+a,m+b}^{(j)}, \quad (1)$$

and the inverse relation

$$p_{n,m}^{(j)}=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{(-1)^{a+b}}{a!b!}\int_0^1 d\eta_{n+1}\cdots\int_0^1 d\eta_{n+a} \times\int_0^1 d\eta_{n+a+m+1}\cdots\int_0^1 d\eta_{n+a+m+b}q_{n+a,m+b}^{(j)}. \quad (2)$$

Hence, if either $p_{n,m}^{(j)}$ or $q_{n,m}^{(j)}$ is known the other may in theory be determined. It is found, however, to be more expedient to derive diffusion equations for each of the $p_{n,m}^{(j)}$ and $q_{n,m}^{(j)}$ and to solve them directly.

The (n, m)th factorial moment $T_{n,m}^{(j)}(\eta; x)$ is defined by

$$T_{n,m}^{(j)}(\eta;x) =\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{(n+a)!}{a!}\frac{(m+b)!}{b!}\varphi_{n+a,m+b}^{(j)}(\eta;x), \quad (3)$$

[1] W. H. Furry, Phys. Rev. **52**, 569 (1937).
[2] N. Arley, Proc. Roy. Soc. (London) **A168**, 519 (1938).
[3] H. Euler, Z. Physik **110**, 450 (1938).
[4] Nordsieck, Lamb, and Uhlenbeck, Physica **7**, 344 (1940).
[5] W. T. Scott and G. E. Uhlenbeck, Phys. Rev. **62**, 497 (1942).
[6] N. Arley, *On the Theory of Stochastic Processes* (John Wiley & Sons, Inc., New York, 1949).
[7] H. J. Bhabha, Proc. Roy. Soc. (London) **A202**, 301 (1950).
[8] H. J. Bhabha and A. Ramakrishnan, Proc. Indian Acad. Sci. **32**, 141 (1950).
[9] L. Janossy, Proc. Phys. Soc. (London) **A63**, 241 (1950).
[10] L. Janossy and H. Messel, Proc. Phys. Soc. (London) **A63**, 1101 (1950).
[11] H. Messel, Proc. Phys. Soc. (London) **A64**, 807 (1951).
[12] W. T. Scott, Phys. Rev. **82**, 893 (1951).
[13] H. Messel, Proc. Phys. Soc. (London) (to be published, 1952).
[14] H. Messel and R. B. Potts, Proc. Phys. Soc. (London) (to be published, 1952).
[15] H. Messel and R. B. Potts, Proc. Phys. Soc. (London) (to be published, 1952).
[16] H. Messel and J. W. Gardner, Phys. Rev. **84**, 1256 (1951).

where $\varphi_{n+a,\,m+b}{}^{(j)}(\eta;x)$ is the probability that after a depth x a primary (j) of unit energy has given rise to $n+a$ electrons and $m+b$ photons with energies $>\eta$ and any numbers of electrons and photons with energies $<\eta$. The inverse relation is expressed by

$$\varphi_{n,\,m}{}^{(j)}(\eta;x)=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{(-1)^{a+b}}{a!b!n!m!}T_{n+a,\,m+b}{}^{(j)}(\eta;x). \quad (4)$$

Here again, if either $\varphi_{n,\,m}{}^{(j)}$ or $T_{n,\,m}{}^{(j)}$ is known the other can be found by a double summation, but this may be exceedingly difficult to perform.

The function $\varphi_{n,\,m}{}^{(j)}$ is related to the $p_{n,\,m}{}^{(j)}$ by

$$\varphi_{n,\,m}{}^{(j)}=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{1}{a!b!n!m!}\int_{\eta}^{1}d\eta_1\cdots\int_{\eta}^{1}d\eta_n\int_{0}^{\eta}d\eta_{n+1}\cdots$$
$$\times\int_{0}^{\eta}d\eta_{n+a}\int_{\eta}^{1}d\eta_{n+a+1}\cdots$$
$$\times\int_{\eta}^{1}d\eta_{n+a+m}\int_{0}^{\eta}d\eta_{n+a+m+1}\cdots$$
$$\times\int_{0}^{\eta}d\eta_{n+a+m+b}p_{n+a,\,m+b}{}^{(j)}. \quad (5)$$

The moments $T_{n,\,m}{}^{(j)}$ may be expressed in terms of the distribution function $q_{n,\,m}{}^{(j)}$ by means of the simple relation (see reference 15),

$$T_{n,\,m}{}^{(j)}(\eta;x)=\int_{\eta}^{1}d\eta_1\cdots\int_{\eta}^{1}d\eta_{n+m}q_{n,\,m}{}^{(j)}$$
$$\times(\eta_1,\cdots,\eta_n;\eta_{n+1},\cdots,\eta_{n+m};x). \quad (6)$$

It is this relation which will be used for obtaining the $T_{n,\,m}{}^{(j)}$. Once the solution for $q_{n,\,m}{}^{(j)}$ is known, the moments are obtained by an elementary integration over the energy variables.

For the cross sections, the Bethe-Heitler expressions in the full-screening approximation will be used; $w^{(1)}(\eta_k,\eta_l)d\eta_l dx$ will denote the probability that an electron of energy $\eta_k+\eta_l$ emits in a distance dx a photon of energy in the range η_l, $\eta_l+d\eta_l$ and $w^{(2)}(\eta_k,\eta_l)d\eta_l dx$ the probability for pair production. For the total cross sections we write

$$\alpha^{(j)}=\int_0^1 w^{(j)}(\eta_k,\eta_l)d\eta_l. \quad (7)$$

For $j=1$, this integral diverges (the infrared catastrophe) and to obtain numerical results for the $p_{n,\,m}{}^{(j)}$ it is necessary to impose a cutoff at the lower limit of integration. In the solutions for the $q_{n,\,m}{}^{(j)}$ and the $T_{n,\,m}{}^{(j)}$ the divergences cancel out.

III. THE SOLUTION FOR $p_{n,\,m}{}^{(j)}$

By considering all possible last collisions, the diffusion equation satisfied by the $p_{n,\,m}{}^{(j)}$ is obtained in the form

$$(\partial/\partial x+n\alpha^{(1)}+m\alpha^{(2)})p_{n,\,m}{}^{(j)}$$
$$=\sum_{C_1^n}\sum_{C_1^m}p_{n,\,m-1}{}^{(j)}(\eta_1',\cdots,\eta_{n-1}',\eta_n'+\eta_{n+m}';$$
$$\eta_{n+1}',\cdots,\eta_{n+m-1}';x)w^{(1)}(\eta_n',\eta_{n+m}')$$
$$+\sum_{C_2^n}p_{n-2,\,m+1}{}^{(j)}(\eta_1',\cdots,\eta_{n-2}';\eta_{n+1},\cdots,$$
$$\eta_{n+m},\eta_{n-1}'+\eta_n';x)w^{(2)}(\eta_{n-1}',\eta_n'), \quad (8)$$

with the initial conditions

$$p_{n,\,m}{}^{(j)}(x=0)=\delta_{n+j,\,2}\delta_{m+1,\,j}\delta(1-\eta_1). \quad (9)$$

In (4), the sum over C_1^n and C_2^n signify summations over all possible choices of η_n' and η_{n-1}', η_n' respectively from the η_k, $k=1,\cdots,n$ and the sum over C_1^m signifies summation over all possible choices of η_{n+m}' from the η_{n+l}, $l=1,\cdots,m$.

If we define the Laplace transform of $p_{n,\,m}{}^{(j)}$ as

$$P_{n,\,m}{}^{(j)}(\lambda)=\int_0^{\infty}e^{-\lambda x}p_{n,\,m}{}^{(j)}(x)dx, \quad (10)$$

then (8) may be transformed into the matrix equations

$$\left[\lambda\mathbf{E}_N+\sum_{k=1}^{N}\boldsymbol{\alpha}_N(k)\right]\mathbf{P}_N(\eta_1,\cdots,\eta_N;\lambda)$$
$$=\sum_{C_2^N}\mathbf{w}_{N-1}(\eta_{N-1},\eta_N)\mathbf{P}_{N-1}(\eta_1,\cdots,$$
$$\eta_{N-2},\eta_{N-1}+\eta_N;\lambda),\quad N>1 \quad (11)$$

and

$$[\lambda\mathbf{E}_1+\boldsymbol{\alpha}_1(1)]\mathbf{P}_1(\eta_1;\lambda)=\mathbf{E}_1\delta(1-\eta_1),\quad N=1. \quad (12)$$

$\mathbf{E}_N$ is the unit matrix of order 2^N. The $\boldsymbol{\alpha}_N(k)$ are given by the direct product of N matrices

$$\boldsymbol{\alpha}_N(k)=\mathbf{E}_1\times\cdots\times\underset{k\text{th place}}{\begin{bmatrix}\alpha^{(1)} & 0\\ 0 & \alpha^{(2)}\end{bmatrix}}\times\cdots\times\mathbf{E}_1. \quad (13)$$

The $\mathbf{P}_N$ is a $2^N\times 2$ matrix the columns of which correspond to $P^{(1)}$ and $P^{(2)}$ and the rows are ordered by writing $\eta_1\cdots\eta_N$ as a binary number with digits 1 and 2, standing for an electron and photon respectively. The $\mathbf{w}_{N-1}$ is a $2^N\times 2^{N-1}$ matrix in which the nonzero elements $w^{(j)}(\eta_k,\eta_l)$ are ordered according to the following rules:

(1) If in the binary number $\eta_1\cdots\eta_N$, $\eta_k=1$ and $\eta_l=1$, then all the elements of the row corresponding to this number are zero except for the term $2w^{(2)}(\eta_k,\eta_l)$, which is placed in the first even-numbered column in which this term has not already appeared.

(2) If $\eta_k=1$ and $\eta_l=2$, then all the elements of the row are zero except for the term $w^{(1)}(\eta_k, \eta_l)$, which is placed in the first odd-numbered column in which this term has not already appeared.

(3) If $\eta_k=2$ and $\eta_l=1$, then all the elements of the row are zero except for the term $w^{(1)}(\eta_l, \eta_k)$, which is placed in the first odd-numbered column in which this term has not already appeared.

(4) If $\eta_k=2$ and $\eta_l=2$, all the elements of the row are zero.

According to these rules there will be two non-zero elements in each odd-numbered column and one in each even-numbered column.

Equation (11) is a simple matrix recurrence relation, the solution of which is immediately given by[17]

$$\mathbf{P}_N(\eta_1, \cdots, \eta_N; \lambda)=\left\{\prod_{l=N-1}^{1}\sum_{C_2^{l+1}}\left[\lambda\mathbf{E}_{l+1}+\sum_{k=1}^{l+1}\boldsymbol{\alpha}_{l+1}(k)\right]^{-1}\mathbf{w}_l(\eta_l, \eta_{l+1}+\cdots+\eta_N)\right\}\times[\lambda\mathbf{E}_1+\boldsymbol{\alpha}_1(1)]^{-1}\delta(1-\eta_1-\cdots-\eta_N), \quad (14)$$

where the sum over C_2^{l+1} signifies summation over all possible choices of η_l, $\eta_{l+1}+\cdots+\eta_N$ from $\eta_1, \eta_2, \cdots, \eta_l, \eta_{l+1}+\cdots+\eta_N$. The $p_{n,m}^{(i)}$ are now obtained by taking an inverse Laplace transform.

The solution given in (14) should be compared with that given for the corresponding functions appearing in nucleon cascade theory.[13,14] The delta function in (14) merely ensures the conservation of energy in the shower, which is a consequence of the full-screening cross sections used and the neglect of ionization losses.

In order to illustrate our notation, the case $N=3$ will now be discussed in detail.

From (14)

$$\begin{aligned}\mathbf{P}_3(\eta_1, \eta_2, \eta_3; \lambda) &=\left\{\prod_{l=2}^{1}\sum_{C_2^{l+1}}\left[\lambda\mathbf{E}_{l+1}+\sum_{k=1}^{l+1}\boldsymbol{\alpha}_{l+1}(k)\right]^{-1}\times\mathbf{w}_l(\eta_l, \eta_{l+1}+\cdots+\eta_3)\right\}\times[\lambda\mathbf{E}_1+\boldsymbol{\alpha}_1(1)]^{-1}\delta(1-\eta_1-\eta_2-\eta_3)\\ &=\sum_{C_2^3}[\lambda\mathbf{E}_3+\boldsymbol{\alpha}_3(1)+\boldsymbol{\alpha}_3(2)+\boldsymbol{\alpha}_3(3)]^{-1}\mathbf{w}_2(\eta_2, \eta_3)\\ &\quad\times[\lambda\mathbf{E}_2+\boldsymbol{\alpha}_2(1)+\boldsymbol{\alpha}_2(2)]^{-1}\mathbf{w}_1(\eta_1, \eta_2+\eta_3)\\ &\quad\times[\lambda\mathbf{E}_1+\boldsymbol{\alpha}_1(1)]^{-1}\delta(1-\eta_1-\eta_2-\eta_3).\end{aligned}$$

From the binary numbers

$$\eta_1\eta_2\eta_3=111,\ 112,\ 121,\ 122,\ 211,\ 212,\ 221,\ 222,$$

the explicit form for $\mathbf{P}_3(\eta_1, \eta_2, \eta_3; \lambda)$ is

$$\mathbf{P}_3=\begin{bmatrix} P_{3,0}^{(1)}(\eta_1,\eta_2,\eta_3;\lambda) & P_{3,0}^{(2)}(\eta_1,\eta_2,\eta_3;\lambda)\\ P_{2,1}^{(1)}(\eta_1,\eta_2;\eta_3;\lambda) & P_{2,1}^{(2)}(\eta_1,\eta_2;\eta_3;\lambda)\\ P_{2,1}^{(1)}(\eta_1,\eta_3;\eta_2;\lambda) & P_{2,1}^{(2)}(\eta_1,\eta_3;\eta_2;\lambda)\\ P_{1,2}^{(1)}(\eta_1;\eta_2,\eta_3;\lambda) & P_{1,2}^{(2)}(\eta_1;\eta_2,\eta_3;\lambda)\\ P_{2,1}^{(1)}(\eta_2,\eta_3;\eta_1;\lambda) & P_{2,1}^{(2)}(\eta_2,\eta_3;\eta_1;\lambda)\\ P_{1,2}^{(1)}(\eta_2;\eta_1,\eta_3;\lambda) & P_{1,2}^{(2)}(\eta_2;\eta_1,\eta_3;\lambda)\\ P_{1,2}^{(1)}(\eta_3;\eta_1,\eta_2;\lambda) & P_{1,2}^{(2)}(\eta_3;\eta_1,\eta_2;\lambda)\\ P_{0,3}^{(1)}(\eta_1,\eta_2,\eta_3;\lambda) & P_{0,3}^{(2)}(\eta_1,\eta_2,\eta_3;\lambda) \end{bmatrix}.$$

From the rules (1)–(4) above and the binary numbers η_1, $\eta_2+\eta_3=11,\ 12,\ 21,\ 22$ the matrix $\mathbf{w}_1(\eta_1, \eta_2+\eta_3)$ is

$$\mathbf{w}_1(\eta_1, \eta_2+\eta_3)=\begin{bmatrix} 0 & 2w^{(2)}(\eta_1,\eta_2+\eta_3)\\ w^{(1)}(\eta_1,\eta_2+\eta_3) & 0\\ w^{(1)}(\eta_2+\eta_3,\eta_1) & 0\\ 0 & 0 \end{bmatrix}$$

and from the binary number $\eta_1\eta_2\eta_3$

$$\mathbf{w}_2(\eta_2, \eta_3)=\begin{bmatrix} 0 & 2w^{(2)}(\eta_2,\eta_3) & 0 & 0\\ w^{(1)}(\eta_2,\eta_3) & 0 & 0 & 0\\ w^{(1)}(\eta_3,\eta_2) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 2w^{(2)}(\eta_2,\eta_3)\\ 0 & 0 & w^{(1)}(\eta_2,\eta_3) & 0\\ 0 & 0 & w^{(1)}(\eta_3,\eta_2) & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{w}_2(\eta_1, \eta_2)=\begin{bmatrix} 0 & 2w^{(2)}(\eta_1,\eta_2) & 0 & 0\\ 0 & 0 & 0 & 2w^{(2)}(\eta_1,\eta_2)\\ w^{(1)}(\eta_1,\eta_2) & 0 & 0 & 0\\ 0 & 0 & w^{(1)}(\eta_1,\eta_2) & 0\\ w^{(1)}(\eta_2,\eta_1) & 0 & 0 & 0\\ 0 & 0 & w^{(1)}(\eta_2,\eta_1) & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

[17] Scott (see reference 12) has recently considered the fluctuation problem for a simplified model of the electron-photon shower. Although he was able to obtain a recurrence relation for the distribution function he was unable to solve it. His recurrence relation is a very special case of (11) and its solution is (in his notation)

$$r_N=\prod_{l=N-1}^{1}\sum_{C_2^{l+1}}(E_l+E_{l+1}+\cdots+E_N)^{-1},$$

where the sum over C_2^{l+1} signifies summation over all possible ways of choosing E_l, $E_{l+1}+\cdots+E_N$ from $E_1, \cdots, E_l, E_{l+1}+\cdots+E_N$.

In a similar way one obtains $\mathbf{w}_2(\eta_1, \eta_3)$. It should be noted that $\mathbf{w}_2(\eta_1, \eta_2)$ cannot be obtained from $\mathbf{w}_2(\eta_2, \eta_3)$ by merely changing the variables in the elements of the matrix.

From (13) $\lambda\mathbf{E}_3+\boldsymbol{\alpha}_3(1)+\boldsymbol{\alpha}_3(2)+\boldsymbol{\alpha}_3(3)$ is the diagonal matrix

$$\begin{bmatrix} \lambda+3\alpha^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda+2\alpha^{(1)}+\alpha^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda+2\alpha^{(1)}+\alpha^{(2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda+\alpha^{(1)}+2\alpha^{(2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda+2\alpha^{(1)}+\alpha^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda+\alpha^{(1)}+2\alpha^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda+\alpha^{(1)}+2\alpha^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda+3\alpha^{(2)} \end{bmatrix}$$

and similarly one obtains $\lambda\mathbf{E}_2+\boldsymbol{\alpha}_2(1)+\boldsymbol{\alpha}_2(2)$ and $\lambda\mathbf{E}_1+\boldsymbol{\alpha}_1(1)$. The inverses of these matrices are obtained by replacing the diagonal elements by their reciprocals.

By carrying out the matrix product one obtains, for instance,

$$P_{3,0}^{(1)}(\eta_1, \eta_2, \eta_3; \lambda)=\frac{\sum_{C_2^3} 2w^{(2)}(\eta_2, \eta_3)w^{(1)}(\eta_1, \eta_2+\eta_3)}{(p+3\alpha^{(1)})(p+\alpha^{(1)}+\alpha^{(2)})(p+\alpha^{(1)})} \times\delta(1-\eta_1-\eta_2-\eta_3).$$

Similar expressions are obtained for $P_{2,1}^{(2)}(\eta_1, \eta_2; \eta_3; \lambda)$ and $P_{1,2}^{(1)}(\eta_1; \eta_2, \eta_3; \lambda)$. The solution for $p_{3,0}^{(1)}(\eta_1, \eta_2, \eta_3; x)$ follows by taking the inverse Laplace transform of $P_{3,0}^{(1)}(\eta_1, \eta_2, \eta_3; \lambda)$.

IV. THE SOLUTION FOR $q_{n,m}^{(j)}$ AND $T_{n,m}^{(j)}$

The "last-collision" diffusion equation satisfied by $q_{n,m}^{(j)}$ is given by

$$\begin{aligned}(\partial/\partial x+n\alpha^{(1)}+m\alpha^{(2)})&q_{n,m}^{(j)}(\eta_1, \cdots, \eta_n; \eta_{n+1}, \cdots, \eta_{n+m}; x)\\ =\sum_{C_1^n}\sum_{C_1^m} &q_{n,m-1}^{(j)}(\eta_1', \cdots, \eta_{n-1}', \eta_n'+\eta_{n+m}'; \eta_{n+1}', \cdots, \eta_{n+m-1}'; x)w^{(1)}(\eta_n', \eta_{n+m}')\\ +\sum_{C_2^n} &q_{n-2,m+1}^{(j)}(\eta_1', \cdots, \eta_{n-2}'; \eta_{n+1}, \cdots, \eta_{n+m}, \eta_{n-1}'+\eta_n'; x)w^{(2)}(\eta_{n-1}', \eta_n')\\ +\sum_{C_1^n}\int_0^1 &q_{n-1,m+1}^{(j)}(\eta_1', \cdots, \eta_{n-1}'; \eta_{n+1}, \cdots, \eta_{n+m}, u; x)2w^{(2)}(u-\eta_n', \eta_n')du\\ +\sum_{C_1^n}\int_0^1 &q_{n,m}^{(j)}(\eta_1', \cdots, \eta_{n-1}', u; \eta_{n+1}, \cdots, \eta_{n+m}; x)w^{(1)}(\eta_n', u-\eta_n')du\\ +\sum_{C_1^m}\int_0^1 &q_{n+1,m-1}^{(j)}(\eta_1, \cdots, \eta_n, u; \eta_{n+1}', \cdots, \eta_{n+m-1}'; x)w^{(1)}(u-\eta_{n+m}', \eta_{n+m}')du \qquad (15)\end{aligned}$$

where $q_{0,0}^{(j)}(x)=1$. The initial conditions for $q_{n,m}^{(j)}$ are given by

$$q_{n,m}^{(j)}(x=0)=\delta_{n+j,2}\delta_{m+1,j}\delta(1-\eta_1)$$

and

$$q_{0,0}^{(j)}(x=0)=1. \qquad (16)$$

We define the n-fold Mellin and single Laplace transform of $q_{n,m}^{(j)}$ by

$$Q_{n,m}^{(j)}(s_1, \cdots, s_n; s_{n+1}, \cdots, s_{n+m}; \lambda) =\int_0^1 d\eta_1\cdots\int_0^1 d\eta_{n+m}\int_0^\infty dx\eta_1^{s_1}\cdots \times\eta_{n+m}^{s_{n+m}}e^{-\lambda x}q_{n,m}^{(j)} \qquad (17)$$

and

$$W^{(j)}(s_1, s_2) =\int_0^1\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{s_1}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{s_2}w^{(j)}(\eta_1, \eta_2)d\eta_2. \qquad (18)$$

By taking the Mellin-Laplace transform of (15) we get

$$\begin{aligned}(\lambda+n\alpha^{(1)}+m\alpha^{(2)})&Q_{n,m}^{(j)}(s_1, \cdots, s_n; s_{n+1}, \cdots, s_{n+m}; \lambda)-\delta_{n+j,2}\delta_{m+1,j}\\ =\sum_{C_1^n}\sum_{C_1^m} &Q_{n,m-1}^{(j)}(s_1', \cdots, s_{n-1}', s_n'+s_{n+m}'; s_{n+1}', \cdots, s_{n+m-1}'; \lambda)W^{(1)}(s_n', s_{n+m}')\\ +\sum_{C_2^n} &Q_{n-2,m+1}^{(j)}(s_1', \cdots, s_{n-2}'; s_{n+1}, \cdots, s_{n+m}, s_{n-1}'+s_n'; \lambda)W^{(2)}(s_{n-1}', s_n')\\ +\sum_{C_1^n} &Q_{n-1,m+1}^{(j)}(s_1', \cdots, s_{n-1}'; s_{n+1}, \cdots, s_{n+m}, s_n'; \lambda)2W^{(2)}(s_n', 0)\\ +\sum_{C_1^n} &Q_{n,m}^{(j)}(s_1, \cdots, s_n; s_{n+1}, \cdots, s_{n+m}; \lambda)W^{(1)}(s_n', 0)\\ +\sum_{C_1^m} &Q_{n+1,m-1}^{(j)}(s_1, \cdots, s_n, s_{n+m}'; s_{n+1}', \cdots, s_{n+m-1}'; \lambda)W^{(1)}(0, s_{n+m}'). \qquad (19)\end{aligned}$$

The first two terms on the right hand side of (19) are similar in form to those in (8) and may be expressed in matrix notation as before. By transferring the last three terms of the right hand side of (19) to the left hand side, the equation may be written in the following matrix form:

$$\left[\lambda\mathbf{E}_N+\sum_{k=1}^{N}\mathbf{A}_N(s_k)\right]\mathbf{Q}_N(s_1,\cdots,s_N;\lambda)$$
$$=\sum_{C_2^N}\mathbf{W}_{N-1}(s_{N-1},s_N)\mathbf{Q}_{N-1}(s_1,\cdots,s_{N-2},s_{N-1}+s_N;\lambda),\quad N>1 \qquad (20)$$

and

$$[\lambda\mathbf{E}_1+\mathbf{A}_1(s_1)]\mathbf{Q}_1(s_1;\lambda)=\mathbf{E}_1,\quad N=1. \qquad (21)$$

The matrices $\mathbf{Q}_N$ and $\mathbf{W}_{N-1}$ are constructed in a manner identical to that used for $\mathbf{P}_N$ and $\mathbf{w}_{N-1}$ in (11). The $\mathbf{A}_N(s_k)$ are given by the direct product of N matrices:

$$\mathbf{A}_N(s_k)=\mathbf{E}_1\times\cdots\times\underset{k\text{th place}}{\begin{bmatrix}A_1(s_k) & A_2(s_k)\\ A_3(s_k) & A_4(s_k)\end{bmatrix}}\times\cdots\times\mathbf{E}_1, \qquad (22)$$

where, as in reference 10,

$$\begin{aligned}A_1(s)&=A(s+1)=\alpha^{(1)}-W^{(1)}(s,0)\\ A_2(s)&=-B(s+1)=-2W^{(2)}(s,0)\\ A_3(s)&=-C(s+1)=-W^{(1)}(0,s)\\ A_4(s)&=D=\alpha^{(2)}.\end{aligned} \qquad (23)$$

Note that the divergent term $\alpha^{(1)}$ has cancelled out in $A_1(s)$ above.

The solution of Eq. (20) is

$$\mathbf{Q}_N(s_1,\cdots,s_N;\lambda)=\left\{\prod_{l=N-1}^{1}\sum_{C_2^{l+1}}\left[\lambda\mathbf{E}_{l+1}+\sum_{k=1}^{l+1}\mathbf{A}_{l+1}(s_k)\right]^{-1}\mathbf{W}_l(s_l,s_{l+1}+\cdots+s_N)\right\}\times[\lambda\mathbf{E}_1+\mathbf{A}_1(s_1+\cdots+s_N)]^{-1}, \qquad (24)$$

where the sum over C_2^{l+1} signifies summation over all possible choices of s_l, $s_{l+1}+\cdots+s_N$ from $s_1,\cdots,s_l$, $s_{l+1}+\cdots+s_N$, and where for $k=l+1$, the matrix $\mathbf{A}_{l+1}(s_k)$ stands for $\mathbf{A}_{l+1}(s_{l+1}+\cdots+s_N)$. For example, when $l=2$, the expression inside the brackets in (24) is

$$[\lambda\mathbf{E}_2+\mathbf{A}_3(s_1)+\mathbf{A}_3(s_2)+\mathbf{A}_3(s_3+\cdots+s_N)]^{-1}\times\{\mathbf{W}_2(s_2,s_3+\cdots+s_N)+\mathbf{W}_2(s_1,s_3+\cdots+s_N)+\mathbf{W}_2(s_1,s_2)\}.$$

By taking the inverse transform of (17) we immediately obtain the solution for the $q_{n,m}^{(i)}$

$$q_{n,m}^{(i)}(\eta_1,\cdots,\eta_n;\eta_{n+1},\cdots,\eta_{n+m};x)=\frac{1}{(2\pi i)^{n+m+1}}\int_{u_1-i\infty}^{u_1+i\infty}ds_1\cdots\times\int_{u_{n+m}-i\infty}^{u_{n+m}+i\infty}ds_{n+m}\int_{\lambda_0-i\infty}^{\lambda_0+i\infty}d\lambda e^{\lambda x}\times\eta_1^{-(s_1+1)}\cdots\eta_{n+m}^{-(s_{n+m}+1)}Q_{n,m}^{(i)}; \qquad (25)$$

and from (6) the analytical solution for the moments is

$$T_{n,m}^{(i)}(\eta;x)=\frac{1}{(2\pi i)^{n+m+1}}\int_{u_1-i\infty}^{u_1+i\infty}\frac{ds_1}{s_1}\cdots\times\int_{u_{n+m}-i\infty}^{u_{n+m}-i\infty}\frac{ds_{n+m}}{s_{n+m}}\int_{0-i\infty}^{\lambda_0+i\infty}d\lambda e^{\lambda x}\times\eta^{-(s_1+\cdots+s_{n+m})}Q_{n,m}^{(i)}. \qquad (26)$$

This completes the analytical solution of the fluctuation problem in approximation A; as all functions of interest may be obtained from the $p_{n,m}^{(i)}$, $q_{n,m}^{(i)}$ and $T_{n,m}^{(i)}$ we have complete knowledge of the number behavior of the shower.

IV. DISCUSSION

In the solution (14) for $\mathbf{P}_N$, the elements of the inverse diagonal matrices are reciprocals of linear functions of λ; and as these are independent of the summations over the energy variables, the inverse Laplace transform may be easily taken. Apart from the depth-dependent factor, the solution for $p_{n,m}^{(i)}$ is a sum of products of various combinations of the cross-sections $w^{(1)}$ and $w^{(2)}$. The complexity of this result reduces its usefulness for numerical calculations, especially as the physically interesting distribution function $\varphi^{(i)}$ is obtained from the $p^{(i)}$ by a difficult double summation.

The solution (24) for $\mathbf{Q}_N$ is of direct physical interest because it leads to the general analytical solution for all the moments, including the correlations between electrons and photons. The diffusion equation for the moments as previously derived[10] from the Janossy G-equation was not in a form amenable to direct solution. The relation (6), however, enabled us to by-pass the G-equation and obtain the solution for the moments by a simple integration over the energy variables of $q_{n,m}^{(i)}$. The solution (26) for the moments contains the special cases $n=1$, $m=0$ and $n=1$, $m=1$ already calculated by Janossy and Messel.[10] Bhabha[18] has recently given a recurrence equation for his correlation functions similar to (19) without, however, obtaining a solution.

In this paper only the fluctuation problem neglecting ionization loss (approximation A) has been discussed. Any results which can be obtained for the problem including ionization losses (approximation B) will be even more complicated in nature and less amenable to numerical calculations than those obtained above. The numerical evaluations in approximation A for $p^{(i)}$, $q^{(i)}$, $\varphi^{(i)}$ and the moments $T^{(i)}$ will be discussed in a later publication together with those for the nucleon cascades.

[18] H. J. Bhabha, Proc. Indian Acad. Sci. **32**, 154 (1950).