A Phase Shift Analysis of Neutron-Deuteron Scattering*

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A simultaneous phase shift analysis of *n-d* and *p-d* angular cross sections is shown to be consistent with the charge-independence hypothesis. Disagreement is found with the theoretical phase shifts of Buckingham and Massey.

1. INTRODUCTION

EXPERIMENTAL angular cross sections for *n-d* scattering at 4.5 and 5.5 Mev have been determined by Wantuch[1] at Brookhaven. The *d-p* cross section at 10.4 Mev has been determined by Allred and Rosen[2] at Los Alamos. Breit has considered these data for consistency with charge-independence of nuclear forces.[3] He assumed the *p-d* cross section to be the square of the Coulomb plus nuclear scattering amplitudes and the nuclear scattering amplitude to be the square root of the *n-d* cross section. The difference between *p-d* and *n-d* cross sections, so calculated, he found to be of the same order of magnitude as the observed difference in the experimental curves.

We shall also treat the Wantuch and Allred-Rosen data, but more quantitatively by means of a thorough phase shift analysis. Our purpose is twofold, however; not only to test the charge-independence hypothesis, but also to make a direct comparison with the theoretical *n-d* phase shifts of Buckingham and Massey.[4] Noncentral and velocity dependent forces are presumed to be negligible throughout the present analysis.

A previous attempt has been made to check phase shifts with the Buckingham and Massey theory by Critchfield,[5] using the *p-d* angular cross sections at 1.5 to 3.5 Mev of Sherr, Blair, *et al.*[6] In that analysis it is supposed that *p-d* and *n-d* phase shifts should be comparable on the basis of charge-independence. Unfortunately, in order to reduce the cross sections to unique phase shifts, Critchfield was forced to make auxiliary assumptions using the Buckingham and Massey theory itself. In particular, he assumed that the quartet and doublet S-wave phase shifts are equal and negative.[7] His convention (which we shall also adopt) is that phase shifts lie in the first and fourth quadrants only.

In our analysis we want to avoid such auxiliary assumptions. In fact, it is just this result of the Buckingham and Massey theory, that the S-wave phase shifts are equal and negative, that seems most interesting to check against experiment. It is impossible, however, to analyze *n-d* or *p-d* cross sections individually in terms of phase shifts without some auxiliary assumptions. In the *p-d* case large uncertainties in small angle measurements allow a fit of the data with a whole spectrum of sets of nuclear phase shifts. In the *n-d* case the matter is even worse because if, say, $2L$ phase shifts are involved (including quartet and doublet), the highest harmonic that can occur in the cross section is a P_{L-1}^2, which contains a P_{2L-2}. One derives one equation on the phase shifts for each harmonic from P_0 to P_{2L-2}, a total of $2L-1$, one less than the number of phase shifts. Hence, the *n-d* phase shifts are not even in principle determinable from the cross section.

In our procedure we shall try to avoid the indeterminacy of the individual phase shift analyses by considering the *n-d* and *p-d* cross sections conjointly. First we shall analyze the *n-d* cross section into the whole spectrum of sets of phase shifts consistent with it, and then see what portion, if any, of this spectrum is also consistent with the *p-d* cross section (actually *d-p*) under the assumption of charge-independence. In this way we hope to avoid the need for auxiliary information.

The problem arises of inferring from charge-independence a precise relationship between the *p-d* and *n-d* phase shifts themselves. The usual statement is, of course, that the *p-d* and *n-d* phase shifts ought to be approximately equal. A theoretical study of this problem has been carried out by one of us,[8] with the result that the phase shifts may be taken as equal (at the energies being considered) provided a suitable compensation is made for Coulomb effects. In particular, it is shown that the usual device of comparing the cross

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[1] E. Wantuch, Phys. Rev. **84**, 169 (1951).
[2] L. Rosen and J. C. Allred, Phys. Rev. **82**, 777 (1951).
[3] G. Breit, Phys. Rev. **80**, 1110 (1950).
[4] R. A. Buckingham and H. S. W. Massey, Proc. Roy. Soc. (London) **A179**, 123 (1942).
[5] C. L. Critchfield, Phys. Rev. **73**, 1 (1948).
[6] Sherr, Blair, Kratz, Bailey, and Taschek, Phys. Rev. **72**, 662 (1947).
[7] Buckingham and Massey found this to be true irrespective of energy and the nature of exchange forces.
[8] A. L. Latter, Ph.D. thesis, University of California, Los Angeles, California, 1951.

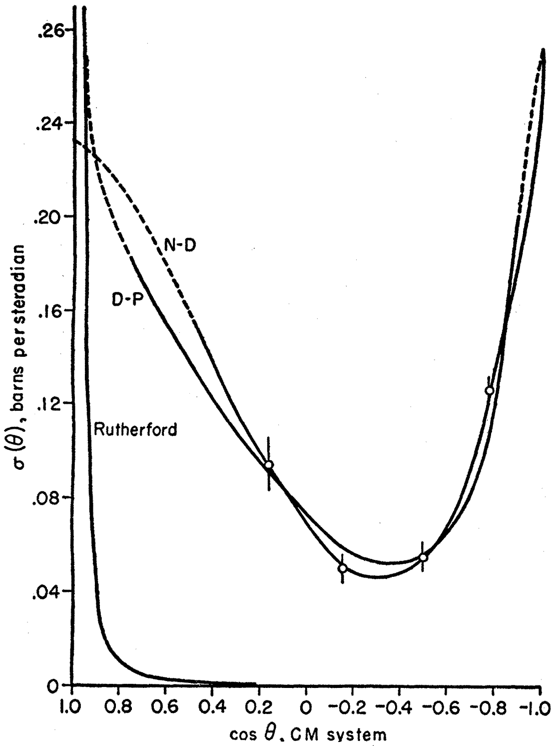

FIG. 1. Differential cross sections: 4.7-Mev *n-d*, 10.4-Mev *d-p*, and 5.2-Mev Rutherford.

sections at different energies is a good approximation. More precisely, the *p-d* energy should be about 0.5 Mev (lab) higher than the *n-d*, corresponding roughly to the *p-d* Coulomb barrier.

Since the *d-p* cross section is at 10.4 Mev, which is equivalent to *p-d* at 5.2 Mev, the *n-d* cross section should be taken at 4.7 Mev. The *n-d* at 4.7 Mev can be found by interpolation from the 4.5- and 5.5-Mev curves. Figure 1 shows the version of the data to be used in the calculations below. Experimental uncertainties in the *n-d* curve are indicated as usual by cross-lines. The dashed portion of the curve is an extrapolation made to fit the total cross section, 1.56 barns, known independently from the work of Nuckolls *et al.*[9] (No angular data were obtained in the Wantuch experiment below about 60°.)

Also shown in Fig. 1 is the 10.4-Mev *d-p* cross section. No errors are indicated because, for the most part, the data are good to 3 percent, which for our purposes is presumed to be exact. Dashed portions of the curve represent regions of insufficient data filled in by guesswork.

2. *n-d* PHASE SHIFTS

Six partial waves are expected to contribute to the cross section: *S*, *P* and *D*, quartet and doublet. This number may be estimated on the basis of the kR value for the interaction. For a neutron or proton scattered by a deuteron at 5 Mev, with a radius R of, say, 4×10^{-13} cm for the deuteron, the kR value is 1.3. As usual,

$$k^2=(2\mu/\hbar^2)E,$$

where μ is the reduced mass ($\frac{2}{3}$ the mass of a nucleon)

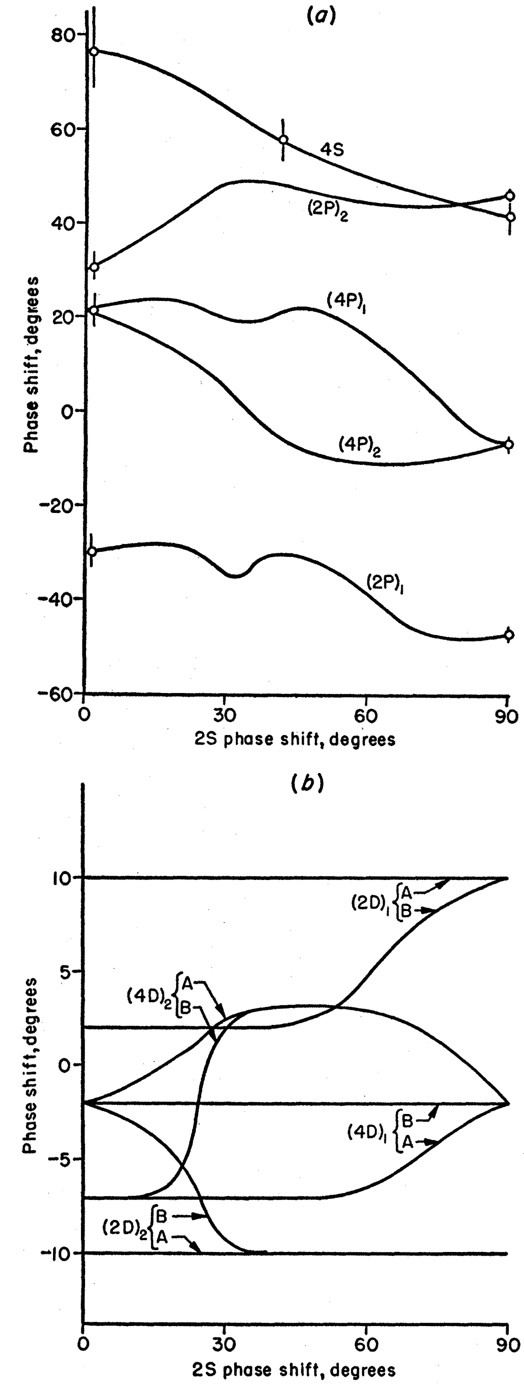

FIG. 2(a). Quartet *S* and *P*, and doublet *P*, phase shifts corresponding to assumed values of doublet *S*, for the 4.7-Mev *n-d* cross section; (b). Quartet and doublet *D*-wave phase shifts.

[9] Nuckolls, Bailey, Bennett, Bergstralh, Richards, and Williams, Phys. Rev. **70**, 805 (1946).

TABLE I. Right side of Eq. (8) for $\theta=54.7°$, to be compared with left side which equals -0.026 ± 0.012 (empirical value).

u	1++	2++	1+−	2+−	1−+	2−+	1−−	2−−
0	−0.014*	−0.029*	−0.029*	−0.013	+0.022	+0.007	+0.007	+0.022
0.2	−0.017*	−0.030*	−0.028*	−0.008	+0.021	+0.002	+0.010	+0.023
0.4	−0.019*	−0.031*	−0.027*	−0.004	+0.021	−0.002	+0.013	+0.024
0.6	−0.021*	−0.027*	−0.027*	+0.001	+0.021	−0.008	+0.015	+0.021
0.7	−0.023*	−0.023*	−0.028*	+0.006	+0.022	−0.013	+0.017	+0.017
0.8	−0.022*	−0.022*	−0.029*	+0.007	+0.022	−0.013	+0.015	+0.015
0.9	−0.017*	−0.020*	−0.027*	+0.006	+0.021	−0.013	+0.011	+0.014
0.95	−0.012	−0.020*	−0.024*	+0.005	+0.017	−0.011	+0.005	+0.013
1.00	−0.001	−0.020*	−0.020*	−0.001	+0.013	−0.005	−0.005	+0.013

and E is the relative energy in the center-of-mass system.

A Fourier analysis of the n-d cross section confirms the estimate that the D wave is the highest to enter. The result is

$$\sigma(\theta)=a_0+a_1P_1(\theta)+a_2P_2(\theta)+a_3P_3(\theta)+a_4P_4(\theta), \quad (1)$$

where $a_0=0.124\pm0.004$, $a_1=0.051\pm0.008$, $a_2=0.118\pm0.006$, $a_3=-0.057\pm0.004$, and $a_4=0.013\pm0.001$. Errors were estimated from the error lines on the n-d curve in Fig. 1, except for a_0, which was derived from the total cross section of Nuckolls *et al.*

Let $\delta_L{}^Q$ and $\delta_L{}^D$ denote the Lth quartet and doublet phase shifts, respectively. In terms of these,

$$\sigma(\theta)=(2/3k^2)|g_n{}^Q|^2+(1/3k^2)|g_n{}^D|^2, \quad (2)$$

where

$$\begin{aligned} g_n{}^Q&=\exp(i\delta_0{}^Q)\sin\delta_0{}^Q+3\exp(i\delta_1{}^Q)\sin\delta_1{}^Q P_1(\theta)\\ &\quad+5\exp(i\delta_2{}^Q)\sin\delta_2{}^Q P_2(\theta),\\ g_n{}^D&=\exp(i\delta_0{}^D)\sin\delta_0{}^D+3\exp(i\delta_1{}^D)\sin\delta_1{}^D P_1(\theta)\\ &\quad+5\exp(i\delta_2{}^D)\sin\delta_2{}^D P_2(\theta). \end{aligned} \quad (3)$$

It is convenient to introduce the following notation:

$$\begin{aligned} x&=\sin\delta_0{}^Q, & u&=\sin\delta_0{}^D,\\ y&=\sin\delta_1{}^Q, & v&=\sin\delta_1{}^D,\\ z&=\sin\delta_2{}^Q, & w&=\sin\delta_2{}^D. \end{aligned} \quad (4)$$

If we compare the expressions (1) and (2) for $\sigma(\theta)$ and equate coefficients of P_0, P_1, $\cdots P_4$, we obtain the five equations which the six quantities in (4) must satisfy.

$$\begin{aligned}
&\text{(a)}\ 2x^2+u^2+3(2y^2+v^2)=3.54\pm0.12\\
&\text{(b)}\ 2[xy(1-x^2)^{\frac12}(1-y^2)^{\frac12}+x^2y^2]\\
&\qquad+[uv(1-u^2)^{\frac12}(1-v^2)^{\frac12}+u^2v^2]=0.44\pm0.04\\
&\text{(c)}\ 2[xy(1-x^2)^{\frac12}(1-z^2)^{\frac12}+x^2z^2]\\
&\qquad+[uw(1-u^2)^{\frac12}(1-w^2)^{\frac12}+u^2w^2]+\tfrac{3}{5}(2y^2+v^2)\\
&\qquad\qquad=0.33\pm0.02 \quad (5)\\
&\text{(d)}\ 2[yz(1-y^2)^{\frac12}(1-z^2)^{\frac12}+y^2z^2]\\
&\qquad+[vw(1-v^2)^{\frac12}(1-w^2)^{\frac12}+v^2w^2]\\
&\qquad\qquad=-0.094\pm0.007\\
&\text{(e)}\ 2z^2+w^2=0.030\pm0.005.
\end{aligned}$$

We have set $k=3.17\times10^{12}$ cm^{-1} corresponding to 4.7-Mev n-d.

To solve Eq. (5) we ignore the uncertainties in the constants and treat the equations as exact. The effect of the uncertainties will be discussed later. Since there are six unknowns but only five equations, we shall have a one-parameter family of solutions; and because the totality of solutions encompasses the entire permissible range of u from -1 to $+1$, it is convenient to choose u as the parameter. The equations can then be solved by a simple perturbation technique. The quantities z and w are expected to be small and are set equal to zero in (c). Then (a), (b), and (c) become a set of three equations in three unknowns, with u regarded as a parameter. These equations can be easily solved and the solution introduced into (d), and then (d) and (e) solved for z and w. The values of z and w so obtained are then applied to improving (c) and the whole process iterated.

The solutions of Eq. (5) are plotted in Fig. 2. As shown, both S-waves have positive phase shifts, but this is not the only possibility since Eq. (5) is invariant under a complete change of signs of the set x, y, z or u, v, w, or both concomitantly. Hence, there are really four sets of solutions implied by Fig. 2. Note that for a given value of the $2S$ phase shift, there is only one value for the quartet S-wave phase shift ($4S$) but two possible P-wave phase shifts denoted by subscripts 1 and 2. For each set of P-wave phase shifts, there are two possible D-wave phase shifts, denoted by A and B. Error lines are not indicated for the D-wave phase shifts because this part of the analysis was rather crude. For the S and P waves, errors were estimated by solving Eq. (5) with the constants varied over the limits of the experimental certainty.

3. COMPARISON WITH d-p CROSS SECTION

Let $\bar{\sigma}(\theta)$ be the 10.4-Mev d-p angular cross section (center-of-mass system). The wave number $\bar{k}$ is given by

$$\bar{k}^2=(2\mu/\hbar^2)\bar{E},$$

where $\bar{E}=\frac{2}{3}(5.2)$ Mev. Setting n-d and p-d nuclear phase shifts equal, we have approximately

$$\bar{k}^2\bar{\sigma}(\theta)=\tfrac{2}{3}|g_c+g_n{}^Q|^2+\tfrac{1}{3}|g_c+g_n{}^D|^2, \quad (6)$$

with $g_n{}^Q$ and $g_n{}^D$ as defined by Eq. (3), and

$$g_c=-(\eta/1-\cos\theta)\exp[i\eta\ln(2/1-\cos\theta)],\quad \eta=e^2\mu/\hbar^2\bar{k}. \quad (7)$$

TABLE II. Right side of Eq. (8) for $\theta=125.3°$, to be compared with left side which equals -0.003 ± 0.005 (empirical value).

u	1++	2++	1+−	2+−	1−+	2−+	1−−	2−−
0	+0.001**	+0.004	+0.004	0	−0.004	0	0	−0.004
0.2	0**	+0.003	+0.005	−0.001	−0.005	+0.001	0	−0.003
0.4	−0.002**	+0.001**	+0.005	−0.001	−0.005	+0.001	+0.002	−0.001
0.6	−0.003**	−0.001**	+0.004	−0.003	−0.004	+0.003	+0.003	+0.001
0.7	−0.003**	−0.003**	+0.004	−0.005	−0.004	+0.005	+0.003	+0.003
0.8	−0.003**	−0.003**	+0.004	−0.005	−0.004	+0.005	+0.003	+0.003
0.9	−0.004**	−0.003**	+0.003	−0.006	−0.003	+0.006	+0.004	+0.003
0.95	−0.004	−0.003**	+0.003	−0.006	−0.003	+0.006	+0.004	+0.003
1.00	−0.006	−0.001**	+0.001**	−0.006	+0.001	+0.006	+0.006	+0.001

The approximation in Eq. (6) is in setting $\arg\Gamma(L+1+i\eta)$ equal to zero, which makes little difference in our results because η is small ($=0.069$).

Subtracting Eq. (2) from (6), we find

$$\bar{\sigma}(\theta)-(k^2/\bar{k}^2)\sigma(\theta)-\sigma_c(\theta) = (2/3\bar{k}^2)\,\mathrm{Re}[g_c^*(2g_n^Q+g_n^D)], \quad (8)$$

where $\bar{k}^2\sigma_c(\theta)=|g_c(\theta)|^2$. The left side of (8) is known directly from experiment. The right side assumes different values depending upon the choice of phase shifts. Our procedure is to sample all the phase shifts of Fig. 2 and see what sets give agreement with the left side. In principle, this might be done at several angles leading eventually to a unique determination of the phase shifts. However, since the D-wave analysis of Fig. 2(b) is very unreliable, and the $(2L+1)$-weighting favors the importance of the D-wave in the cross section, our procedure would lead to indefinite results at angles where the D-wave makes a large contribution.

There are two angles where the D-wave does not enter at all, 54.7° and 125.3° (the roots of P_2). For the reasons given above, we shall confine our attention to these two angles. Table I shows the results for $\theta=54.7°$. The tabulated values represent the right-hand side of Eq. (8) for various possible choices of the phase shifts given in Fig. 2(a). The phase shifts used were those corresponding to the most probable experimental values of the cross section, but the error incurred this way is negligible for our purposes. Note that any value of $|u|$ from 0 to 1 is permitted. The P-wave solutions are bifurcated, the forks being 1 and 2. Other solutions are generated by sign changes of x, y, z and u, v, w, independently. Thus for each $|u|$ we have eight cases to consider, of which (2+−) is typical. The symbols mean: the second fork of the P-wave, the positive solution for x, and the negative solution for u. The values in the table are to be compared with the left-hand side, which is −0.026. Actually the uncertainty in the n-d data makes this comparison value -0.026 ± 0.012. Those entries that are in agreement with these limits have been marked by an asterisk.

The same procedure is followed for $\theta=125.3°$. The right side of Eq. (8) is shown in Table II and is to be compared this time with -0.003 ± 0.005. Those values that are allowed at this angle as well as at 54.7° are distinguished now by a double asterisk.

The most significant fact about Table II is that there are any double asterisks at all. Their occurrence may be construed as evidence in favor of the charge-independence hypothesis, since our comparison of p-d and n-d cross sections has depended primarily on this feature of nuclear forces. On the other hand, it is gratifying that the possibilities can be so well delimited by using data at two angles only.[10]

With improved n-d data the phase shifts could probably be determined uniquely. But from Table II alone we can already point out a sharp disagreement with the theory of Buckingham and Massey. For ordinary forces their values of $2S$ and $4S$ is about −64° at 4.7 Mev, and −71° for "mixed exchange," whereas according to Table II the S-wave phase shifts are almost certainly positive. To be sure, Buckingham and Massey have considered only a few particular types of exchange, and it is possible that a different choice of nuclear parameters would lead to better agreement.

We wish to thank Dr. R. J. Finkelstein and Dr. D. S. Saxon for helpful discussion.

[10] Other angles were considered but no further elimination occurred.