The Theory of the Impulse Approximation

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An integral equation is derived for the scattering of a neutron by a bound proton. This equation has the impulse approximation as the first approximation to its solution. The connection between the impulse approximation and the Fermi approximation is discussed and clarified.

I. INTRODUCTION

RECENTLY, Chew[1] has proposed a method, the impulse approximation, for treating problems in which a free particle collides with bound particles. An essential feature of the method is the neglect of binding forces during the collision. However, this simplification alone is, in general, not enough to enable one to solve the problem. If there are N bound scatterers, then one has reduced the problem to an $N+1$ body problem with N interactions, and this reduced problem is itself not susceptible to exact treatment unless $N=1$. In this paper we will not be concerned with the further approximations which are necessary if $N\neq 1$, but will confine ourselves to $N=1$. For this case we will derive the impulse approximation as a well-defined first approximation to the exact solution and will have available an iteration scheme for improving this approximation.

In particular we shall consider a system consisting of a neutron and a proton. The neutron-proton potential will be denoted by V, and the proton will be assumed to be bound by a potential U due to a fixed center of force.[2] The form into which we shall put the solution to this problem will be dictated by our desire to describe the scattering of the neutron by the bound proton in terms of the free neutron-proton scattering. Guided by this we shall in the next section relate the solution of our problem to the solution of a certain integral equation. The impulse approximation will then appear naturally as a first approximation to the solution of this equation. In the final section the connection between the impulse approximation and the Fermi approximation is discussed and clarified.

II. DERIVATION OF THE INTEGRAL EQUATION

Let $\Psi(E)$ be the wave function describing the scattering of the neutron by the bound proton. E is the total energy. $\Psi(E)$ satisfies an integral equation which we can write symbolically[3] as

$$\Psi(E)=\Phi_0(E)+(E^+-T-U)^{-1}V\Psi(E). \tag{1}$$

Here T is the kinetic energy operator and $\Phi_0(E)$ is the incident wave. $\Phi_0(E)$ is therefore the product of a plane wave for the neutron and a bound state wave function for the proton. To compute the differential scattering cross section one needs[3] the matrix element

$$W=(\Phi_F(E),\ V\Psi(E)), \tag{2}$$

where $\Phi_F(E)$ is the wave function of the final state.

In order to relate the bound scattering to the free scattering it seems natural to write[4]

$$\Psi(E)=\sum_{E'}\psi(E')\alpha(E',E), \tag{3}$$

where the $\alpha(E',E)$ are to be determined and where the $\psi(E')$ describe free neutron-proton scattering at an energy E'.[5] $\psi(E')$ thus satisfies the integral equation

$$\psi(E')=\varphi(E')+(E'^+-T)^{-1}V\psi(E'), \tag{4}$$

where, of course, $\varphi(E')$ is simply the product of two plane waves, one for the neutron and one for the proton. To find the integral equation which determines $\alpha(E',E)$ we insert the expansion (3) into the second term on the right-hand side of (1). We then get

$$\Psi(E)=\Phi_0(E)+\sum_{E'}(E^+-T-U)^{-1}V\psi(E')\alpha(E',E)$$

which we rewrite as

$$\Psi(E)=\Phi_0(E)+\sum_{E'}(E'^+-T)^{-1}V\psi(E')\alpha(E',E)$$
$$+\sum_{E'}[(E^+-T-U)^{-1}-(E'^+-T)^{-1}]V\psi(E')\alpha(E',E).$$

But from (4) we can write this as

$$\Psi(E)=\Phi_0(E)+\sum_{E'}(\psi(E')-\varphi(E'))\alpha(E',E)$$
$$+\sum_{E'}[(E^+-T-U)^{-1}-(E'^+-T)^{-1}]V\psi(E')\alpha(E',E),$$

which from (3) is just

$$\sum_{E'}\varphi(E')\alpha(E',E)=\Phi_0(E)+\sum_{E'}[(E^+-T-U)^{-1}$$
$$-(E'^+-T)^{-1}]V\psi(E')\alpha(E',E).$$

Finally, we derive the integral equation[6]

$$\alpha(E',E)=\alpha_i(E',E)+\sum_{E''}\mathcal{G}(E,E',E'')\alpha(E'',E). \tag{5}$$

Here

$$\alpha_i(E',E)=(\varphi(E'),\Phi_0(E)), \tag{6}$$

[1] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952).
[2] This same problem is discussed in the papers of reference 1.
[3] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[4] By the symbol $\Sigma_{E'}$ we mean integration over the two momenta involved in $\psi(E')$.
[5] For simplicity of presentation we have omitted any "pick-up" terms [M. Goldberger and G. F. Chew, Phys. Rev. **77**, 470 (1950)].
[6] Had we included "pick-up" terms this one integral equation would be replaced by two integral equations coupling the simple scattering amplitudes with the "pick-up" amplitudes.

represents a first approximation to $\alpha(E', E)$ which we may hope to improve by iteration. When inserted into (2) and (3), $\alpha_i(E', E)$ gives just the impulse approximation to W. The kernel of our integral equation is given by

$$\mathcal{G}(E, E', E'') = (\varphi(E'), [(E^+ - T - U)^{-1} - (E''^+ - T)^{-1}] V\psi(E'')), \quad (7)$$

and, in principle, is a known function. $(E''^+ - T)^{-1}$ is, of course, just the familiar Green's function for two free particles while $(E^+ - T - U)^{-1}$ is the Green's function for a free neutron and a bound proton. Both are readily expressible in terms of one-particle eigenfunctions.

In connection with the solution of (5) by iteration, it is of interest to note that this corresponds to the solution of (1) by successive impulse approximations. That is, let us write

$$\Psi(E) = \sum_{E'} \psi(E')(\varphi(E'), \Phi_0(E)) + \Psi_1(E),$$

then in a manner analogous to our derivation of (5) one finds that $\Psi_1(E)$ satisfies

$$\Psi_1(E) = \Phi_1(E) + (E^+ - T - U)^{-1} V\Psi_1(E),$$

where

$$\Phi_1(E) = \sum_{E'} [(E^+ - T - U)^{-1} - (E'^+ - T)^{-1}] V\psi(E')(\varphi(E'), \Phi_0(E)).$$

Now put

$$\Psi_1(E) = \sum_{E'} \psi(E')(\varphi(E'), \Phi_1(E)) + \Psi_2(E),$$

and one finds

$$\Psi_2(E) = \Phi_2(E) + (E^+ - T - U)^{-1} V\Psi_2(E).$$

Continuing the process one finds after n steps

$$\Psi(E) = \sum_{E'} \psi(E')\alpha^{(n)}(E', E) + \Psi_{n+1}(E),$$
$$\Psi_{n+1}(E) = \Phi_{n+1}(E) + (E^+ - T - U)^{-1} V\Psi_{n+1}(E),$$

and one readily verifies that, not unexpectedly, $\alpha^{(n)}(E', E)$ is identical with the result of n iterations of (5).

III. CONNECTION WITH THE FERMI APPROXIMATION

Chew has shown that the impulse approximation yields a good approximation in the problem of the scattering of slow neutrons by protons bound in molecules. However, the only proof of this which has been offered is simply the demonstration that in the slow neutron limit the impulse approximation to W becomes identical with the Fermi approximation which is known[7] to be an excellent approximation in such problems. No real reason has been given as to why the impulse approximation is such a good approximation for this case.

By use of the integral equation (5)[8] we are now able to clarify this situation. Indeed (5) is very closely related to the integral equation used by Lippmann and Schwinger[3] in their discussion of slow neutron scattering by molecules; their equation in turn being a generalization of the equation used by Breit.[7] Following Lippmann and Schwinger, we will now make it plausible that the term in (5) which involves $\mathcal{G}$ will make only a small contribution to $\alpha(E', E)$ for this problem. To do this we note that (i) $\mathcal{G}$, because it has V as a factor, is concerned only with happenings during the collision; (ii) the significant values of E'' differ from E by something of the order of the proton's binding energy; and (iii) the proton's binding energy, the potential energy of the proton in the molecule, and E are all very small when compared with V. This means that during the collision the kinetic energy of the neutron-proton system will become so large that one may very nearly replace each Green's function in $\mathcal{G}$ by T^{-1}, thus making the contribution of the $\mathcal{G}$ term essentially zero.

This completes our discussion of the impulse approximation. The one major point which we have not considered is the problem of determining the error in the impulse approximation at high energies. This problem has been discussed already by Wick and Chew, and at this time we shall remark only that the expression on which they base their estimate of the error is easily shown to be identical with the result of a first iteration of (5).

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[7] G. Breit, Phys. Rev. **71**, 215 (1947).

[8] Quite apart from the question of the validity of the impulse approximation, the probability of "pick-up" is obviously negligible for molecular problems so that (5) provides a suitable basis for the discussion of the applicability of the impulse approximation to these problems.