Angular Correlation of First and Third Gamma-Rays

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In a nuclear cascade involving three or more gamma-rays it is both desirable and feasible to measure the angular correlation of nonconsecutive gammas either to confirm the results of correlating consecutive radiations or to resolve possible ambiguities. The angular correlation functions for the observation of the first and third gammas in a triple cascade have been calculated with the use of Racah functions for cases of physical interest. While the anisotropies tend to be somewhat less than those from the correlation of consecutive gamma-rays, they are easily detectable by present experimental techniques.

IN investigations of nuclear decay schemes one frequently finds three or more gamma-rays in cascade. The angular correlation of any two consecutive gammas has been developed in detail[1-3] and has proven a valuable experimental tool in determining multipolarity of the radiation and angular momenta of the nuclear levels. In the case of three gamma-rays in cascade it is, in general, possible to measure the angular correlation of the first and third as easily as the correlation of the first and second or second and third. If the order of emission of the gamma-rays is not known from other evidence and the radiations can be distinguished by their energies, it is clearly possible to measure three correlations by taking the gamma-rays in pairs. A comparison of the measurement with the calculated results will then serve to establish the order of emission. If evidence whereby the order of emission may be established is available, then the one-three correlation will serve as a useful check on the correlation experiments involving consecutive gamma-rays, and in some instances it will make possible the resolution of ambiguities in the interpretation of the correlation of consecutive gammas. Two such cases are given below. The numerical results for this one-three correlation form the subject of this paper.

The theory of the correlation of three (or more) radiations has been discussed in some detail in a previous paper.[4] It was shown that when the second of three radiations was not observed, all interference terms were removed by taking the z axis along either the first or the third gamma. The angular correlation function for the emission of pure multipoles can then be written in the form

$$w(\vartheta)=\sum_{m_3M_2M_1}(j_3L_3m_3\pm 1\,|\,j_2m_2)^2(j_2L_2m_2M_2\,|\,j_1m_1)^2 \times(j_1L_1m_1M_1\,|\,j_0m_0)^2F_{L_1}{}^{M_1}(\vartheta). \quad (1)$$

Here and in the remainder of this paper j_0 will be used to denote the ground state (or final state) of the cascade while j_1, j_2, and j_3 denote the excited states in order of increasing energy. The m's and M's give the projection of the j's and L's, respectively, on the quantization axis. As the vector addition coefficients indicate, a 2^{L_i} pole quantum is emitted in the transition $j_i\rightarrow j_{i-1}$. The $F_L{}^M(\vartheta)$ is the relative angular distribution function as defined by Falkoff and Uhlenbeck.[2] The summation over these magnetic quantum numbers may be carried out as in Appendix I of reference 4. Dropping irrelevant scale factors the result is

$$w(\vartheta)=\textstyle\sum_\nu(L_1L_11-1\,|\,\nu0)(L_3L_31-1\,|\,\nu0) \times W(j_1j_1L_1L_1;\,\nu j_0)W(j_1j_1j_2j_2;\,\nu L_2) \times W(j_2j_2L_3L_3;\,\nu j_3)P_\nu(\cos\vartheta). \quad (2)$$

The summation index ν is restricted to even non-negative integers and may not exceed the smallest of $2L_1$, $2L_3$, $2j_1$ and $2j_2$. It is not limited by L_2, the angular momentum of the unobserved intermediate gamma-ray. The three W functions are Racah functions; their properties have been given by Racah.[5,6] It will be noted that $w(\vartheta)$ is invariant under the transformation $L_1\leftrightarrow L_3$, $j_0\leftrightarrow j_3$, $j_1\leftrightarrow j_2$ corresponding to the Hermitian character of the matrix elements. This ambiguity, which also characterizes the correlation of the consecutive radiations, could be eliminated if one j, the ground-state spin, say, were known.

For conciseness define a set of Δ's by

$$j_i=\Delta_i+j_{i-1}, \quad (3)$$

and let $L_i=1$ when $\Delta_i=0,\pm1$ and $L_i=2$ when $\Delta_i=\pm2$. Higher values of $|\Delta_i|$ are not considered here. This limits consideration to cascades involving no multipoles higher than quadrupoles and assumes that in a given transition the lowest multipole possible participates which latter is the practical case. The restriction to dipoles and quadrupoles may be relaxed as discussed in reference 7, Sec. V-e.[7]

With these restrictions and with the normalization $\langle w(\vartheta)\rangle_{\text{Av}}=1$ (averaged over all directions), the ratios of Racah functions needed are given in Table I.

* This document is based on work performed for the AEC.
[1] D. R. Hamilton, Phys. Rev. **58**, 122 (1940).
[2] D. L. Falkoff and G. E. Uhlenbeck, Phys. Rev. **79**, 323 (1950).
[3] S. P. Lloyd, Phys. Rev. **83**, 716 (1951).
[4] Biedenharn, Arfken, and Rose, Phys. Rev. **83**, 586 (1951).
[5] G. Racah, Phys. Rev. **62**, 438 (1942).
[6] L. C. Biedenharn, Tables of Racah Functions, Oak Ridge National Laboratory Report No. 1098.
[7] Rose, Biedenharn, and Arfken, Phys. Rev. **85**, 5 (1952).

Some simplification can be obtained by noting that $A_1A_{-1}=1/10$, $A_{\pm2}=2(5/7)^{\frac{1}{2}}A_{\pm1}$, and $A_{24}A_{-24}=1/126$. If one wishes to obtain the individual Racah functions,[6] they can be obtained from these ratios by using the result

$$W(jjLL;0K)=(-1)^{L+j-K}(2L+1)^{-\frac{1}{2}}(2j+1)^{-\frac{1}{2}}, \quad (4)$$

TABLE I. Ratios of Racah functions for one-three correlation.

$$A_0\equiv\frac{W(j\,j\,1\,1;2\,j)}{W(j\,j\,1\,1;0\,j)}=-\left[\frac{(2j-1)(2j+3)}{10j(j+1)}\right]^{\frac{1}{2}}$$

$$A_1\equiv\frac{W(j\,j\,1\,1;2\,j-1)}{W(j\,j\,1\,1;0\,j-1)}=\left[\frac{(j+1)(2j+3)}{10j(2j-1)}\right]^{\frac{1}{2}}$$

$$A_{-1}\equiv\frac{W(j\,j\,1\,1;2\,j+1)}{W(j\,j\,1\,1;0\,j+1)}=\left[\frac{j(2j-1)}{10(j+1)(2j+3)}\right]^{\frac{1}{2}}$$

$$A_2\equiv\frac{W(j\,j\,2\,2;2\,j-2)}{W(j\,j\,2\,2;0\,j-2)}=\left[\frac{2(j+1)(2j+3)}{7j(2j-1)}\right]^{\frac{1}{2}}$$

$$A_{-2}\equiv\frac{W(j\,j\,2\,2;2\,j+2)}{W(j\,j\,2\,2;0\,j+2)}=\left[\frac{2j(2j-1)}{7(j+1)(2j+3)}\right]^{\frac{1}{2}}$$

$$B_0\equiv\frac{W(j\,j\,j\,j;2\,1)}{W(j\,j\,j\,j;0\,1)}=\frac{j^2+j-3}{j(j+1)}$$

$$B_1\equiv\frac{W(j\,j\,j+1\,j+1;2\,1)}{W(j\,j\,j+1\,j+1;0\,1)}=\frac{1}{j+1}\left[\frac{j(j+2)(2j+5)(2j-1)}{(2j+1)(2j+3)}\right]^{\frac{1}{2}}$$

$$B_2\equiv\frac{W(j\,j\,j+2\,j+2;2\,2)}{W(j\,j\,j+2\,j+2;0\,2)}=\frac{1}{2j+3}\left[\frac{j(j+3)(2j+7)(2j-1)}{(j+1)(j+2)}\right]^{\frac{1}{2}}$$

$$A_{24}\equiv\frac{W(j\,j\,2\,2;4\,j-2)}{W(j\,j\,2\,2;0\,j-2)}=\frac{1}{3}\left[\frac{(j+1)(2j+3)(j+2)(2j+5)}{14j(2j-1)(j-1)(2j-3)}\right]^{\frac{1}{2}}$$

$$A_{-24}\equiv\frac{W(j\,j\,2\,2;4\,j+2)}{W(j\,j\,2\,2;0\,j+2)}=\frac{1}{3}\left[\frac{j(2j-1)(j-1)(2j-3)}{14(j+1)(2j+3)(j+2)(2j+5)}\right]^{\frac{1}{2}}$$

$$B_{04}\equiv\frac{W(j\,j\,j\,j;4\,1)}{W(j\,j\,j\,j;0\,1)}=\frac{j^2+j-10}{j(j+1)}$$

$$B_{14}\equiv\frac{W(j\,j\,j+1\,j+1;4\,1)}{W(j\,j\,j+1\,j+1;0\,1)}=\frac{1}{j+1}\left[\frac{(j-1)(j+3)(2j-3)(2j+7)}{(2j+1)(2j+3)}\right]^{\frac{1}{2}}$$

$$B_{24}\equiv\frac{W(j\,j\,j+2\,j+2;4\,2)}{W(j\,j\,j+2\,j+2;0\,2)}$$

$$=\frac{[j(j-1)(j+3)(j+4)(2j-3)(2j-1)(2j+7)(2j+9)]^{\frac{1}{2}}}{(j+1)(j+2)(2j+3)[(2j+1)(2j+5)]^{\frac{1}{2}}}$$

provided $|j-L|\leqslant K\leqslant j+L$. The Clebsch-Gordan coefficients that appear in $w(\vartheta)$ are

$$(111-1|20)/(111-1|00)=2^{-\frac{1}{2}},$$

$$(221-1|20)/(221-1|00)=-(5/14)^{\frac{1}{2}}$$

and

$$(221-1|40)/(221-1|00)=-(8/7)^{\frac{1}{2}}.$$

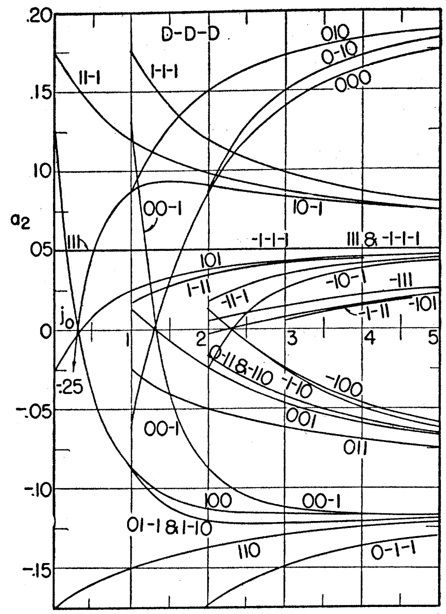

FIG. 1. The coefficient a_2 for dipole-dipole-dipole transitions (D-D-D) as a function of ground-state spin, j_0. The curves are labeled with the values of the Δ's, $w(\vartheta)=1+a_2P_2(\cos\vartheta)$.

From these and Table I the coefficients of $P_2(\cos\vartheta)$ and $P_4(\cos\vartheta)$ in the Legendre polynomial expansion

$$w(\vartheta)=1+a_2P_2(\cos\vartheta)+a_4P_4(\cos\vartheta) \quad (5)$$

have been calculated for the various one-three gamma-cascades subject to the above-mentioned restrictions. Since almost all of the gamma-ray cascades of interest terminate at the ground state and since the spin of the ground state will frequently be known, the angular cor-

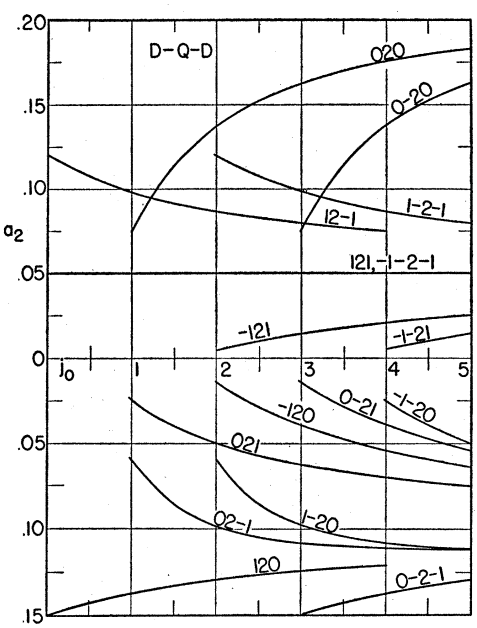

FIG. 2. The coefficient a_2 for dipole-quadrupole-dipole (D-Q-D) transitions.

relation functions are tabulated for the various possible values of the ground-state spin, j_0. The results are exhibited in Figs. 1, 2, 3, and 4. Cascades in which quadrupoles would always be competing with crossover dipole radiation have not been included since the competition is generally so unfavorable to the cascade transition. An example of this is the case $j\rightarrow j+2\rightarrow j+1\rightarrow j+3$ or the inverse transition scheme (Δ_1, Δ_2, $\Delta_3=2$, -1, 2 or -2, 1, -2).

When $L_i=|\Delta_i|$ and the j's are monotonic, the coefficients a_2 and a_4 are independent of j_0. A similar situation exists for two γ-rays in consecutive cascade as has been pointed out by Lloyd.[3] That this holds for the one-three correlation may be verified using Table I.

For convenience the values of a_2 and a_4 have been plotted against j_0 for all the sets of Δ's. These curves

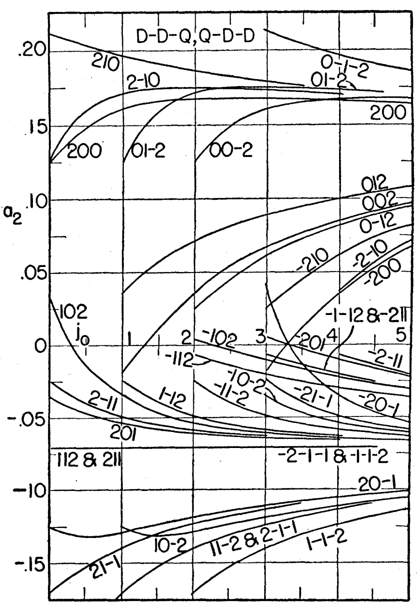

FIG. 3. The coefficient a_2 for dipole-dipole-quadrupole (D–D–Q and Q–D–D) transitions.

show all possibilities subject to the three restrictions (1) $L_i\leqslant 2$, (2) $L_i=1$ when $\Delta_i=0, \pm 1$, and (3) no crossover dipole gamma-rays competing with quadrupoles. Of course the curves have physical meaning only for integral and half-integral values of j_0.

It is perhaps of interest to note that in at least two cases the one-three gamma-gamma angular correlation

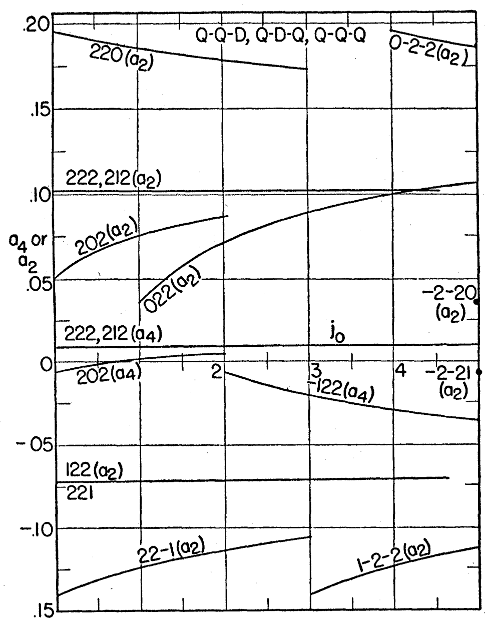

FIG. 4. The coefficient a_2 for quadrupole-quadrupole-dipole (Q–Q–D and D–Q–Q) transitions. The coefficients a_2 and a_4 for quadrupole-dipole-quadrupole (Q–D–Q) and quadrupole-quadrupole-quadrupole (Q–Q–Q) transitions.

can give new information that cannot be obtained from the correlation of consecutive gammas. An example is the pair of level schemes 0–1–1–2 and 0–2–1–1 where the numbers designate j-values. In each case the coefficient of $P_2(\cos\vartheta)$ for the first two gammas is -0.2500. In each case the coefficient a_2 for the last two gammas is -0.0250. It is impossible even with infinite resolution to distinguish the two decay schemes by correlating consecutive gamma-rays. However, if one correlates the first and third gammas the results for a_2 are -0.0250 and 0.1250 for the respective level schemes. The two cases may be easily distinguished. The pair of level schemes $\frac{1}{2}$–$\frac{3}{2}$–$\frac{3}{2}$–5/2 and $\frac{1}{2}$–5/2–$\frac{3}{2}$–$\frac{3}{2}$ show the same ambiguity under the correlation of consecutive gamma-rays and may also be resolved by the one-three correlations. These cases are, of course, rather exceptional. In general, with the increased number of parameters in the one-three correlation ($3L$'s, $4j$'s) it may be expected to yield more ambiguities than the consecutive gamma-correlation. The extent of the ambiguities may be estimated from the graphs.