Renormalized S-Matrix for Scalar Electrodynamics

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By a consistent use of the concepts of mass and charge renormalization Dyson has demonstrated the possibility of constructing a divergence-free S-matrix for scattering problems in spinor-electrodynamics, valid to all orders in the fine-structure constant. In this paper a proof is given of the possibility of renormalizing theories of charged scalar (or pseudoscalar) particles in the presence of the electromagnetic field. It is found that in addition to the renormalization of mass and charge, an infinite constant of direct interaction $\delta\lambda$ has to be introduced in a term $\delta\lambda\phi^{*2}\phi^2$ added to the Hamiltonian, in order to cancel consistently all divergences arising from the Møller scattering of one spinless particle by another. This, combined with the fact that theories of charged scalar and pseudoscalar mesons in scalar interaction with the nucleons can be renormalized with the same additional term in the Hamiltonian, seems to be of some significance.

1. INTRODUCTION

BY establishing the equivalence of Tomonaga-Schwinger formalism for spinor electrodynamics with that of Feynman, Dyson[1] has formulated an S-matrix theory for scattering problems in spinor electrodynamics. In this theory any real process taking place with specific virtual processes can be represented by a graph, the contribution to the matrix element for this process being an integral in the writing of which each line and each vertex of the aforementioned graph contributes a single factor. By analyzing the divergences in these integrals and giving precise rules for separating divergent parts from the (physically significant) convergent parts of these integrals, Dyson has further demonstrated that all divergences in the theory can be absorbed in the unobservable mass and charge renormalization of the theory.

For scalar electrodynamics it has been possible[2] to develop corresponding graphical methods and thus to analyze the possible divergences of the theory. It is the purpose of this paper to consider the adequacy of the concepts of renormalization for dealing with these divergences.

If $\phi(x)$ and $A_\mu(x)$ represent the meson and electromagnetic fields, respectively, the interaction Hamiltonian for mesons interacting with the electromagnetic field is

$$H_0(x)=\frac{ie}{\hbar c}A_\mu(x)\left[\phi^*(x)\frac{\partial\phi}{\partial x_\mu}-\frac{\partial\phi^*(x)}{\partial x_\mu}\phi(x)\right]-\left(\frac{ie}{\hbar c}\right)^2\phi^*(x)\phi(x)\delta_{\mu\nu}A_\mu(x)A_\nu(x)-\left(\frac{ie}{\hbar c}\right)^2\phi^*(x)\phi(x)[A_\mu(x)n_\mu]^2. \quad (1)$$

$A_\mu(x)$ creates and annihilates photons, while $\phi^*(x)$ and $\phi(x)$ are interpreted as creation and annihilation charge operators, so that ignoring the surface-dependent terms in the Hamiltonian, we have effectively[3]

$$\left\langle P\frac{\partial\phi(x)}{\partial x_\mu}\frac{\partial\phi^*(y)}{\partial y_\nu}\right\rangle_0=\tfrac{1}{2}\hbar c\frac{\partial^2}{\partial x_\mu\partial y_\nu}\Delta_F(x-y).$$

In Feynman graphs, the terms in the Hamiltonian linear in $A_\mu(x)$ lead to 3-vertices, with 2 meson lines and 1 photon line incident, while the term bilinear in $A_\mu(x)$ leads to 4-vertices, with 2 meson and 2 photon lines incident.

The contribution of this graph to the matrix[2] element appears as an integral in momentum space; in the integrand there appear[4]

(i) constant factors $\phi^*(p)$, $\phi(p)$, or $A_\mu(p)$, corresponding to each external line of the graph.
(ii) a factor $\delta_{\mu\nu}(2\pi)^{-3}D_F(p)$ for each internal photon line.
(iii) a factor $\hbar c(2\pi)^{-3}\Delta_F(p)$ for each internal meson line.
(iv) a factor $ie(\hbar c)^{-2}(2\pi)^4(p+p')_\mu\delta(p-p'+q)$ for each 3-vertex. The suffix μ gives the polarization of the photon line.
(v) a factor $-ie(\hbar c)^3(2\pi)^4\delta_{\mu\nu}\delta(p-p'+q-q')$ for each 4-vertex, where p is the momentum vector of the incoming meson, p' that of the meson leaving the 4-vertex, and q, q' refer to the photon lines incident at the 4-vertex.
(vi) the whole integral is multiplied by the number of different ways the operators can be paired off by interchanging the roles of photon operators $A_\mu(x)$ and $A_\nu(x)$ at the 4-vertices.

Thus, in general, it is possible to make a formal distinction between the factors $\delta_{\mu\nu}$ and $\delta_{\nu\mu}$ for a 4-vertex. If this distinction is not made, the integral obtained by following rules (i) to (v) must be multiplied by a "weight factor" 2 for each 4-vertex. Some exceptions

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[1] F. J. Dyson, Phys. Rev. 75, 1736 (1949), hereafter referred to as D II.
[2] P. T. Matthews, Phys. Rev. 80, 282 (1950); F. Rohrlich, Phys. Rev. 80, 666 (1950). The author is indebted to P. T. Matthews for sending him a copy of F. Rohrlich's work prior to publication.
[3] P. T. Matthews, Phys. Rev. 76, 1657 (1949).
[4] These rules were already implicit in Feynman's work [R. P. Feynman, Phys. Rev. 76, 769 (1949)].

occur, however, when the weight factor is not 2, and these have been discussed in detail by Rohrlich.[2] For theoretical considerations, however, we shall always distinguish (if possible) between factors $\delta_{\mu\nu}$ and $\delta_{\nu\mu}$ for 4-vertices and no weight factor will be used.

The genuine primitive divergents[2] are of the same type as in spinor electrodynamics, namely, meson and photon self-energies and vertex parts (parts with 2 external meson lines and 1 external photon line), with the addition of logarithmically divergent graphs with two external photon and meson lines (C parts), and four external meson lines (M parts). The self-energy (S) and vertex (V) parts modify the factors from single lines or single 3-vertices, while the divergent C parts can be regarded as modifications of 4-vertices. Define $\sum_M^*(p)$ as the function arising from adding together all integrals corresponding to proper[5] meson self-energy parts, $\Pi^*(p)$ the corresponding sum for photon self-energy parts, $\Lambda_\mu(p, p')$ the function arising from adding the integrals corresponding to proper V parts, $\theta_{\mu\nu}(p, p', q)$ as the sum arising from adding integrals corresponding to proper C parts, and

$$D_F'=D_F+D_F\Pi^*D_F'$$
$$\Delta_F'=\Delta_F+\Delta_F\textstyle\sum_M^*\Delta_F'$$
$$\Gamma_\mu(p, p')=(p+p')_\mu+\Lambda_\mu(p, p')$$
$$C_{\mu\nu}(p, p', q)=\delta_{\mu\nu}+\theta_{\mu\nu}(p, p', q).^6$$

Every graph G, other than a primitive divergent graph, has a uniquely defined "skeleton" which is the graph obtained by omitting all self-energy parts from the lines, vertex parts from the 3-vertices, and C parts from the 4-vertices. A graph which is its own skeleton will be called "irreducible," and a graph not containing M parts inside it "simple." With the foregoing definition of irreducibility, an irreducible graph may still contain divergent M parts.

For simple irreducible primitive divergent graphs, the forms of the functions $\sum_M^*$, Π^*, Λ_μ, and $\theta_{\mu\nu}$ are given as follows by invariance consideration:

$$\textstyle\sum_M^*(p)=A+B(p^2+\kappa^2)+\Delta_c(p)(p^2+\kappa^2),$$
$$\Pi^*(p)=Cp^2+D_c(p)p^2,$$
$$\Lambda_\mu(p, p')=L(p+p')_\mu+\Lambda_{\mu c}(p, p'),$$
$$\theta_{\mu\nu}(p, p', q)=R\delta_{\mu\nu}+\theta_{\mu\nu c}(p, p', q),$$

where the suffix c stands for the convergent parts of the functions on the left-hand side of the equations.

A, B, C, L, and R are divergent constants, while the definitions of the convergent parts are such that

$$\Delta_c(p_0)=0$$

when

$$p_0^2+\kappa^2=0$$

and

$$D_c(0)=0$$

$\Lambda_{\mu c}(p, p')$ can be written in the form[7]

$$\Lambda_{\mu c}(p, p')=M(p+p')_\mu(p^2+\kappa^2+p'^2+\kappa^2)+F_\mu(p, p') \quad \text{(2a)}$$

where

$$F_\mu=\partial F_\mu/\partial p_\nu=\partial F_\mu/\partial p_\nu'=0 \quad \text{for} \quad p=p'=p_0$$

and in general we can write $\theta_{\mu\nu c}$ as

$$\theta_{\mu\nu c}(p, p', q)=N(p+p')_\mu(p+p')_\nu+F_{\mu\nu}(p, p', q) \quad \text{(2b)}$$

with

$$F_{\mu\nu}(p_0, p_0, 0)=0.$$

We notice that M is finite and, furthermore, that with the foregoing (unique) definition, $\Lambda_{\mu c}(p_0, p_0)\equiv 0$. Similarly, N in (2b) is a finite constant. The precise significance of these particular separations will become clear in Sec. III when we establish relations between $\sum_M^*$, Λ_μ, and $\theta_{\mu\nu}$.

It is possible to obtain a divergence free S-matrix if we can show:

(a) That all infinities associated with self-energy of a free-field meson can be canceled by bringing into the interaction Hamiltonian by means of a unitary transformation,[8] the mass renormalization term $-\delta\kappa^2\phi^*\phi$. The free meson field now propagates with the term $\kappa^2\phi^*\phi$ in the "free" Hamiltonian ($\kappa=mc/h$ where m is the observed meson mass).

(b) That all M divergences occurring anywhere in the theory can be consistently compensated by suitably choosing a constant $\delta\lambda$ in a term $\delta\lambda\phi^{*2}\phi^2$ which is added to the interaction Hamiltonian.[9] After the additions (a) and (b) the Hamiltonian is

$$H_1(x)=H_0(x)-\delta\kappa^2\phi^*\phi+\delta\lambda\phi^{*2}\phi^2. \quad (3)$$

The additional terms give rise to new graphs containing 2- and 4-vertices, with 2 and 4 meson lines incident, respectively. Graphs containing no such 2- or 4-vertices will be called "original."

Unlike spinor electrodynamics $H_0(x)$ contains two constants, e and e'^2, with $e=e'$. The graphs arising entirely from e-vertices (3-vertices) contain S divergences and V divergences; those from e^2 vertices (4-vertices) contain S divergences and C divergences. It would be possible to absorb these divergences in the renormalization of charge if we could show (c).

(c) That by a suitable choice of constants Z,

$$\left.\begin{aligned} &D_F'=Z_3D_{F1}'(e_1)\\ &\Delta_F'=Z_2\Delta_{F1}'(e_1)\\ &\Gamma_\mu=Z_1^{-1}\Gamma_{\mu 1}(e_1) \quad \text{and} \quad C_{\mu\nu}=Z_4^{-1}C_{\mu\nu 1}(e_1)\end{aligned}\right\} \quad (4)$$

where the renormalized (and the observed) charge of the meson

$$e_1=Z_1^{-1}Z_2Z_3^{\frac{1}{2}}e=Z_4^{-\frac{1}{2}}Z_2^{\frac{1}{2}}Z_3^{\frac{1}{2}}e'.$$

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[5] The external lines of any proper part contain no self-energy insertions. In general a proper part is one which cannot be divided into two pieces joined by a single line.

[6] The factor for a 4-vertex is $-e^2\delta_{\mu\nu}$. For $\theta_{\mu\nu}$ as defined here we obtain the e-factor $(-1)^r e^{2r} e^{2s}$ where r is the number of 4-vertices and $2s$ that of 3-vertices, and then divide by $-e^2$. This gives $C_{\mu\nu}$ (to replace $\delta_{\mu\nu}$) as $=\delta_{\mu\nu}+\theta_{\mu\nu}$.

[7] This separation into convergent and divergent parts was given by Dyson (private communication).

[8] F. J. Dyson, Phys. Rev. **75**, 486 (1949); P. T. Matthews, Phil. Mag. **185**, XLI (1950).

[9] The necessity of this condition was pointed out by P. T. Matthews (reference 2).

(d) If $H_0(x)$ is gauge-invariant, so that $e=e'$, it has to be shown that the Z's defined satisfy a further condition,

$$Z_4^{-1}=Z_2Z_1^{-2}.$$

D_F', Δ_F', Γ_μ, $C_{\mu\nu}$ are the functions already defined, except that they now include the graphs with the new 2- and 4-vertices, as well as the original graphs. $C_{\mu\nu 1}=\delta_{\mu\nu}+\theta_{\mu\nu c}$ where $\theta_{\mu\nu c}$ is the function arising by adding together the (absolutely) convergent parts of each integral corresponding to each "original" C part. $\Gamma_{\mu 1}$, Δ_{F1}', D_{F1}' are defined in a similar manner from sums of convergent parts of the original integrals concerned. The procedure for obtaining the (absolutely) convergent part of an integral will be presently explained. It is to be noticed that the "convergent functions," on the right-hand side of the relations (4) appear to be expressed as functions of the renormalized charge e_1.

The general procedure for isolating divergences[10] from an n-fold integral was given in reference 2 in Sec. III. In this procedure the concept of true divergence plays an important part. In general if we subtract from any divergent integral over the variables $t_1\cdots t_n$, the true divergence over t_1 subintegration $D(t_1)\times$ the reduced integral (R) over $t_2t_3\cdots t_n+$ the true divergence over $t_2\times$ the reduced integral over $t_1t_3\cdots t_n+\cdots+$ the true divergence over $t_1t_2\times$ the reduced integral over $t_3\cdots t_n+\cdots+\cdots+\cdots+$ finally the true divergence over $t_1t_2\cdots t_n$, the remainder is an integral which is absolutely convergent and is the convergent part of the n-fold divergent integral we started with. Each reduced integral corresponds to a graph obtained from the graph under consideration by omitting that part of the graph whose true divergence multiplies this particular reduced integral. If a subintegration is "superficially convergent," its true divergence is zero; while the true divergence over a subintegration $t_a\cdots t_b\cdots t_i\cdots t_jt_p\cdots t_q\cdots t_y\cdots t_z$ where $(t_a\cdots t_b)$, $(t_i\cdots t_j)$, $(t_p\cdots t_q)\cdots$ are k groups of variables belonging to nonoverlapping parts of the graph, equals $(-i)^{k-1}$ times the product of the true divergences over each of the k groups, $(t_a\cdots t_b)$, $(t_i\cdots t_j)$, $\cdots$ (Remark b, I, Sec. III[11]). It is in terms of the convergent parts of the integrals defined as above that the functions $C_{\mu\nu 1}$, $\Gamma_{\mu 1}$, etc. are defined. The true divergence of a subintegration corresponding to a meson self-energy graph is characterized by two divergent constants A, B; while all other true divergences are characterized by one divergent constant. The factors multiplying these constants [$(p^2+\kappa^2)$, for example, which occur multiplying B in the meson self-energy case] are absorbed in the reduced integral.

Considering the "original" graphs, the finite and physically significant expressions $C_{\mu\nu 1}$, $\Gamma_{\mu 1}\cdots$ are defined, as already stated, in the first place by an apparently arbitrary dropping (subtracting off) of the infinite terms (true divergences $\times$ the reduced integrals) from the infinite expressions $C_{\mu\nu}$, Γ_μ, etc. Some of the divergent terms thus subtracted can be interpreted under (a) and (b) as direct cancellations with terms from $\delta\kappa^2$ and $\delta\lambda$ arising in graphs which are not original so that, as we shall show, both these divergences as well as the "non-original" graphs need never be considered. The establishment of (c) and (d) shows that the remaining divergent terms isolated can equally well be interpreted as the extraction of infinite constant multiplicative factors Z, from $C_{\mu\nu}$, Γ_μ, $\cdots$, so that each of these functions, instead of appearing as a sum of infinite $(D\times R)+$finite terms, now appears, as in Eq. (4), as a product of a (divergent) constant (Z) multiplying finite and physically significant terms. After this is accomplished, these Z factors can be completely absorbed in renormalizing the charge, so that all infinities occurring in the theory can be eliminated.

The main difficulty of the proof lies in establishing (e). In each of Eqs. (4) the functions appearing on the left-hand side are completely known, while so are the finite parts of these functions $C_{\mu\nu 1}$, $\Gamma_{\mu 1}$, $\cdots$. Let us assume that the relations (4) hold, with the factors Z for the present unknown. Consider an irreducible C graph, T_{IRR}, for example, such that in its lines and vertices, self-energy, vertex, and C parts can be inserted without any one of these insertions overlapping with any other insertion. Each insertion is thus completely localizable and does not simultaneously modify more than one vertex or line. The result of these insertions analytically is that we replace in the integral for the irreducible graph Δ_F by Δ_F', D_F by D_F', $(p+p')_\mu$ by Γ_μ and $\delta_{\mu\nu}$ by $C_{\mu\nu}$. It was shown (Remark c, I, Sec. III), that if, for example, a meson line p has a self-energy insertion with associated momenta $t_at_b\cdots t_k$, and if this self-energy insertion defines no overlap with any other part of the graph, then we can arrange our general subtraction procedure so that we isolate divergences corresponding to the subintegrations t_a, $t_at_b\cdots$, etc., first. After the true divergence corresponding to $t_at_b\cdots t_k$, itself is removed, the integration over this set is left absolutely convergent and it can be performed unambiguously so that the variables t_a, t_b, $\cdots t_k$ no longer need be considered explicitly from this stage onward. Since this procedure of removing divergent terms corresponding to this self-energy insertion is precisely the one we would adopt for obtaining Δ_{F1}' from Δ_F', and, since we have assumed that, in the latter case (Eq. (4)), the divergent terms separated are

[10] The proof given by Rohrlich for the possibility of renormalization (reference 2) for scalar electrodynamics is invalidated because the general procedure for isolating divergences was not available. A "hierarchy" of divergences was somewhat arbitrarily defined and a prescription given according to which a divergence higher in the hierachy should be removed first. This procedure is not equivalent to the one given above and when overlaps occur, does not leave behind an absolutely convergent integral after the proposed subtractions. In a note added in proof (reference 2) Rohrlich has observed that the problem of "b-divergences" (overlaps) has not received proper treatment in his paper. The present paper deals with these difficulties.

[11] Abdus Salam, Phys. Rev. **82**, 217 (1951), hereafter referred to as I.

completely interpreted as the extraction of the divergent factor Z_2 in $\Delta_F'=Z_2\Delta_{F1}'(e_1)$, we can make this same replacement for Δ_F' for this internal line p in the C graph under consideration. It is emphasized again that this is true if and only if no overlap occurs.

Going back to consider T_{IRR}, by inserting all S, V, and C parts in its lines and vertices we obtain an entire class of C parts $T_{\mu\nu}$ "derived" from T_{IRR}. Analytically we replace in the integral for T_{IRR}, Δ_F by $Z_2\Delta_{FI}'(e_1)$, D_F by $Z_3D_{F1}'(e_1)$, $(p+p')_\mu$ by $Z_1^{-1}\Gamma_{\mu 1}$, and $\delta_{\mu\nu}$ by $Z_4^{-1}C_{\mu\nu 1}$.

A C part of order $e^{2s+2r-2}$ with s 4-vertices and $2r$ 3-vertices contains $2r+s-1$ meson and $r+s-1$ photon lines. The foregoing replacements give an e factor

$$e^{2r+2s-2}Z_4^{-s}Z_1^{-2r}Z_2^{2r+s-1}Z_3^{r+s-1}=Z_4^{-1}e_1^{2s+2r-2}.$$

Δ_{F1}', D_{F1}', $\cdots$ are themselves power series in e_1, and their behavior for large p is precisely the same as that of Δ_F, D_F, etc. (D II Sec. VII), so that the new integral for $T_{\mu\nu}$ is again logarithmically divergent. Making a separation of the finite and the infinite part,

$$T_{\mu\nu}=Z_4^{-1}(T_d(e_1)\delta_{\mu\nu}+T_{\mu\nu c}(e_1)). \qquad (5)$$

Equation (5) gives in a compact form the sum of integrals from the entire class of graphs $T_{\mu\nu}$, "derived" (uniquely) from the one graph T_{IRR} by insertions. This form for $T_{\mu\nu}$ will be referred to as the fundamental form. It is to be emphasized that the content of $T_{\mu\nu}$ is not altered in the fundamental form by expressing it thus in terms of the renormalized charge; and that the equality of both sides is exact considered up to any order in e_1 and e (expressed in terms of e_1).

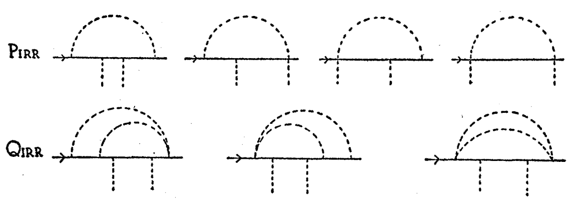


FIG. 1.

If it were possible to make unambiguous nonoverlapping insertions in all the irreducible C parts, this procedure would have given us all the reducible and irreducible C parts. Analytically from each T_{IRR} we would have obtained $T_{\mu\nu}$ in the form (5) so that summing over all C parts and adding $\delta_{\mu\nu}$ for the 4-vertex itself we would obtain

$$C_{\mu\nu}=\delta_{\mu\nu}+Z_4^{-1}(R_d\delta_{\mu\nu}+\theta_{\mu\nu c}(e_1)).$$

But if Eqs. (4) hold, $C_{\mu\nu}$ ought to equal $Z_4^{-1}C_{\mu\nu 1}$. Thus

$$Z_4^{-1}C_{\mu\nu 1}=Z_4^{-1}(\delta_{\mu\nu}+\theta_{\mu\nu c}(e_1)).$$

This gives us an equation with just one unknown, Z_4, which could then be determined (and so the consistency of the relations (4) established.

However, as soon as overlaps occur the whole idea of an unambiguous insertion in a line or a vertex loses validity. The reduction of a reducible graph cannot be defined unambiguously and, conversely, the replacement of all vertex and line factors of irreducible parts by $C_{\mu\nu}$, Γ_μ, etc., leads to the counting of certain graphs more than once. [This is a definite redundance, quite apart from the "weight factors" occurring under rule (vi).] Thus, a proof on the aforementioned lines can no longer be given.

For C parts it is always possible to make unambiguous insertions in all the lines and 3-vertices. However there are certain classes of C parts in which a C part inserted in one 4-vertex appears simultaneously as a C part inserted at some other vertex.

For vertex parts the complexity of overlaps increases greatly and $C-V$ overlaps can occur. For self-energies, besides these overlaps, vertex parts overlap with vertex parts. Also, if any meson (or photon) line is opened in a self-energy graph, this leads to parts with four external lines which (except for photon-photon scattering graphs) always diverge. M parts, which have not been considered so far, can complicate the picture still further by producing simultaneously $M-M$, $M-C$, or $M-S$ overlaps.

A great simplification can be effected by applying a powerful technique first introduced by Ward[12] in spinor electrodynamics. Ward's technique can be extended to reduce the complexity of the overlaps to be considered. However, if a primitive divergent is logarithmically divergent and still suffers from overlaps, this technique in general fails. The general procedure to be followed then for obtaining a proof of (4) and for the construction of the relevant functions is that of categorization, as developed in I, Sec. IV. By analyzing the overlaps, such categories of reducible graphs are defined in which certain insertions of S, V, and C parts cause no overlap, so that the substitutions $C_{\mu\nu}$, Γ_μ, etc., can be made in a defined way. By considering the implications of the subtraction rules, I, Sec. III, it is then possible to derive recurrence relations for such graphs in terms of graphs of lower order in the same category and their true divergences. The required values for the Z's are obtained by substituting Eqs. (4) into these recurrence relations and demanding that they lead back to the fundamental form for the type of part under consideration. These various techniques are illustrated in Secs. 2, 3, and 4 and in Appendix I where it is assumed that condition (b) can be satisfied. A proof of this is given in Sec. 5.

2. C PARTS

The most general C part consists of an open polygon formed by the meson line entering (and leaving) the graph, this line (the base line of the graph) being joined by photon lines to one or more (possibly interconnected)

[12] J. C. Ward, Proc. Phys. Soc. (London) **64**, 54 (1951). The author is deeply indebted to J. C. Ward for sending him a copy of his work prior to publication.

closed loops of meson lines with a structure inside them. If the external photon lines (with polarization vectors μ and ν) belong to the base line, the graph will be called a base-line graph. For base-line C parts it is possible to distinguish between $\theta_{\mu\nu}$ and $\theta_{\nu\mu}$. For such graphs $\theta_{\mu\nu}(p, p', q)$ is defined such that μ occurs topologically "before" ν and carries momentum q.

In order to construct the function $C_{\mu\nu}$ (and the corresponding convergent function $C_{\mu\nu 1}$) it is possible as explained in Sec. I, to take irreducible C parts and to make unambiguous insertions in their lines and 3-vertices, replacing in the corresponding integrals Δ_F by Δ_F', D_F by D_F', and $(p+p')_\mu$ by Γ_μ. Furthermore, it is possible to replace the factor $\delta_{\mu\nu}$ by $C_{\mu\nu}$ for the 4-vertices of all irreducible C parts except for the 4-vertices in the graphs P_{IRR} and Q_{IRR} shown in Fig. 1. (There is another class of graphs R which are defined in Appendix II and in the 4-vertices of which the foregoing insertion is also not valid.) Taking all other irreducible C parts and making the foregoing insertions gives all reducible C parts "derived" from them. If $T_{\mu\nu}$ denotes the sum of corresponding integrals, by counting the number of lines and vertices in the irreducible C parts in which insertions are being made, we have

$$T_{\mu\nu}=Z_4^{-1}[T_d(e_1)\delta_{\mu\nu}+T_{\mu\nu c}(e_1)], \qquad (6)$$

where $T_d(e_1)$ is the sum of the true divergences from all such graphs.

In this section the graphs P_{IRR} and a linear chain of graphs "derived" from them (the whole class being called P) will be treated in detail. For Q and R we assume the result in Appendix II, namely, that with a proper choice of Z_4 the corresponding functions $Q_{\mu\nu}$ and $R_{\mu\nu}$ can also be expressed in the fundamental form of Eq. (6). In fact, at this stage $T_{\mu\nu}$ will be understood to contain $Q_{\mu\nu}$ and $R_{\mu\nu}$.

Consider the graph in Fig. 2, of order e.[7] We can obtain (2^5-1) other topologically distinct graphs from it by shrinking $1\to 1$, $2\to 2$, $3\to 3$, $4\to 4$, $5\to 5$ in all possible ways. If $4\to 4$, for example, the resulting 4-vertex will be (uniquely) numbered 4. In Fig. 2 the letters $a, b, \cdots$ give the polarization of the photon. The order of the numbers $1, 2, 3\cdots$ and the letters $a, b, c, \cdots$ are important. If 2 is a 4-vertex, it could be denoted equally well by δ_{ab} or by δ_{ba} so that the number of formally different graphs obtained from the graph in Fig. 2 by joining $1\to 1$, $2\to 2$, $\cdots$ is 3^5-1, when the graphs with factors δ_{ab} and δ_{ba} are treated as different.

Let $P_{\mu\nu}$ stand for the graphs obtained by drawing chain graphs of the type illustrated in Fig. 2 in all orders in the powers of e, and also the graphs obtained by joining $1\to 1$, $2\to 2$, $\cdots$, in all possible ways.

Let $[n]$ stand for such graphs (3^n in number) of order e^{2n}, $[n|1]$ stand for all graphs $[n]$ such that the first vertex is necessarily a 4-vertex. These are 3^n in number. Similarly, let $[n|i,j]$ stand for all graphs(3^{n-1} in number) which have their ith and jth vertices necessarily δ_{pq} and δ_{rs} 4-vertices (the order of photon polarization-vectors p, q and r, s is to be noted), irrespective of the character of the other vertices, and so on for $[n|i,j,\cdots]$. Let $X_{\mu\nu}$ stand for all those graphs of all orders (and the sum of corresponding integrals) from among the P graphs which have their last end vertex a $\delta_{z\nu}$ 4-vertex, irrespective of the character of other vertices, $Y_{\mu\nu}$ stand for all graphs with the first and the last end-vertices, $\delta_{\mu a}$ and $\delta_{\mu a}$ vertices, and $Z_{\mu\nu}$ for those with the first end-vertex a 4-vertex $\delta_{\mu a}$, again irrespective of the character of the other vertices. Thus, Y, for example, equals

$$[1|1,2]+[2|1,3]+\cdots+\cdots+[n|1,n+1]+\cdots.$$

With the foregoing definitions we prove Lemma 1.

Lemma 1

$$P_{\mu\nu}=P_d(\delta_{\mu\tau}+X_{\mu\tau})(\delta_{\tau\rho}-P_dY_{\tau\rho}+P_d^2Y_{\tau\rho}^2-\cdots)(\delta_{\rho\nu}+Z_{\rho\nu})+P_{\mu\nu c}. \qquad (7)$$

Here P_d is the sum of the true divergences from the graphs P. In particular if P_n stands for the sum of the true divergences from $[n]$, then $P_d=P_1+P_2+P_3+\cdots$.

For the proof we make use of the property of a 4-vertex occurring in this type of linear chain of graphs P to split the chain into two parts.

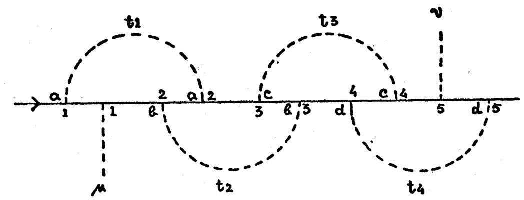


FIG. 2.

Consider, for example, the graphs in Fig. 3. It is obvious that the double integral over t_1t_2 corresponding to Fig. 3(a) is the product of two single integrals over t_1 and t_2, so that symbolically $(a)=(b)\times(c)$. In the sequel we shall further specialize this splitting off of graphs. A split will be made for a δ_{ab} vertex but it will never be made for a δ_{ba} 4-vertex.

$[n]$ is an n-fold integral over variables $t_1t_2\cdots t_n$, each one of the subintegrations being logarithmically divergent. If $ij\cdots q$ is a group of indices from the set $123\cdots n$ (where $ij\cdots$ run consecutively) then isolating the divergence over the subintegration $t_it_j\cdots t_q$ gives $P_{(q-i+1)}\times$the corresponding reduced integral. This reduced integral contains a 4-vertex of the type with a factor δ_{ab} (and not δ_{ba}) corresponding to the divergence which has been separated and this immediately splits the chain (if, of course, this 4-vertex is not one of the two possible end 4-vertices).

To obtain the coefficient of $(P_d)^f$ in (7), consider all subsets of indices $123\cdots n$, such that each subset divides naturally into f groups, the indices in each group running consecutively, the different groups being such that if k is the first and i the last index of two consecutive groups, then $k-i\geqslant 1$. The true divergence for a subintegration corresponding to these subsets consists of $(-1)^{f-1}P_\alpha P_\beta\cdots(f\text{ factors})$, (I, Remark b, Sec. III) corresponding to the number (f) of nonoverlapping graphs, the true divergences of which are in fact thus being simultaneously removed. The reduced integral itself necessarily contains f 4-vertices (again of the type δ_{ab} and not δ_{ba}), which may split the graph according to the scheme XY^{f-1}, Y^{f-1}, $Y^{f-1}Z$, or $XY^{f-1}Z$. By considering all possible subsets and the groups in them of the type mentioned we readily establish the lemma.

We can show this by considering, as an example, the coefficient of P_d^2 in (7) in detail. Only graphs up to $[n]$ are considered. In our symbolic notation $[1|1,2]\times[n-1|1]$ would be equal to $[n|1,2]$. We consider all subsets of indices which fall into two nonoverlapping groups. If the first of these two groups starts with index 1, we have the following terms to separate:[13]

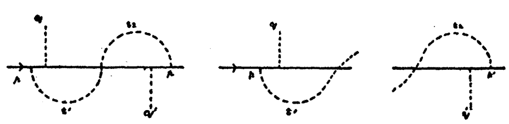
$$\left.\begin{array}{l}
-P_1[1|1,2]\times \\
\quad\begin{bmatrix} P_1[n-3|1]+P_2[n-4|1]+P_3[n-5|1]+\cdots+P_{n-2} \\ \qquad +P_1[n-4|1]+P_2[n-5|1]+\cdots+P_{n-3} \\ \qquad\qquad +P_1[n-5|1]+\cdots+P_{n-4} \\ \qquad\qquad\qquad +\cdots+\cdots \\ \qquad\qquad\qquad\qquad +P_1 \end{bmatrix} \\
-P_2[1|1,2]\times \\
\quad\begin{bmatrix} P_1[n-4|1]+P_2[n-5|1]+\cdots+P_{n-3} \\ \qquad +P_1[n-5|1]+\cdots+P_{n-4} \\ \qquad\qquad +\cdots+\cdots \\ \qquad\qquad\qquad +P_1 \end{bmatrix} \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
-P_{n-2}[1|1,2]P_1
\end{array}\right\} \quad (8)$$

The terms (8) sum up to

$$\begin{aligned}&=-P_d[1|1,2]P_d(1+[1|1]+[2|1]+[3|1]+\cdots+[n-3|1]) \\ &=-P_d^2[1|1,2](\delta_{\mu\nu}+Z_{\mu\nu})\end{aligned}$$

if the equality of both sides is supposed to hold to the order e^{2n}. Also, there are terms such as

$$\begin{array}{l}
-P_1[2|1,3]\begin{bmatrix} P_1[n-4|1]+P_2[n-5|1]+\cdots+P_{n-3} \\ \qquad +P_1[n-5|1]+\cdots+P_{n-4} \\ \qquad\qquad +\cdots+\cdots \\ \qquad\qquad\qquad +P_1 \end{bmatrix} \\
-P_2[2|1,3][\qquad P_1[n-5|1]+\cdots\cdots\cdots] \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}$$



(a) (b) (c)

FIG. 3.

which sum up to $-P_d^2[2|1,3](\delta_{\mu\nu}+Z_{\mu\nu})$. We similarly obtain $-P_d^2[3|1,4](\delta_{\mu\nu}+Z_{\mu\nu})$ and so on, concluding with $-P_1^2[n-2|1,n-1]$. The sum of all these terms is $-P_d^2 Y_{\mu\rho}(\delta_{\rho\nu}+Z_{\rho\nu})$.

If we consider the two groups of indices such that the first group does not contain the index 1, it may be easily verified that the sum of corresponding contributions is $-P_d^2 X_{\mu\tau}Y_{\tau\rho}(\delta_{\rho\nu}+Z_{\rho\nu})$ so that the entire sum is $-P_d^2(\delta_{\mu\tau}+X_{\mu\tau})Y_{\tau\rho}(\delta_{\rho\nu}+Z_{\rho\nu})$ precisely as in (7).

We now notice an important property of these graphs P. Whereas the insertion of any graph P_{ab} for δ_{ab} leads only to a graph already belonging to P, an insertion of P_{ba} for δ_{ba} gives a new graph unambiguously.

In order to obtain all reducible graphs "derived" from the graphs in Fig. 1. We proceed as follows:

(i) Draw all graphs P of all orders in e^2.

(ii) Insert at all 4-vertices δ_{ji}, all reducible and irreducible C parts including P^*_{ji}; analytically, replace δ_{ji} in the integrals by C_{ji}.

(iii) Insert at all 4-vertices δ_{ij}, the graphs T_{ij}; i.e., replace δ_{ij} by $\delta_{ij}+T_{ij}$. In the lines and the 3-vertices, all self-energy and vertex parts are inserted. The result after these insertions is $P_{\mu\nu}^*$.

It is easy to verify that no graph occurs more than once provided we adhere to the convention of distinguishing between δ_{ij} and δ_{ji} and further that all graphs "derived" from the graphs P_{IRR} do appear in the foregoing scheme of categorization, so that

$$C_{\mu\nu}=\delta_{\mu\nu}+T_{\mu\nu}+P_{\mu\nu}^*. \quad (9)$$

Since all the insertions (i) to (iii) are unambiguous and localizable, we can replace Δ_F by $\Delta_F'=Z_2\Delta_{F1}'$, D_F by $D_F'=Z_3D_{F1}'(e_1)$, $(p+p')_\mu$ by $Z_1^{-1}\Gamma_{\mu1}(e_1)$, δ_{ji} by $C_{ij}=Z_4^{-1}C_{ji1}(e_1)$ while δ_{ij} is replaced by

$$\begin{aligned}\delta_{ij}+T_{ij}&=\delta_{ij}+Z_4^{-1}(T_d\delta_{ij}+T_{ijc}(e_1)) \\ &=Z_4^{-1}[(T_d+Z_4-1)\delta_{ij}+\delta_{ij}+T_{ijc}(e_1)] \\ &=Z_4^{-1}[S\delta_{ij}+S_{ij1}(e_1)].\end{aligned}$$

(definition of S)

The effect of these substitutions is to give Z_4^{-1} times the correct number of Z factors to renormalize the charge while all integrals now appear explicitly as functions of e_1. While at each δ_{ji} vertex, δ_{ji} has been replaced by C_{ji1}, each δ_{ij} vertex gives rise to two terms; one in which the integral corresponding to the graph is merely multiplied by S, the other in which δ_{ij} is replaced by $S_{ij1}(e_1)$.

Denote by $P_{\mu\nu}^\times$ the sum of integrals corresponding to all graphs thus formed, for whose 4-vertices appear the factors S_{ij1} or C_{ji1} while for the lines and 3-vertices Δ_{F1}, D_{F1}, $\Gamma_{\mu1}$ appear. $X_{\mu\nu}^\times$ denotes the corresponding sum for all graphs in which only the last end-vertex is an unchanged 4-vertex $\delta_{z\nu}$ while all other 4-vertices are replaced by factors S_{ij1} or C_{ji1}. $Y_{\mu\nu}^\times$ similarly denotes the sum of integrals for all graphs with only the first and the last end-vertices unchanged 4-vertices, and $Z_{\mu\nu}^\times$ the sum for graphs with only the first end-vertex an unchanged 4-vertex.

With these definitions we prove the following lemma.

Lemma 2

$$\begin{aligned}P_{\mu\nu}^*=Z_4^{-1}[S(\delta_{\mu\tau}+X_{\mu\tau}^\times)(\delta_{\tau\rho}+SY_{\tau\rho}^\times \\ +S^2Y_{\tau\rho}^{\times2}+\cdots)(\delta_{\rho\nu}+Z_{\rho\nu}^\times)-S\delta_{\mu\nu}+P_{\mu\nu}^\times]. \quad (10)\end{aligned}$$

Defining $[n|l,m]^\times$, etc., to represent all graphs $[n|l,m]$ for the lines and vertices of which replacements Δ_{F1}', D_{F1}', $\Gamma_{\mu1}$, C_{ji1} or S_{ij1} have been made, except at the lth and mth 4-vertices where the factors δ_{pq} and δ_{rs} remain unchanged, the proof of the lemma follows by noticing that

$$\begin{aligned}[n]^*=[n]^\times&+S([n|1]^\times+[n|2]^\times+\cdots) \\ &+S^2([n|1,2]^\times+[n|1,3]^\times+\cdots \\ &+[n|i,l]^\times+\cdots)+S^3([n|1,2,3]^\times+\cdots \\ &+[n|i,l,k]^\times+\cdots)+\cdots+\cdots \\ &+S^{n+1}[n|1,2,3,\cdots,n+1]^\times \quad (11)\end{aligned}$$

[13] To verify that a term like $-P_1[1|1,2]P_3[n-5|1]$, for example, takes account of all the 3^{n+1} graphs in $[n]$, one can make a check as follows: $[n-5|1]$ contains 3^{n-5} graphs, P_3 is the true divergence of 3^{3+1} graphs in $[3]$, $[1|1,2]$ accounts for 3^0 and P_1 for 3^2 graphs. Thus, the foregoing term expresses the sum of relevant contributions from all the 3^{n+1} graphs in $[n]$.

By splitting the graphs and arranging the sumations as in Lemma 1, the result (10) is established.

Since Δ_{F1}', D_{F1}', etc., have the same behavior for large values of p, as the corresponding functions Δ_F, D_F (DII, Sec. VII), an immediate consequence of Lemma 1 is Lemma 3.

Lemma 3

$$P_{\mu\nu}^{\times}=P_d^{\times}(\delta_{\mu\tau}+X_{\mu\tau}^{\times})(\delta_{\tau\rho}-P_d^{\times}Y_{\tau\rho}^{\times}+P_d^{\times 2}Y_{\tau\rho}^{\times 2}-\cdots)(\delta_{\rho\nu}+Z_{\rho\nu}^{\times})+P_{\mu\nu c}^{\times}. \quad (12)$$

We have now reached the stage in our inductive argument, when we can establish the consistency of our procedure by an explicit choice of the unknown constant Z_4. We desire to choose Z_4 such that

$$C_{\mu\nu}=\delta_{\mu\nu}+T_{\mu\nu}+P_{\mu\nu}^{*}=Z_4^{-1}C_{\mu\nu 1}=Z_4^{-1}(\delta_{\mu\nu}+T_{\mu\nu c}+P_{\mu\nu c}^{\times}). \quad (13)$$

From (6), (11), and (12), however,

$$\delta_{\mu\nu}+T_{\mu\nu}+P_{\mu\nu}^{*}=\delta_{\mu\nu}+Z_4^{-1}[(S-T_d)\delta_{\mu\nu}+P_{\mu\nu c}^{\times}+T_{\mu\nu c}]+Z_4^{-1}(\delta_{\mu\tau}+X_{\mu\tau}^{\times})[S/(1-SY^{\times})+P_d^{\times}/(1+P_d^{\times}Y^{\times})](\delta_{\rho\nu}+Z_{\rho\nu}^{\times}). \quad (14)$$

Let $S+P_d^{\times}=Z_4+T_d+P_d^{\times}-1=0$, then $[S/(1-SY^{\times})+P_d^{\times}/(1-P_d^{\times}Y^{\times})]\equiv 0$ and simultaneously Eq. (13) is satisfied.

Thus, with the choice $Z_4=1-T_d-P_d^{\times}$ we finally establish that we can express $C_{\mu\nu}=Z_4^{-1}C_{\mu\nu 1}(e_1)$. Furthermore, by thus expressing Z_4 in terms of the true divergences of all original C parts, we have also expressed $P_{\mu\nu}^{*}$ in Lemma 2 in the fundamental form

$$P_{\mu\nu}^{*}=Z_4^{-1}[P_d^{\times}(e_1)\delta_{\mu\nu}+P_{\mu\nu c}^{\times}(e_1)]. \quad (15)$$

3. VERTEX PARTS AND MESON SELF-ENERGY GRAPHS

We now desire to show that it is possible to choose the constants Z_1 and Z_2 such that $\Delta_F'=Z_2\Delta_{F1}'$ and $\Gamma_\mu=Z_1^{-1}\Gamma_{\mu 1}$ and furthermore that $Z_4^{-1}=Z_2Z_1^{-2}$. The proof can be made to depend on that in Sec. II for C parts, by employing a technique due to Ward. It will be found that at least for Δ_{F1}', it is not even necessary to obtain its value by evaluating the absolutely convergent parts of all self-energy integrals. For this purpose we utilize the following differential identities,[14]

$$\Lambda_\mu(p,p)=-(1/2\pi i)(\partial/\partial p_\mu)\textstyle\sum^{*} \quad (16)$$

$$(\partial/\partial p+\partial/\partial p')_\nu\Lambda_\mu(p,p')=\theta_{\mu\nu}(p,p',p'-p)+\theta_{\nu\mu}(p,p',0). \quad (17)$$

To prove these identities, we notice the differential relation,

$$-(1/2\pi i)(\partial/\partial p_\mu)\Delta_F(p)=\Delta_F(p)2p_\mu\Delta_F(p),$$

which correctly describes the insertion of an external photon line (with its energy-momentum set equal to zero) in a meson line with momentum p. A second differentiation with respect to ν describes not only the insertion of another photon 3-vertex on the same meson line, but also the complication of the first 3-vertex into a 4-vertex, with the proper weight factor.

To prove (16) we notice first that the sum of the contributions to $\Lambda_\mu(p,p)$ from all (reducible or irreducible) non-base-line vertex parts vanishes identically. Let the momentum p be always associated with the base line. If t_i represent the momentum variables associated with closed meson loops (connected to the base line and to each other), then the sum of the contributions to $\Lambda_\mu(p,p)$ from non-base-line vertex parts is

$$\sum_i\int(\partial/\partial t_i)[F_\mu(p,t_\alpha)]dt_\alpha$$

which vanishes identically because F_μ is uniformly small for large values of momenta t_α.

Similarly, for a C part, if the photon ν has its energy-momentum zero, and does not belong to the base line, the sum of the contributions for such graphs to $\theta_{\mu\nu}(p,p',p'-p)$ and $\theta_{\nu\mu}(p,p',0)$ identically vanishes.

Thus, (a) the only vertex parts giving a contribution to $\Lambda_\mu(p,p)$ are the base-line vertex parts,

(b) The only C parts contributing to $\theta_{\mu\nu}(p,p',p'-p)+\theta_{\nu\mu}(p,p',0)$ are those for which ν belongs to the base line.

If now we associate p with the base line for meson self-energy graphs, a differentiation gives precisely all the base-line vertex parts and no photon line is differentiated. This establishes (16).

Similarly on associating p, p' with the base line in vertex parts, the operator $[\partial/\partial p+\partial/\partial p']$ would give all the C parts with ν belonging to the base line and these are the only ones contributing to the right-hand side of (17), as shown. If μ itself does not belong to the base line, but to one of the closed meson loops, some photon line joining this loop to the base line must carry momentum $(p-p'+t_a)$, while a part of the loop itself may have momentum variables $(p-p'+t_b)$. The operator $[\partial/\partial p+\partial/\partial p']$ insures that neither this photon line, nor any part of the closed loop is differentiated. This establishes (17).

The foregoing proofs depend on a very particular choice of momenta; p, p' must be associated with the base line. This choice can sometimes lead to difficulties in isolating divergences. This question will be examined further in Sec. V, when we deal with M parts.

We now show by an inductive application of the identities (16) and (17) that these differential relations give rise to relations between true divergences and also hold for the convergent parts of the functions concerned as well. These convergent parts are the same parts of the divergent integrals as are obtained by our subtraction procedure.

[14] Abdus Salam, Phys. Rev. **79**, 910 (1950). These (or similar) identities were derived independently by F. J. Dyson (private communication), F. Rohrlich (reference 2). Rohrlich's derivation of the relations between divergent constants from them was, however, incomplete because difficulties connected with overlaps were not noticed.

We treat the case of vertex parts in detail. It is obvious that the C parts produced by applying the operator $\partial/\partial p+\partial/\partial p'$ to an irreducible vertex part are themselves irreducible. (An extra external line always improves the chances of irreducibility and reduces overlaps.) Thus, from (2), for irreducible graphs,

$$2L\delta_{\mu\nu}+(\partial/\partial p+\partial/\partial p')_\nu\Lambda_{\mu c}(p,p') \\ =2R\delta_{\mu\nu}+\theta_{\mu\nu c}(p,p',p'-p)+\theta_{\nu\mu c}(p,p',0). \quad (18)$$

Setting $p=p'=p_0$ shows that $L=R$ and also that the differential relation holds for convergent parts as well.

Consider now a reducible vertex part. The integral corresponding to it is $\Lambda_\mu=\Lambda_{\mu c}+$the true divergence constant $L\times(p+p')_\mu+$true divergent constants $D\times$the reduced integrals, where each reduced integral represents a vertex part of a lower order. Therefore $(\partial/\partial p+\partial/\partial p')_\nu\Lambda_\mu=(\partial/\partial p+\partial/\partial p')_\nu\Lambda_{\mu c}+$divergent constants $D\times(\partial/\partial p+\partial/\partial p')_\nu$ applied to the reduced integrals.

Consider now the C parts produced by this operator from the given vertex part. We can show that for such graphs $\theta_{\mu\nu}+\theta_{\nu\mu}=\theta_{\mu\nu c}+\theta_{\nu\mu c}+2\delta_{\mu\nu}\times$the true divergence constant $R+$the same true divergence constants D as above$\times$precisely such C parts as are obtained by

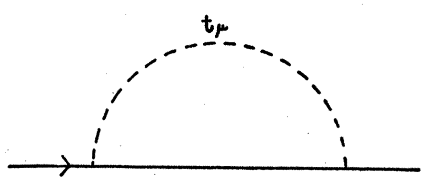


FIG. 4.

applying the differentiation operator to the aforementioned reduced vertex graphs.

A term by term comparison would thus allow us to write an equation of type (18), from which the equality of L and R and a differential relation between $\Lambda_{\mu c}$ and $\theta_{\mu\nu c}$ can be deduced.

The proof of this proceeds inductively. Since the differentiation operator affects only the base line in the given vertex part V, the true divergences for those parts of this graph which do not contain portions of the base line are the same as the true divergences for the relevant C parts obtained by differentiating V, while the reduced integrals in the two cases are themselves connected by the differential relation. Thus it is only for those subintegrations which extend over a part of the graph containing portions of the base line that a proof is needed. If, in the vertex part under consideration, such subintegrations correspond to inserted self-energy, vertex part or C part divergences, the differentiation operator converts them into, respectively, vertex part, or C part divergences, while the last type of divergence becomes superficially convergent. To bring out the points involved we consider, as an example, the case of a vertex part V, with its base line containing as an insertion the irreducible self-energy graph S (subintegration t_r) in Fig. 4.

Disregarding the mass renormalizing constant A from this self-energy graph S, the relevant divergent term is $B'\times V_{\text{Red}}$ where B' is charge-renormalizing constant from S, and V_{Red} is obtained from V by removing this self-energy insertion from the base line altogether.

Let us now suppose that the number of vertices on the base line of V is n, of which k are 3-vertices. The differentiation operator produces from the graph V, precisely $(n-1+2k)$ C parts. Of these 5 are such that they represent the replacing of S in the base line of V, by the 5 (irreducible) vertex parts which a differentiation of S yields, while in the remaining $(n-6+2k)$ C parts, the insertion S remains unchanged in the base line. Therefore the divergence separated corresponding to the subintegration t_r from this class of $(n-1+2k)$ C parts is $=B'\times C_{\text{Red}}'+L'\times C_{\text{Red}}''$. C_{Red}' are the $(n-6+2k)$ reduced C parts obtained by omitting S altogether from the base line, L' is the sum of the true divergent constants from the 5-vertex parts obtained by differentiating S, while C_{Red}'' is the reduced C part (just one graph) obtained by omitting any of the 5-vertex parts, and replacing it by a 3-vertex. Two of the graphs C_{Red}' are identical with the graph C_{Red}'', i.e., the reduced graphs from those two C graphs obtained by inserting the external photon line ν, just "before" and just "after" S, on the base line of V.

Now from Eq. (2) and (16), $B'=-L'$ (all graphs concerned in this relation are irreducible and the factor $1/2\pi i$ is absorbed in the definition of the reduced integral C_{Red}''); consequently, the term separated corresponding to the subintegration t_r is $=B'\times C_{\text{Red}}$ where C_{Red} represents $(n+2k-7)$ (different) C graphs; but these are precisely the graphs which we obtain by applying the differentiation operator $(\partial/\partial p+\partial/\partial p')$ to the graph V_{Red}, as can be checked by noticing that V_{Red} has $n-2$ vertices on the base line, of which $k-2$ are 3-vertices and therefore the number of C parts obtained by differentiating V_{Red} is $n-3+2(k-2)=n+2k-7$.

The proof is thus arranged, by considering each of the true divergence$\times$the reduced integral separated from the vertex part V. It is shown that for the corresponding C parts, the true divergences$\times$reduced integrals can be grouped such that the true divergences are equal by an inductive application of (16) and (17), and, furthermore, that the reduced integrals from C parts are precisely those which can be obtained by applying the differentiation operator to the reduced graphs. Since the argument proceeds by considering all subintegrations, overlaps are automatically taken care of.

The proof, therefore, of the required relations for a vertex part of order e^{2n+2} depends on establishing relations between true divergent constants of corresponding S, V, and C parts up to order e^{2n}. As the induction starts with irreducible graphs for which such relations are obviously true from the very definitions, the foregoing result follows. Thus, finally (putting $p=p=p_0$), we have

$$(\partial/\partial p+\partial/\partial p')_\nu\Lambda_{\mu c}(p,p') = \theta_{\mu\nu c}(p,p',p'-p)+\theta_{\nu\mu c}(p,p',0). \quad (19)$$

We shall now show that Z_1 can be chosen such that

$$\Gamma_\mu=(p+p')_\mu+\Lambda_\mu(p,p')=Z_1^{-1}[(p+p')_\mu+\Lambda_{\mu c}(p,p',e_1)]$$

where by definition $\Lambda_{\mu c}(p_0,p_0)=0$. Let

$$\theta_{\mu\nu}(p,p',p'-p)+\theta_{\nu\mu}(p,p',0)=2G_{\mu\nu}(p'+p,p'-p).$$

Then defining

$$r^\lambda=(p+p')\lambda+2p_0(1-\lambda),\quad s^\lambda=(p'-p)\lambda$$

we rewrite (17) as

$$(\partial/\partial r_\nu^\lambda)\Lambda_\mu(r^\lambda,s^\lambda)=G_{\mu\nu}(r^\lambda,s^\lambda)$$

but

$$(\partial/\partial\lambda)\Lambda_\mu(r^\lambda,s^\lambda)=(p+p'-2p_0)_\nu G_{\mu\nu}(r^\lambda,s^\lambda) +(p'-p)_\nu\partial\Lambda_\mu/\partial s_\nu^\lambda;$$

consequently,

$$\Lambda_\mu(p,p')-\Lambda_\mu(p_0,p_0) =\int_0^1 d\lambda(p+p'-2p_0)_\nu G_{\mu\nu}(r^\lambda,s^\lambda) +\int_0^1 d\lambda(p'-p)_\nu\partial\Lambda_\mu/\partial s_\nu^\lambda. \quad (20)$$

Also, from (19),

$$\Lambda_{\mu c}(p,p')=\int_0^1 d\lambda(p+p'-2p_0)_\nu G_{\mu\nu c}(r^\lambda,s^\lambda) +\int_0^1 d\lambda(p'-p)_\nu\partial\Lambda_{\mu c}/\partial s_\nu^\lambda. \quad (21)$$

We have already shown, however, that Z_4 can be chosen such that

$$\theta_{\mu\nu}(p,p',q)=\delta_{\mu\nu}(Z_4^{-1}-1)+Z_4^{-1}\theta_{\mu\nu c}(p,p',q,e_1).$$

Therefore, (20) can be rewritten as

$$\left[(p+p')_\mu+\Lambda_\mu(p,p')-\int_0^1 d\lambda(p'-p)_\nu\partial/\partial s_\nu^\lambda(r_\mu^\lambda+\Lambda_\mu)\right] -[(2p_0)_\mu+\Lambda_\mu(p_0,p_0)]=Z_4^{-1}\left[(p+p')_\mu +\int_0^1 d\lambda(p+p'-2p_0)_\mu G_{\mu\nu c}\right]-Z_4^{-1}[(2p_0)_\mu]. \quad (22)$$

Since by definition, $\Gamma_{\mu 1}=(p+p')_\mu+\Lambda_{\mu c}(p,p',e_1)$, we have, from (22) using (21)

$$\Gamma_\mu(p,p')-\int_0^1 d\lambda(p'-p)_\nu\partial\Gamma_\mu/\partial s_\nu^\lambda =Z_4^{-1}\left[\Gamma_{\mu 1}(p,p')-\int_0^1 d\lambda(p'-p)_\nu\partial\Gamma_{\mu 1}/\partial s_\nu^\lambda\right]. \quad (23)$$

This leads us to infer the required result, namely, that $\Gamma_\mu=Z_4^{-1}\Gamma_{\mu 1}(e_1)$.

Knowing the value of $G_{\mu\nu c}$ we cannot obtain that of $\Lambda_{\mu c}(p,p')$ from Eq. (21), but that of $\Lambda_{\mu c}(p,p)$ can be readily found by a simple integration. Since

$$\textstyle\sum_c^*(p)=-2\pi i\int_0^1 d\lambda(p-p_0)_\mu\Lambda_{\mu c}(p^\lambda,p^\lambda) \quad (24)$$

where $p^\lambda=p\lambda+p_0(1-\lambda)$, it is not needful for the derivation of $\sum_c^*(p)$ to consider the meson self-energy graphs any longer. The proof that $\Delta_F'=Z_4\Delta_{F1}'(e_1)$ follows from Eq. (16) in a similar manner to (23); in this case $\sum_M^*(p_0)$ is canceled by combining it with the contribution to the self-energy arising from the mass-renormalization term $-\delta\kappa^2\phi^*\phi$ in the Hamiltonian. We have also incidentally shown that $Z_1=Z_2=Z_4$, which satisfies condition (*d*) of the introduction.

4. THE FUNCTION D_F'

For photon self-energy graphs, we extend a formal technique introduced by Ward and define the functions, $\Delta_\mu(p)$ and $\Phi_{\mu\nu}(p)$, by equations,

$$-1/2\pi(\partial/\partial p_\mu)\Pi^*=\Delta_\mu(p) \quad (25)$$

$$(\partial/\partial p_\mu)\Delta_\mu(p)=\Phi_{\mu\nu}(p). \quad (26)$$

Also, let

$$W_\mu(p)=2ip_\mu+\Delta_\mu(p) \quad (27)$$

$$X_{\mu\nu}(p)=2i\bar{\delta}_{\mu\nu}+\Phi_{\mu\nu}(p). \quad (28)$$

The bar in $\bar{\delta}_{\mu\nu}$ distinguishes it from the $\delta_{\mu\nu}$ for a 4 *C*-vertex. These functions $W_\mu(p)$, and $X_{\mu\nu}$ stand in analogy to Γ_μ and $C_{\mu\nu}$ while Δ_μ and $\Phi_{\mu\nu}$ are analogous to Λ_μ and $\theta_{\mu\nu}$.

By integration we obtain from the foregoing

$$\Delta_\mu(p)=\int_0^1 d\lambda\Phi_{\mu\nu}(\lambda p)p_\nu \quad (29)$$

and

$$\Pi^*(p)=-2\pi\int_0^1 d\lambda\Delta_\mu(\lambda p)p_\mu. \quad (30)$$

Thus, in order to obtain $\Pi^*(p)$ we only need construct the function $\Phi_{\mu\nu}(p)$ which is logarithmically divergent. Dropping its divergent terms, the integrations in (29) and (30) give the desired convergent part of Π^*. In actual fact Π^* is not a scalar but a tensor[15] and its correct form (for the case of a simple irreducible photon self-energy part) is given by

$$\Pi_{\rho\kappa}^*=(\delta_{\rho\kappa}p^2-p_\rho p_\kappa)(C+D_c(p^2)).$$

The terms $-p_\rho p_k$ are omitted on account of charge conservation.[16]

[15] Julian Schwinger, Phys. Rev. **76**, 790 (1949), Appendix.
[16] R. P. Feynman, Phys. Rev. **76**, 781 (1949).

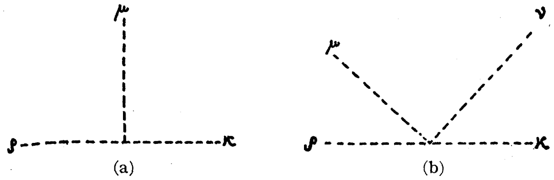


FIG. 5.

Thus, the function $\Phi_{\mu\nu}$ is in fact of the form:

$$\Phi_{\mu\nu,\,\rho\kappa}=[C+D_c(p^2)][2\bar{\delta}_{\mu\nu}\delta_{\rho\kappa}-\delta_{\rho\mu}\delta_{\kappa\nu}-\delta_{\rho\nu}\delta_{\kappa\mu}]+(\delta_{\rho\kappa}p^2-p_\rho p_\kappa)(\partial^2/\partial p_\mu\partial p_\nu)D_c(p^2).$$

However, if the terms $-\delta_{\rho\mu}\delta_{\kappa\nu}-\delta_{\rho\nu}\delta_{\kappa\mu}$ and $-p_\rho p_\kappa$ are dropped after the evaluation of $\Phi_{\mu\nu,\,\rho\kappa}$, we have

$$\Phi_{\mu\nu,\,\rho\kappa}=2\bar{\delta}_{\mu\nu}\delta_{\rho\kappa}(C+D_c(p^2))+\delta_{\rho\kappa}(\partial^2/\partial p_\mu\partial p_\nu)D_c(p^2)=\delta_{\rho\kappa}\Phi_{\mu\nu}.$$

In order to evaluate the function $\Phi_{\mu\nu}$ we must examine the implications of our formal differentiation $\partial/\partial p$. In general a photon self-energy graph may have both its external photon lines, μ and ν, belonging either to the same meson loop or to two distinct interconnected meson loops. In the latter case the momentum p must run along some photon lines joining the two distinct loops. The differentiation $\partial/\partial p$ which applied to a meson line carrying momentum p graphically signifies insertion of an external photon line (with energy-momentum zero), when applied to a photon line gives a new type of "vertex," illustrated in Fig. 5 with the vertex factor $2ip_\mu$ since $[-1/2\pi i(\partial/\partial p_\mu)D_F=D_F\cdot 2ip_\mu\cdot D_F]$. Thus to obtain the graphs corresponding to $\Phi_{\mu\nu}$ we first adopt some convention for the path of the variable p through the photon self-energy graphs. After "differentiating" all the photon self-energy graphs twice, we obtain the graphs corresponding to $\Phi_{\mu\nu}$ which contain vertices of the type in Fig. 5(a) and (b). The number and complexion of these graphs is governed by the convention[17] we adopt for the path of p but as the integrations (29) and (30) show, the choice of a convention does not matter as far as the evaluation of $\Pi^*(p)$ is concerned. After drawing all graphs $\Phi_{\mu\nu}$ we select the irreducible $\Phi_{\mu\nu}$, the criterion for irreducibility being the same as in Sec. 1, with the obvious extension that if a photon self-energy graph inside an internal photon line is differentiated, in order to obtain the irreducible skeleton, this is replaced by either of the vertices in Fig. 5 as the case may be. The advantage, as we shall see, in considering the function $\Phi_{\mu\nu}$ rather than Π^* itself is that the overlaps occurring for $\Phi_{\mu\nu}$ are relatively simple and it is easier to categorize the graphs corresponding to $\Phi_{\mu\nu}$ rather than Π^*. It may be emphasized that Δ_μ and $\Phi_{\mu\nu}$ are "fictitious" primitive divergents, being entirely defined by Eqs. (25) and (26) and have no real place in the theory. The convention for the path of p adopted here is illustrated in Fig. 6. Single arrows give the direction of charge, while double arrows follow p. In Fig. 6(a), p runs along the edge of the diagram; in (b) the path of p is "complementary," in the sense that if in (a), a portion of a closed loop is "differentiated," (b) insures the "differentiation" of the remaining portion of this same closed loop. Thus

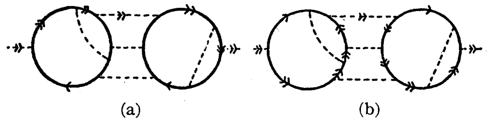


FIG. 6.

$$(-1/2\pi)\partial/\partial p_\mu[(a)+(b)]=2\Delta_\mu(p).$$

To illustrate the possible types of errors, let us notice that the graph in Fig. 7 is not reducible, because what appears as a C part with external photon lines ρ and μ is not in fact a C part comprised in the functions $C_{\mu\rho}$ so that this graph cannot be treated as reducible. When selecting the irreducible graphs for $\Phi_{\mu\nu}$ from the totality of graphs obtained, the following consideration is helpful. We have shown in Sec. 2, that $\theta_{\mu\rho}(p,p',0)$

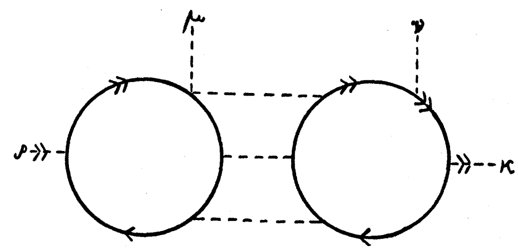


FIG. 7.

$=\theta_{\mu\rho}{}^b(p,p',0)$, where $\theta_{\mu\rho}{}^b$ denotes the sum of the integrals corresponding to C parts with the photon line μ (with energy-momentum zero) necessarily belonging to the base line. Thus, if in a set of reducible $\Phi_{\mu\nu}$, only base-line C parts $\theta_{\mu\rho}{}^b$ appear, the irreducible skeleton for such $\Phi_{\mu\nu}$ contains the factor $\delta_{\mu\rho}$ and to obtain $\Phi_{\mu\nu}$ this factor can be replaced by $C_{\mu\rho}$ (the function for all and not merely base-line C parts) without incurring any error. The foregoing choice of "complementary" convention for p was designed, such that if a portion of a closed meson loop acts as the base line for graphs $\theta_{\mu\rho}$ (say) contained in a set of graphs $\Phi_{\mu\nu}$, the convention should insure (by differentiating the entire loop) that all relevant base line $\theta_{\mu\rho}{}^b$ do appear. The entire set $\theta_{\mu\rho}{}^b$ can then be replaced by a 4-vertex $\delta_{\mu\rho}$ and a (unique) skeleton thus defined.[18]

By categorizing the relevant graphs and using our inductive procedure, we shall establish the following equations (which are completely analogous to the set

[17] The choice of a convention is arbitrary but it is absolutely essential that once a convention is adopted it must be consistently followed.

[18] In this section (and subsequent sections) the distinction of δ_{ij} and δ_{ji} will be understood and the arguments will proceed in terms of 4-vertices, invoking the "weight factors" if necessary.

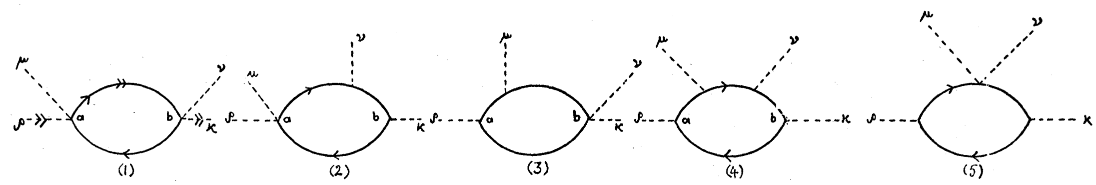


FIG. 8.

of Eqs. (4)):

$$\left.\begin{aligned} X_{\mu\nu}&=Z_3^{-1}X_{\mu\nu1}(e_1)\\ W_\mu&=Z_3^{-1}W_{\mu1}(e_1)\\ D_F'&=Z_3D_{F1}'(e_1).\end{aligned}\right\} \quad (31)$$

Except for the graphs in Fig. 8, it is possible to make unambiguous insertions (4) and (31) in all the lines and vertices of the irreducible $\Phi_{\mu\nu}$ (the vertices now including those in Fig. 5), so as to obtain all derived graphs and their corresponding integrals.

For the graphs in Fig. 8, however, insertion of certain types of C parts causes overlaps. Barring these, and considering all other irreducible graphs for $\Phi_{\mu\nu}$, we can express the sum of their integrals and of the reducible graphs derived from them in the fundamental form.

In Appendix I we will show, by a procedure of categorization, that the sum of the integrals from the graphs in Fig. 8 and all graphs derived from them can also be given in the fundamental form. Adding together the contribution from these and $2i\bar{\delta}_{\mu\nu}$ for the 4-vertex in Fig. 5(b), we obtain

$$X_{\mu\nu}=2i\bar{\delta}_{\mu\nu}+Z_3^{-1}[2iC(e_1)\bar{\delta}_{\mu\nu}+\Phi_{\mu\nu c}(e_1)].$$

At this stage a choice of $Z_3=1-C(e_1)$ establishes the relation, $X_{\mu\nu}=Z_3^{-1}X_{\mu\nu1}(e_1)$. From this and (29) and (30) we prove as in Sec. 2 that $D_F'=Z_3D_{F1}'(e_1)$.

5. *M* PARTS

In this section the problem of M parts is dealt with. We show that with a correct choice of $\delta\lambda$ in a term $\delta\lambda\phi^{*2}\phi^2$ in the Hamiltonian, we cannot only consistently cancel all M divergences in the theory (whether they arise inside the graphs representing other primitive divergents or from what would otherwise be non-divergent graphs), we can also arrange that the graphs with 4-M vertices introduced into the theory by this term introduce no new infinities. It is easy to verify that if the $\delta\lambda$ term constituted the only interaction term the number of primitive divergents in the theory would be precisely two; namely, the graphs with two external meson lines (meson self-energy graphs) and those with four external meson lines (M parts).

The choice of $\delta\lambda$ is made in three steps.

(1) Consider all irreducible simple M parts. If M_d represents the sum of their true divergences, a choice of $\delta\lambda$ such that $\delta\lambda+M_d=0$ cancels all divergences arising from these graphs.

(2) For the case of irreducible nonsimple M parts, unlike scalar meson-nucleon theories, joining two simple M parts may lead to an $M-M$ overlap. An example is shown in Fig. 9. However, whether or not these overlaps occur, our subtraction procedure gives the result that the correct choice of $\delta\lambda$ is once again given by $\delta\lambda+M_d=0$, where $\delta\lambda$ is the sum of true divergences arising from all irreducible M parts, whether simple or otherwise. In spite of the overlap, the manner of proof is exactly similar to that given in detail in I, Sec. A, and is not repeated.

(3) To obtain all the original M parts from irreducible M parts, we make the usual insertions in all the lines and the vertices. Assuming (a) that the relations (3) hold and (b) that an insertion does not cause a further overlap, we immediately see by counting up the number of lines and vertices (including M vertices) that the correct final value for $\delta\lambda$ is given by $Z_2^{-2}M_d(e_1)+\delta\lambda=0$ where $M_d(e_1)$ is the sum of the true divergences from all M parts. Condition (b) is satisfied for all except the three irreducible graphs in Fig. 10.

In these graphs, insertion of certain types of C parts causes $C-M$ overlaps. By a procedure of categorization it is not difficult to prove that the contribution made by such graphs to $\delta\lambda$ is also found as $-Z_2^{-2}M_d$, where M_d is the sum of their true divergences.

To prove (a) we remark that the foregoing proof for the cancellation of M-divergences extends to the case when M parts are contained inside other graphs. In particular for photon self-energy graphs or C parts there exist other graphs in the theory derived by replacing the contained M parts by M vertices in all possible ways. These graphs combine to cancel the M divergences times the corresponding "reduced integrals" so that neither need be considered at any stage of our inductive procedure for the formation of the functions Π^* or $C_{\mu\nu}$.

Since our procedure for deriving relations for operators $\sum^*$ and Λ_μ is not to follow the relevant graphs but to obtain these analytically from $\theta_{\mu\nu}$ we have to show that the explicit neglect of M divergences and of the corresponding graphs with 4-M vertices in $\theta_{\mu\nu}$ does not affect the proof of the identities (16) and (17); in

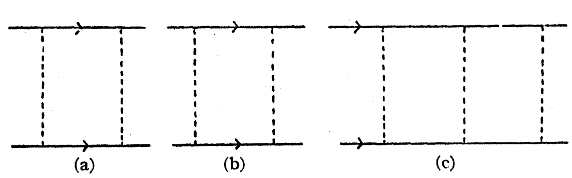


FIG. 9.

other words that this neglect is justified for both sides of the equations represented in (16) and (17).

The proof would be trivial if, for example, the differentiation $\partial/\partial p$ which, from the graphs corresponding to $\sum^*(p)$, produces all graphs relevant to $\Lambda_\mu(p, p)$ (in the manner of the proof in Sec. 3), also took all compensatory self-energy graphs with M vertices to corresponding compensatory vertex graphs. Since the proof of equality of $-(1/2\pi i)(\partial/\partial p)\sum^*$ with $\Lambda_\mu(p, p)$, for example, must depend on letting p run along the base line we have to show that a choice of basic momentum vectors which allows the path of p to lie along the base line also allows us to isolate M divergences.

The entire problem is closely linked with the problem of the general possibility of a "correct" choice of basic variables. This was discussed in detail in a previous paper[19] and only the results will be given here. For vertex parts situations arise when the number of divergences exceeds the number of subintegrations. In such cases, however, the extra divergences are M divergences which prove to be "final," and need not be separated. This happens, however, only on account of the gauge-invariance of the theory.

For meson self-energy graphs there is (as in the meson-nucleon case) another class of M divergences

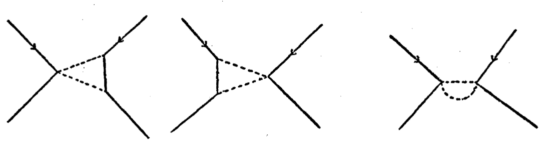


FIG. 10.

which were also called "final"[20] in Sec. B, I. The reduced integral corresponding to such "final" divergences is, by definition, independent of the external momentum p. Their effect is to leave just one type of graph with a 4-M vertex (illustrated in Fig. 6, I) and those derived from it by insertions in the meson line, as "odd" graphs which do not act as compensatory graphs like all other graphs with 4-M vertices, and so have to be considered separately. Here, as in the meson-nucleon case, these graphs only contribute to the mass renormalization constant, making it an explicit function of $\delta\lambda$; a fact finding analytic expression in that a differentiation $\partial/\partial p$ for such graphs gives the result zero.

Similar considerations apply for Π^*, Δ_μ, and $\Phi_{\mu\nu}$. The proof of the finiteness of the S matrix now follows precisely as in D II Sec. VII.

6. CONCLUSION

The only new feature arising for the renormalization of spin zero Bose particles interacting with the electromagnetic field is the introduction into the Hamiltonian of the (infinite) direct interaction term $\delta\lambda\phi^{*2}\phi^2$. In a sense it is satisfactory that this same term can remove the M divergences associated with the scattering of a meson by a meson when these particles interact through an exchange of virtual nucleons. The definition adopted in Sec. V of the divergent part of an M integral, however, makes $M_c(p_0, p_0, p_0)=0$, so that no graph of any order higher than the second in e^2, can contribute to the scattering amplitude of two mesons of equal initial and final momenta. Theoretically it is possible to proceed slightly differently and to introduce into the Hamiltonian, besides the compensatory $\delta\lambda\phi^{*2}\phi^2$, another "real" direct-interaction term $\lambda\phi^{*2}\phi^2$. The graphs involving 4-vertices with $\delta\lambda$ as coefficient are differentiated from those with λ as coefficient, the definition of original graphs being extended to include, besides the graphs involving 3-e- and 4-e^2-vertices, also the graphs containing 4-λ-vertices. The renormalization of the theory (not presented here in detail) proceeds as before, except that the Z factors now appear as functions of both e_1 and λ_1 (the renormalized value of λ) while the graphs with $\delta\lambda$-vertices compensate the additional M divergences introduced by the graphs containing these new λ-vertices as well. Thus $\delta\lambda$ is to be chosen as $\delta\lambda+Z_2^{-2}M_d(e_1, \lambda_1)=0$ while $\lambda_1=Z_2^{-2}\lambda$. The entire theory, after renormalization, appears in terms of two constants λ_1 and e_1. The retention of the condition $M_c(p_0, p_0, p_0)=0$ has the desirable feature, however, that even with this new term, the additional contribution to the Møller scattering amplitude, for mesons of equal initial and final momenta, is not given as a power series in λ_1 but consists merely of λ_1 itself.

The scheme presented here sketches what is theoretically possible. On account of the new feature noted, the physical validity of our renormalization scheme cannot be extrapolated from the fact that a very close agreement with experiment exists for the renormalized theory of spinor electrodynamics. To determine whether a constant λ_1 exists (even if only to find that its value is zero) and whether the answers to any physical problems given by the foregoing scheme approximate to the truth, we must turn again the pages of nature.

ACKNOWLEDGMENTS

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[19] Abdus Salam, Phys. Rev. **83**, 426 (1951).

[20] Opening any one photon line in a meson self-energy graph (or a meson line in a photon self-energy graphs) gives a C part with a corresponding "final" C divergence. Their effect is precisely similar to that of "final" M divergences.

APPENDIX I

Here we consider the graphs in Fig. 8 and all reducible graphs which can be derived from them and prove that their contribution to $\Phi_{\mu\nu}$ can be expressed in the fundamental form.

In the 4-vertices appearing in the graphs in Fig. 8 an insertion of those C parts which have their two external photon lines belonging either to the same 4-vertex (class B) or to two consecutive 3-vertices on the same meson line (class C), causes an overlap. These same C parts inserted at one end 4-vertex of the graphs in Fig. 8, appear simultaneously as an insertion at the other end. The situation is completely analogous to the insertion of vertex parts in end-vertices of photon (or electron) self-energy graphs in spinor electrodynamics (the "b divergences" of D II Sec. VII). Let the class of C parts $B+C$ be called A, while let T denote all C parts other than those in class A. From Sec. 2,

$$T_{\mu\nu}=Z_4^{-1}[T_d\delta_{\mu\nu}+T_{\mu\nu c}(e_1)].$$

We now categorize the graphs derived from those in Fig. 8. The categories obtained are similar to those in Sec. IV, (I). In graphs (1), (2), (3), a and b stand for the end 4-vertices. Graph (1) has both its end-vertices as 4-vertices, (2) and (3) have either the left- or the right-hand vertex (as drawn here) as a 4-vertex while (4) has no 4-vertex at all. Let the graphs (1), (2), (3), and (4) in Fig. 8 belong to category [1]. Insert all *irreducible* A at the 4-vertex $a_{[1]}$ in $[1|1]$ and $[1|2]$, thereby obtaining a set of graphs belonging to category [2]. These graphs can once again be distinguished as $[2|1]$, $[2|2]$, $[2|3]$, and $[2|4]$, according as their end-vertices are 3-vertices or 4-vertices. Insertion of irreducible A at the 4-vertices $a_{[2]}$ in $[2|1]$ and $[2|2]$ then gives all graphs in category [3] and so on. $a_{[n]}$ and $b_{[n]}$ stand for the end 4-vertices appearing in $[n|1]$, $[n|2]$, and $[n|3]$ at every stage.

Given a graph in $[n]$; let A_a^1 denote the true divergent constant arising from the irreducible A which was inserted at $a_{[n-1]}$ to obtain this particular graph in $[n]$; A_a^2, the constant of true divergence from the reducible graph A which could be inserted at $a_{[n-2]}$ to obtain this graph in $[n]$. This last graph is reducible because it contains as an insertion the irreducible graph with true divergence A_a^1. Similar definitions apply for $A_b^1, A_b^2, A_b^3\cdots$, etc. These definitions are analogous to those of L_a^1, L_a^2, $\cdots$, L_b^1, L_b^2, $\cdots$ in Sec. IV, I.

With these definitions we have the following lemma.

Lemma 1

$$\begin{aligned}[n]=&[n|1]+[n|2]+[n|3]+[n|4]\\ =&A_a^1[n-1|1]+A_a^2[n-2|1]+A_a^3[n-3|1]+\cdots\\ &+A_b^1[n-1|1]+A_b^2[n-2|1]+A_b^3[n-3|1]+\cdots\\ &-[A_a^1A_b^1[n-2|1]+A_a^1A_b^2[n-3|1]+\cdots\\ &+A_a^2A_b^1[n-3|1]+\cdots]+A_a^1[n-1|2]+A_a^2[n-2|2]\\ &+\cdots+A_b^1[n-1|3]+A_b^2[n-2|3]+\cdots\\ &+2iF_d\bar{\bar{\delta}}_{\mu\nu}+F_{\mu\nu c}.\quad(1)\text{A}\end{aligned}$$

$2i\bar{\bar{\delta}}_{\mu\nu}F_d$ gives the true divergence of $[n]$, while $F_{\mu\nu c}$ is the convergent part. The proof of the lemma is given exactly on the lines of the proof of Lemma 1 (Sec. IV, I) and is not repeated.

In any graph $[n]$, we can insert self-energy and vertex parts without causing any overlaps, while at all 4-vertices except $a_{[n]}$ and $b_{[n]}$, all C parts can be inserted unambiguously. Correspondingly, we replace Δ_F by $\Delta_F'=Z_2\Delta_{F1}'$, D_F by Z_3D_{F1}', $(p+p')_\mu$ by $Z_1^{-1}\Gamma_{\mu1}$ in all the lines and 3-vertices of the graphs in these categories; while for all 4-vertices except $a_{[n]}$ and $b_{[n]}$, replace $\delta_{\mu\nu}$ by $Z_4^{-1}C_{\mu\nu1}$. For $a_{[n]}$ and $b_{[n]}$, however, only the replacement of $\delta_{\mu\nu}$ by $\delta_{\mu\nu}+T_{\mu\nu}=Z_4^{-1}[(Z_4+T_d-1)\delta_{\mu\nu}+(\delta_{\mu\nu}+T_{\mu\nu c})]$ is analytically valid.

In terms of graphs, we obtain unambiguously by the insertions corresponding to these substitutions all graphs derived from the irreducible graphs in Fig. 8, no graph appearing more than once.

Let $\Sigma[n]^*$ represent the totality of graphs after these replacements, $[n]^\times$ represent graphs $[n]$ with $\Delta_{F1}'(e_1)$, $D_{F1}'(e_1)$, $\Gamma_{\mu1}(e_1)$, and either $C_{\mu\nu1}(e_1)$ or $\delta_{\mu\nu}+T_{\mu\nu c}(e_1)$ appearing for their lines and vertices, $[n]_a^\times$ represent all graphs with the above factors $\Delta_{F1}'(e_1)$, $D_{F1}'(e_1)$, etc., except at the 4-vertex a where factor $\delta_{\mu\nu}$ remains unchanged, and let similar definitions apply for $[n]_b^\times$ and $[n]_{ab}^\times$. Then precisely as in Lemma 2, the entire class of graphs $\Sigma[n]^*$ can be expressed compactly as

$$\Sigma[n]^*=Z_3^{-1}\Sigma\{[n]^\times+(Z_4+T_d-1)([n]_a^\times+[n]_b^\times+(Z_4+T_d-1)[n]_{ab}^\times)\}.\quad(2)\text{A}$$

From Lemma 1, however, by summing up

$$\Sigma[n]^\times=A_d^\times\{\Sigma([n]_a^\times+[n]_b^\times-A_d[n]_{ab}^\times)\}+2iF_d^\times(e_1)+F_{\mu\nu c}^\times,\quad(3)\text{A}$$

where $A_d^\times=A_a^{1\times}+A_a^{2\times}+A_a^{3\times}+\cdots=A_b^{1\times}+A_b^{2\times}+A_b^{3\times}+\cdots$. Here $A_a^{1\times}$, etc., denote the true divergences with Δ_{F1}', D_{F1}', etc., replacing Δ_F, D_F, etc. $F_d^\times(e_1)$ is the sum of the true divergences of $[n]^\times$.

We have shown in Sec. 2 that $Z_4=1-T_d-A_d^\times$. So substituting (3)A in (2)A we obtain

$$\Sigma[n]^*=Z_3^{-1}\{2iF_d^\times(e_1)\bar{\bar{\delta}}_{\mu\nu}+F_{\mu\nu c}^\times(e_1)\}.\quad(4)\text{A}$$

It still remains to consider the graph (5) in Fig. 8. The set consisting of graphs (1), (2), (3), and (4) was completely symmetrical regarding the classes B and C ($B+C=A$) but the graph (5) contains only one 4-vertex $\delta_{\mu\nu}$. This vertex will be called $e_{[1]}$. Inserting irreducible B at $e_{[1]}$ we obtain graphs belonging to $[2|1]$ while insertion of irreducible C at $e_{[1]}$ gives $[2|2]$. These (irreducible B and C) can be inserted again at $e_{[2]}$ in $[2|1]$ to obtain $[3]=[3|1]+[3|2]$ and so on. The graph (5) in Fig. 8, itself constitutes the category $[1|1]$, there being no graph in $[1|2]$. When the divergence in such a graph $[n]$ is being isolated, the "reduced" graph left after the true divergence corresponding to the C part (which C part necessarily belongs to class C) has been removed, will be denoted by $[n|3]$. A graph $[n|3]$ does not itself belong to the class of graphs $[n]$, because for $[n|3]$, $\rho\kappa$ must be a 4-vertex. This new 4-vertex will be called f. Similarly let $[n|4]$ denote the graph obtained by a simultaneous reduction of $[n]$ at both ends, so that in the graph $[n|4]$ both e and f appear as 4-vertices. Again a graph $[n|4]$ does not itself belong to the class of graphs $[n]$.

To obtain all graphs that can be derived from the graph (5), we construct the above categories $[n]$ and then in their lines and vertices make the replacements $\Delta_F=Z_2\Delta_{F1}'(e_1)$, etc., except at the 4-vertex e where $\delta_{\mu\nu}$ is replaced by $Z_4^{-1}[(Z_4+T_d-1)\delta_{\mu\nu}+(\delta_{\mu\nu}+T_{\mu\nu c}(e_1))]$. If $[n]^*$, $[n]^\times$ are defined as those $[n]$ in which, instead of Δ_F, for example, appear, respectively, the factors Δ_F' and Δ_{F1}', while $[n]_e^\times$ are $[n|1]^\times$ except that at the 4-vertex e, $\delta_{\mu\nu}$ is left completely unchanged, then by counting lines and vertices we obtain, as in (2)A

$$\Sigma[n]^*=Z_3^{-1}\{\Sigma([n]^\times+(Z_4+T_d-1)[n]_e^\times)\}.\quad(5)\text{A}$$

Now considering the implications of the subtraction procedure the result corresponding to Lemma 1 for $[n]$ is as follows.

Lemma 2

$$\begin{aligned}[n]=&[n|1]+[n|2]=A^1[n-1|1]+A^2[n-2|1]+A^3[n-3|1]\\ &+\cdots+C^1[n-1|3]+C^2[n-2|3]+C^3[n-3|3]\\ &+\cdots-(A^1C^1[n-2|4]+A^1C^2[n-3|4]+\cdots\\ &+A^2C^1[n-3|4]+\cdots)+2iG_d\bar{\bar{\delta}}_{\mu\nu}+G_{\mu\nu c}.\quad(6)\text{A}\end{aligned}$$

By replacing Δ_F, by Δ_{F1}', etc., in $[n]$ and summing up, we obtain from the foregoing

$$\Sigma[n]^\times=\Sigma\{A_d^\times[n]_e^\times+C_d^\times([n]_f^\times-A_d^\times[n]_{ef}^\times)\}+2iG_d^\times(e_1)\bar{\bar{\delta}}_{\mu\nu}+G_{\mu\nu c}^\times(e_1).\quad(7)\text{A}$$

$[n]_f^\times$ are $[n|3]$ with Δ_{F1}', etc., replacing Δ_F except that the 4-vertex f is completely unchanged, and $[n]_{ef}^\times$ are similarly obtained from $[n|4]$ except that for both e and f, the factors $\delta_{\rho\kappa}$ and $\delta_{\mu\nu}$ are not changed. Substituting for $\Sigma[n]^\times$ in Eq. (5)A,

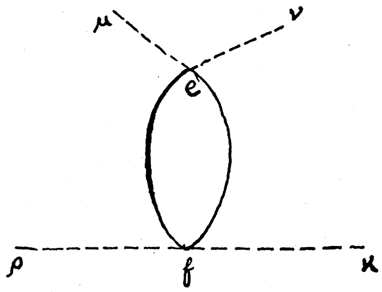

FIG. 11.

and using $Z_4 = 1 - T_d - A_d^\times$ we obtain

$$\Sigma[n]^* = Z_3^{-1}(2i\bar{\delta}_{\mu\nu} G_d^\times(e_1) + G_{\mu\nu c}^\times(e_1)) + Z_3^{-1}(\Sigma C_d^\times\{[n]_f^\times - A_d^\times[n]_{ef}^\times\}). \quad (8)\text{A}$$

We now prove the result:

Lemma 3

$$Z_3^{-1}(\Sigma C_d^\times\{[n]_f^\times - A_d^\times[n]_{ef}^\times\}) \equiv 0. \quad (9)\text{A}$$

For the proof consider the graph in Fig. 11. Construct various categories from it, by inserting irreducible A, successively, at the vertex e only. The characteristic of these graphs then is that f is necessarily a 4-vertex for all of them. After these categories are constructed, make substitutions (4) in all lines and vertices, except in vertex e where $\delta_{\mu\nu}$ is replaced by $Z_4^{-1}[(Z_4+T_d-1)\delta_{\mu\nu} + (\delta_{\mu\nu}+T_{\mu\nu c})]$ and the 4-vertex f which is left completely unchanged. The result is that we obtain as the sum of integrals corresponding to all these graphs, precisely the expression,

$$Z_4 Z_3^{-1}\Sigma([n]_f^\times + (Z_4+T_d-1)[n]_{ef}^\times)(e_1),$$

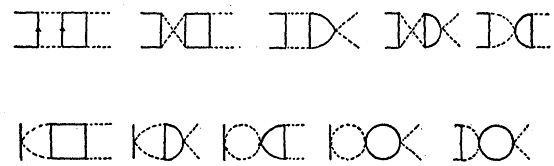

FIG. 12.

but this $= (\partial^2/\partial p_\mu \partial p_\nu)\int F(p+\iota, t^i, e)dt dt^i$ which is identically equal to zero, as can be seen by shifting $p+t \to t'$. Since $Z_4 = 1 - T_d - A_d^\times \neq 0$ the proof is complete.

On account of Lemma 3, (8)A now gives

$$\Sigma[n]^* = Z_3^{-1}[2i\bar{\delta}_{\mu\nu} G_d^\times(e_1) + G_{\mu\nu c}(e_1)], \quad (10)\text{A}$$

which is once again in the fundamental form. The required result is thus established.

APPENDIX II

The graphs R mentioned in Sec. 2 are a set of chain graphs "derived" from the 10 irreducible graphs in Fig. 12. Since C parts have two external photon lines, a chain of meson loops joined to each other by two photon lines can cause $C-C$ and $C-M$ overlaps. The proof that all possible types of $C-C$ overlaps are comprised in the classes P, Q, R is not difficult to give from very simple topological considerations. To show, further, that the sum of integrals from graphs R can be expressed in the fundamental form, a careful scheme of categorization needs to be developed. The general principles on which the proof proceeds is already illustrated in Sec. 2. The details are much more complicated but, as no new principles are involved, we shall not reproduce them here (Fig. 12).