The Intermediate Coupling Theory of the Pseudoscalar Meson-Nucleon Interaction

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Tomonaga's intermediate coupling theory for scalar mesons is extended to pseudoscalar mesons with pseudoscalar coupling. The single free nucleon is discussed. Recoil is included nonrelativistically and the divergences are removed arbitrarily by a cutoff at the nucleon rest mass. Effects involving one virtual pair are allowed for, on the assumption that two or more virtual pairs may be neglected. The calculations are of a preliminary nature and certain possibilities for refining them are discussed. As they stand, they indicate that even for $f^2/4\pi\simeq 10$, a renormalized weak coupling expansion should give a very slowly converging, but not entirely distorted picture of the meson cloud surrounding a nucleon for pseudoscalar coupling.

INTRODUCTION

THE interpretation of recent meson experiments, based on general considerations such as parity conservation and detailed balancing, shows that the π-meson is pseudoscalar.[1] On the other hand, purely theoretical arguments have shown that only the theories of scalar and pseudoscalar mesons with scalar coupling can be made finite to any order in the coupling constant by renormalization, and of these the pseudoscalar is preferable since it contains one less arbitrary constant.[2] This much agreement between pure theory and the experimental facts is encouraging. But it is still not possible to make a detailed comparison of renormalized pseudoscalar meson theory with experiment because renormalization has been carried out in the general framework of an expansion in the coupling constant. Calculations made on the assumption that this constant, f, is small lead to values of f, to give the required orders of magnitude, ($f^2/4\pi\simeq 10$), which invalidate this assumption.[3] This paper attempts to develop methods of calculation which do not depend on the smallness of the coupling constant and thereby to check how much can safely be deduced from the first one or two terms in the small coupling expansion. Preliminary calculations are presented, and the work is mainly of methodological interest.

The simplest system which one can consider is that of a single free nucleon. By "free" we mean a real physical particle capable of interaction with the meson field. For the purpose of calculation it is usual to analyze this state in a representation based on the bare mesons and bare nucleons. By "bare" we mean fictitious mathematical particles which are eigenstates of the bare Hamiltonians with no interaction term. The free nucleon state has nonzero representatives, besides that corresponding to a single bare nucleon, and for this reason the free nucleon is sometimes pictured as a bare nucleon surrounded by a cloud of bare mesons and bare nucleon antinucleon pairs. Our purpose is to examine the probability distribution of this cloud.

More explicitly if ψ is the state vector of the interacting meson and nucleon fields, which represents a single free nucleon, then we wish to calculate the representatives $(\psi|\mathbf{p},\cdots;\mathbf{q},\cdots;\mathbf{k},\cdots)$ where $\mathbf{p},\cdots;\mathbf{q},\cdots;\mathbf{k}\cdots$ is a state vector in Fock space, $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{k}$ being the momenta of the bare nucleons, antinucleons, and mesons, respectively.

These representatives are the solutions to a set of Schrödinger equations which are derived in Sec. 2. For comparison with the later work these equations are then solved by successive approximation in f, according to ordinary weak coupling theory. To first order, this allows for, at most, one meson or one meson and one pair in the cloud, and assumes that by far the largest representative is that for which there are no particles in the cloud at all, $(\psi|\mathbf{p})$.

An alternative approach is Tomonaga's theory of intermediate coupling,[4] which was applied by him to the free nucleon in scalar meson theory with complete neglect of recoil. In this approximation no pairs can occur, but one allows for any number of mesons in the cloud, with the simplifying assumption that they all lie in the same state. An exact solution can be obtained in this form for neutral scalar mesons, and indeed Glauber and Luttinger[5] have developed a neat method for deriving a complete set of exact solutions. These show that if the coupling constants of the order here considered are inserted in scalar theory the largest representative corresponds to a state in which at least one meson is present, and the small coupling approximation breaks down completely.[4]

In Sec. 3, this method is extended to the positive energy subspace of pseudoscalar theory with recoil included nonrelativistically. (That is to say pairs are

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[1] R. E. Marshak, Revs. Modern Phys. 23, 137 (1951).
[2] P. T. Matthews and Abdus Salam, Revs. Modern Phys. 23, 311 (1951).
[3] For nuclear forces $f^2/4\pi\simeq 4$; H. Bethe, Phys. Rev. 76, 191 (1949). For photomeson production $f^2/4\pi\simeq 40$; K. A. Brueckner, Phys. Rev. 79, 645 (1950). While to give the right orders of magnitude for the anomalous nucleon magnetic moments $f^2/4\pi\simeq 10$; K. M. Case, Phys. Rev. 76, 14 (1949).
[4] S. Tomonaga, Prog. Theoret. Phys. 2, 6 (1947). Applications with the same approximations have been made by K. M. Watson and E. W. Hart, Phys. Rev. 79, 918 (1950).
[5] R. J. Glauber and J. M. Luttinger, unpublished. We are deeply indebted to Dr. Glauber for showing us his work on this subject.

arbitrarily excluded, and it is assumed that the momenta of the particles in the cloud are small compared with the nucleon rest mass.) It is shown in subsections 3 (a), (b), and (c) that for this model the Hamiltonian can be split into two parts, the largest of which can be treated exactly by the Glauber-Luttinger method. The remainder is then considered in subsection (d) as a perturbation, using the complete set of exact solutions as a basis.

In Sec. 4 the method is extended to the case in which it is assumed that one pair, but not more, may appear in the cloud. Unlike the model of Sec. 3, this may be regarded as an approximation to the real problem. In treating the pair, only large components of the interaction are considered. A solution similar to that obtained above proves to be still possible, and is compared with that given by weak coupling theory.

The energy, corresponding to the eigenstate considered, includes the self energy of the free nucleon and is divergent. In this paper all divergent integrals are cut off at the nucleon rest mass. This is a questionable assumption in pseudoscalar theory which affects the numerical conclusions. This is discussed in the last section.

We deal so far with a single stationary nucleon in interaction with neutral mesons. This is a purely academic problem. In the appendix a method for extending these considerations to charged mesons is indicated. If the results were further generalized to describe a nucleon of given momentum, the matrix elements of an electromagnetic potential between two such states would lead directly to an estimate of the nucleon magnetic moments. It was felt that the method must be considerably refined before it would be worthwhile carrying out such a calculation, but we have this possibility in mind in the final discussion.

1. STATE VECTORS IN FOCK SPACE

We wish to find the representatives in Fock[5a] space of the state vector which represents a single free nucleon. As a preliminary we consider an assembly of bare bosons which may lie in any of an infinite set of states determined by the momenta $\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_a \cdots$. If the assembly consists of particles in a particular set of states represented by the upper suffices $1, \cdots, r, \cdots, n$, this can be represented by the ket

$$\mathbf{k}^1, \cdots, \mathbf{k}^r, \cdots, \mathbf{k}^n) \quad r=1, 2, \cdots, n \tag{1.1}$$

or alternatively by the normalized ket

$$n_1, \cdots, n_a, \cdots), \tag{1.2}$$

where[6]

$$n_a = \sum_r \delta_{ra} \tag{1.3}$$

is the number of particles in the state a.[7] If we distinguish between the orderings of the factors in the ket (1.1), the number of such kets which corresponds to a single ket (1.2) is

$$n! \prod_a (n_a!)^{-1}. \tag{1.4}$$

Define the normalization of (1.1) by

$$\cdots n_a \cdots) = (n!)^{\frac{1}{2}} \prod_a (n_a!)^{-\frac{1}{2}} \mathbf{k}^1, \cdots, \mathbf{k}^r, \cdots, \mathbf{k}^n), \tag{1.5}$$

where $\mathbf{k}^1, \cdots, \mathbf{k}^n)$ is any particular ket satisfying (1.3). Now since (1.2) is normalized we have the unit matrix

$$\begin{aligned} 1 &= \sum_{\cdots n_a \cdots} \cdots n_a \cdots)(\cdots n_a \cdots \\ &= \sum_{\cdots n_a \cdots} \mathbf{k}^1, \cdots, \mathbf{k}^r, \cdots, \mathbf{k}^n) n! \\ &\qquad \times \prod_a (n_a!)^{-1} (\mathbf{k}^1, \cdots, \mathbf{k}^r, \cdots \mathbf{k}^n \\ &= \sum_n \int \cdots \int \mathbf{k}^1, \cdots, \mathbf{k}^r, \cdots, \mathbf{k}^n) \\ &\qquad \times (\mathbf{k}^1, \cdots, \mathbf{k}^r, \cdots, \mathbf{k}^n \quad d\mathbf{k}^1 \cdots d\mathbf{k}^n \end{aligned} \tag{1.6}$$

by (1.5) and (1.4). We will have frequent occasion to use the unit matrix in its latter form.

Now we have the relations

$$q_a^* | \cdots, n_a, \cdots) = (n_a+1)^{\frac{1}{2}} | \cdots, n_a+1, \cdots), \tag{1.7}$$

and

$$q_a | \cdots, n_a, \cdots) = n_a^{\frac{1}{2}} | \cdots, n_a-1, \cdots). \tag{1.8}$$

Combining (1.5) and (1.7)

$$q^*(\mathbf{k}) | \mathbf{k}^1, \cdots, \mathbf{k}^n) = (n+1)^{\frac{1}{2}} | \mathbf{k}^1, \cdots, \mathbf{k}^n, \mathbf{k}). \tag{1.9}$$

Combining (1.5), (1.8), and (1.3),

$$q(\mathbf{k}_a) | \mathbf{k}^1, \cdots, \mathbf{k}^r, \cdots, \mathbf{k}^n) = n^{-\frac{1}{2}} \sum_r \delta_{ra} | \mathbf{k}^1, \cdots, \mathbf{k}^r, \cdots \mathbf{k}^n, \mathbf{k}_a^{-1}),$$

where the notation on the right-hand side means that $\mathbf{k}^r$ is dropped from the ket $\mathbf{k}^1, \cdots, \mathbf{k}^r, \cdots \mathbf{k}^n)$ if $\mathbf{k}^r = \mathbf{k}_a$. It follows immediately that

$$\sum_a f(\mathbf{k}_a) q(\mathbf{k}_a) | \mathbf{k}^1, \cdots, \mathbf{k}^n) = n^{-\frac{1}{2}} \sum_r f(\mathbf{k}^r) | \mathbf{k}^1, \cdots, \mathbf{k}^{r-1}, \mathbf{k}^{r+2}, \cdots, \mathbf{k}^n) \tag{1.10}$$

For a field of bare fermions[8] we can define a ket by the relation

$$\cdots, n_a, \cdots) = (-1)^P (n!)^{\frac{1}{2}} | \mathbf{p}^1, \cdots, \mathbf{p}^r, \cdots \mathbf{p}^n), \tag{1.11}$$

where $\cdots, n_a, \cdots)$ is again normalized. From this it follows that

$$\mathbf{p}^1, \cdots, \mathbf{p}^r, \cdots, \mathbf{p}^n) = (n!)^{-\frac{1}{2}} a^{*1} a^{*2} \cdots a^{*r} \cdots a^{*n})_0. \tag{1.12}$$

[5a] V. Fock, Z. Physik **75**, 622 (1932).

[6] By δ_{ra} we mean zero except when $k^r = k_a$.

[7] The ket $k^1, \cdots, k^n)$ defined above is equal to $(n!)^{-1} \Sigma_P$ times the ket $k^1, \cdots, k^n)$ defined by Dirac. P. A. M. Dirac, *Quantum Mechanics* (Oxford University Press, London, 1947), Chapter X, Eq. (3).

[8] The notation is that of Dirac (see reference 7, Sec. 65) except that $a^* = \eta$, $a = \eta$ ond our $p^1, \cdots, p^n)$ is equal to $(n!)^{-1} \Sigma_{\pm P}$ times $p^1, \cdots, p^n)$ defined by Dirac.

It can then be deduced that

$$\sum_n \int \cdots \int \quad \mathbf{p}^1, \cdots, \mathbf{p}^n)(\mathbf{p}^1, \cdots, \mathbf{p}^n \quad d\mathbf{p}^1 \cdots d\mathbf{p}^1 = 1. \tag{1.13}$$

(Here the upper suffixes also specify spin and the integrals include sums over spins.) and

$$a^*(\mathbf{p})|\mathbf{p}^1, \cdots, \mathbf{p}^n) = (n+1)^{\frac{1}{2}}|\mathbf{p}, \mathbf{p}^1, \cdots, \mathbf{p}^n),$$

$$\sum_a f(\mathbf{p}_a)a(\mathbf{p}_a)|\mathbf{p}^1, \cdots, \mathbf{p}^r, \cdots, \mathbf{p}^n)$$
$$= n^{-\frac{1}{2}} \sum_r (-1)^{r-1} f(\mathbf{p}^r)|\mathbf{p}^1, \cdots, \mathbf{p}^{r-1}, \mathbf{p}^{r+1}, \cdots \mathbf{p}^n). \tag{1.14}$$

With these relations it is a simple matter to derive the equations for the representatives, $(\psi|\mathbf{p}, \cdots; \mathbf{q}, \cdots; \mathbf{k}, \cdots)$.

2. WEAK COUPLING THEORY

The stationary Schrödinger equation for the interaction of nucleons with pseudoscalar mesons, with pseudoscalar coupling is, in the bra form,

$$(\psi|H_0+H_1-E|=0, \tag{2.1}$$

where E is an eigenvalue and $(\psi$ an eigenstate;

$$H_0 = \int d\mathbf{p} \sum_r \{a_r^*(\mathbf{p})a_r(\mathbf{p}) + b_r^*(\mathbf{p})b_r(\mathbf{p})\} p_0 + \int d\mathbf{k} q^*(\mathbf{k}) q(\mathbf{k}) k_0,$$

$$p_0 = (|\mathbf{p}|^2+\kappa^2)^{\frac{1}{2}}, \quad k_0 = (|\mathbf{k}|^2+\mu^2)^{\frac{1}{2}}; \tag{2.2}$$

and

$$H_1 = f' \int d\mathbf{p} d\mathbf{p}' d\mathbf{k} \sum_{rs} \delta(\text{momentum})(a_r^*(\mathbf{p}') + b_r(\mathbf{p}'))(a_s(\mathbf{p}) + b_s^*(\mathbf{p}))(q^*(\mathbf{k}) + q(\mathbf{k})) \times i u_r^*(\mathbf{p}')\beta\gamma_5 u_s(\mathbf{p}) k_0^{-\frac{1}{2}}, \tag{2.3}$$

$$f' = 2^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}} f. \tag{2.4}$$

a^*, a; b^*, b; and q^*, q are the creation and annihilation operators for bare nucleons, antinucleons, and mesons, respectively. β and γ_5 are the usual Dirac matrices.[9] δ (momentum) is a δ-function of the momentum transfer produced by the operators. The bare Hamiltonian defines a complete set of kets for the bare particles,

$$H_0 - E'|\mathbf{p}_{r}^1, \cdots, \mathbf{p}_{r'}^L; \mathbf{q}_s^1, \cdots \mathbf{q}_{s'}^M; \mathbf{k}^1, \cdots, \mathbf{k}^n) = 0, \tag{2.5}$$

where the suffixes r and s denote spin. These can be normalized as in Sec. 1 and form the basis of a representation in which (2.1) becomes

$$(\psi|H_0+H_1-E|\mathbf{p}_r^1, \cdots \mathbf{p}_{r'}^{L+1}; \times \mathbf{q}_s^1, \cdots, \mathbf{q}_{s'}^L; \mathbf{k}^1, \cdots, \mathbf{k}^n) = 0. \tag{2.6}$$

[9] We take $\hbar = c = 1$, Heaviside units and δ-function normalization. r, s, t $(=1, 2)$ denote spin states.

The state $(\psi$, which represents a single free nucleon, reduces to a single bare nucleon as f tends to zero, and the only nonzero representative would then be $(\psi|\mathbf{p})$. We have inserted in (2.6) the typical ket which is related to $\mathbf{p})$ through the interaction (2.3). Our object is to solve this equation as generally as possible by Tomonaga's method of intermediate coupling.[4]

For comparison with later results we first solve this equation by successive approximation in f to obtain the familiar results of weak coupling theory. To first order in f (2.6) is

$$(p_0-E)(\psi|\mathbf{p}_s) = -f'\Big[\sum_t \int d\mathbf{k}\, i u_+^*(\mathbf{p}-\mathbf{k}_t)\beta\gamma_5 u_+(\mathbf{p}_s) k_0^{-\frac{1}{2}} (\psi|\mathbf{p}-\mathbf{k}_t; \mathbf{k}) + \sum_{a,b} \int\int d\mathbf{p}' d\mathbf{k}\, 2^{\frac{1}{2}} u_+^*(\mathbf{p}_a')\beta\gamma_5 u_-(-\mathbf{p}'-\mathbf{k}_b) \times k_0^{-\frac{1}{2}}(\psi|\mathbf{p}_a', \mathbf{p}_s; -\mathbf{p}'-\mathbf{k}_b; \mathbf{k})\Big], \tag{2.7}$$

$$(p_0^1+k_0-E)(\psi|\mathbf{p}_t^1, \mathbf{k}) = -f' \sum_u i u_+^*(\mathbf{p}^1+\mathbf{k}_u)\beta\gamma_s u_+(\mathbf{p}_t^1) k_0^{-\frac{1}{2}} (\psi|\mathbf{p}^1+\mathbf{k}_u), \tag{2.8}$$

$$(p_0'+p_0''+q_0+k_0-E)(\psi|\mathbf{p}_a', \mathbf{p}_s''; q_b; \mathbf{k}) = -f' 2^{-\frac{1}{2}} [i u_-^*(-\mathbf{p}'-\mathbf{k}_b)\beta\gamma_5 u_+(\mathbf{p}_a') k_0^{-\frac{1}{2}} \times \delta(\mathbf{p}''-\mathbf{p})(\psi|\mathbf{p}_s'') - i u_-^*(-\mathbf{p}''-\mathbf{k}_b) \times \beta\gamma_5 u_+(\mathbf{p}_s'') k_0^{-\frac{1}{2}} \delta(\mathbf{p}'-\mathbf{p})(\psi|\mathbf{p}_a')]. \tag{2.9}$$

These equations are obtained by (1.7), (1.8), (1.13), and (1.14), the bra $(\psi$, being brought through to the left after the operation of the operators. The δ-functions appear because all representatives have the same momentum. To zero order in f'

$$E = p_0, \quad (\psi|\mathbf{p}_s) = 2^{-\frac{1}{2}}. \tag{2.10}$$

(The factor $2^{-\frac{1}{2}}$ is a normalization to allow for the two possible spin states.) Substituting (2.10) into (2.8) and (2.9), determines $(\psi|\mathbf{p}, \mathbf{k})$ and $(\psi|\mathbf{p}', \mathbf{p}''; \mathbf{q}; \mathbf{k})$ to this order. Substituting these expressions back into (2.8) gives the usual expression for the self-energy,

$$\Delta E = E - p_0.$$

The terms arising from $(\psi|\mathbf{p}', \mathbf{p}''; \mathbf{q}; \mathbf{k})$ are

$$(f')^2 \sum_{a,b} \int d\mathbf{p}' d\mathbf{k}\, u_+^*(\mathbf{p}_a')\beta\gamma_5 u_-(-\mathbf{p}'-\mathbf{k}_b) \times k_0^{-1}(p_0'+q_0+k_0)^{-1} [u_-^*(-\mathbf{p}'-\mathbf{k}_b)\beta\gamma_5 u_+(\mathbf{p}_a')\delta(0) - u_-^*(-\mathbf{p}-\mathbf{k}_b)\beta\gamma_5 u_+(\mathbf{p}_a)\delta(\mathbf{p}'-\mathbf{p})]. \tag{2.11}$$

The first term is a vacuum effect due to unrestricted creation of the pair $(\mathbf{p}', \mathbf{q}, \mathbf{k})$ for any values of $\mathbf{p}'$. The second term is the restriction on this because of the presence of the electron $\mathbf{p}$. The state $(\psi$ represents a free nucleon plus the vacuum. Only the second term contributes to the nucleon energy. One thus obtains the usual expression for the self-energy of the nucleon itself.

A similar separation of vacuum terms is required when considering the probability of finding a pair in the nucleon cloud. Thus the squaring of (2.9) gives the probability of a pair and a meson in the vacuum and in the nucleon cloud. The probability for the nucleon cloud alone is given by the cross terms only, namely,

$$R\left[\frac{-(f')^2}{2}\sum_{a,b,c}\frac{u_+^*(\mathbf{p}_a)\beta\gamma_5 u_-(\mathbf{q}_b)u_-(\mathbf{q}_b)\beta\gamma_5 u_+^*(\mathbf{p}_c)}{k_0(p_0+q_0+k_0)^2}\right],$$

where

$$\mathbf{q}=-\mathbf{p}-\mathbf{k}.$$

Note that this separation of vacuum effects is exact in weak coupling theory because the analogous calculations for the vacuum state alone lead to precisely the expressions which have been dropped. (The separation of vacuum effects is dealt with particularly neatly in Feynman's formulation.[10])

3. INTERMEDIATE COUPLING—NO PAIRS

(a) The Energy

We now develop Tomonaga's solution which is independent of the size of f, but with the quite arbitrary restriction that all representatives involving pairs are neglected, (projection on the positive energy subspace). For pseudoscalar theory this cannot be regarded as an approximation to the complete system since we are actually neglecting the large components of the interaction. The typical equation is

$$(\psi|H_0+H_1-E|\mathbf{p}_t^n;\mathbf{k}^1,\cdots,\mathbf{k}^n)=0, \qquad (3.1)$$

where

$$\mathbf{p}^n=\mathbf{P}-\textstyle\sum^n\mathbf{k}^r. \qquad (3.2)$$

$\mathbf{P}$ is the total momentum of the cloud, which will be taken to be zero in the rest of this work. It is legitimate to put in the restriction on the form of the representatives, since, as will be seen immediately below, the Eqs. (3.3) only relate representatives of the same momentum. [Alternatively, one could insert a δ-function in the form for $(\psi|\mathbf{p}_t^n;\mathbf{k}^1,\cdots,\mathbf{k}^n)$.] Substituting for H and H_1 from (2.2) and (2.3) and using (1.7) and (1.8), one obtains an equation in which all operators have been replaced by c-numbers. The bra $(\psi$ can then be brought through to give directly the (stationary Schrödinger) equation for the representatives (wave function);

$$\begin{aligned}(E-p_0^n+\textstyle\sum^n k_0^r)&(\psi|\mathbf{p}_t^n;\mathbf{k}^1,\cdots,\mathbf{k}^n)\\ =f'\Big[(n+1)^{\frac12}\textstyle\sum_s&\int U^*(\mathbf{p}_t^n,\mathbf{p}_s^{n+1})\\ &\times(\psi|\mathbf{p}_s^{n+1};\mathbf{k}^1,\cdots,\mathbf{k}^{n+1})d\mathbf{k}^{n+1}\\ &+n^{-\frac12}\sum_r^n\sum_s U(\mathbf{p}^n+\mathbf{k}_s^r,\mathbf{p}_t^n)\\ &\times(\psi|\mathbf{p}^n+\mathbf{k}_s^r;\mathbf{k}^1,\cdots,\mathbf{k}^{r-1},\mathbf{k}^{r+1},\cdots,\mathbf{k}^n)\Big],\end{aligned} \qquad (3.3)$$

[10] R. P. Feynman, Phys. Rev. **76**, 649, 669 (1949).

where

$$U(\mathbf{p}+\mathbf{k}_s,\mathbf{p}_t)=iu_s^*(\mathbf{p}+\mathbf{k})\beta\gamma_5 u_t(\mathbf{p})(k_0)^{-\frac12}. \qquad (3.4)$$

The energy for the state $(\psi$ is

$$\begin{aligned}W&=(\psi|H_0+H_1|\psi)\\ &=\sum_n^\infty\int(d\mathbf{k})^n\textstyle\sum_t(\psi|H_0+H_1|p_t^n;\mathbf{k}^1,\cdots,\mathbf{k}^n)\\ &\qquad\times(\mathbf{p}_t^n;\mathbf{k}^1,\cdots,\mathbf{k}^n|\psi),\end{aligned} \qquad (3.5)$$

where

$$\int(d\mathbf{k})^n=\int\cdots\int d\mathbf{k}^1\cdots d\mathbf{k}^n.$$

This can be derived immediately from (3.3), giving

$$\begin{aligned}W=\sum_n^\infty\int(d\mathbf{k})^n\sum_t\Big\{&(p_0^n+nk_0^n)(\psi|\mathbf{p}_t^n,\mathbf{k}^1,\cdots,\mathbf{k}^n)^2\\ &+f'\Big[(n+1)^{\frac12}\int\sum_s U^*(\mathbf{p}_t^n,\mathbf{p}_s^{n+1})\\ &\times(\psi|\mathbf{p}_s^{n+1};\mathbf{k}^1,\cdots,\mathbf{k}^{n+1})\\ &\times(\mathbf{p}_t^n;\mathbf{k}^1,\cdots,\mathbf{k}^n)|\psi)dk^{n+1}\\ &+n^{\frac12}\textstyle\sum_s U(\mathbf{p}_s^{n-1},\mathbf{p}_t^n)(\psi|\mathbf{p}_s^{n-1};\mathbf{k}^1,\cdots,\mathbf{k}^{n-1})\\ &\times(\mathbf{p}_t^n;\mathbf{k}^1,\cdots,\mathbf{k}^n|\psi)\Big]\Big\}.\end{aligned} \qquad (3.6)$$

Here we have also replaced

$$\sum_r^n f(\mathbf{k}^r)$$

by $nf(\mathbf{k}^n)$ in the integrands (where f is any function). If we now change the variable of summation in the second term from n to $n+1$, the last two terms are the complex conjugates of each other. Thus

$$\begin{aligned}W=\sum_n^\infty\int(d\mathbf{k})^n\textstyle\sum_t\{&(p_0^n+nk_0^n)(\psi|\mathbf{p}_t^n;\mathbf{k}^1,\cdots,\mathbf{k}^n)^2\\ &+2n^{\frac12}R[f'\textstyle\sum_s U(\mathbf{p}_s^{n-1},\mathbf{p}_t^n)\\ &\times(\psi|\mathbf{p}_s^{n-1};\mathbf{k}^1,\cdots,\mathbf{k}^{n-1})\\ &\times(\mathbf{p}_t^n;\mathbf{k}^1,\cdots,\mathbf{k}^n|\psi)]\}.\end{aligned} \qquad (3.7)$$

We now introduce the Tomonaga approximation which is that

$$(\mathbf{p}_t^n;\mathbf{k}^1,\cdots,\mathbf{k}^n|\psi)=2^{-\frac12}c_n\prod_r^n f(k^r), \qquad (3.8)$$

where $\int f^2(k)d\mathbf{k}=1$, and thus

$$\sum_n^\infty c_n^2=1. \qquad (3.9)$$

$f(k)$ is the wave function in momentum space for a single meson, giving its momentum distribution. $f(k)$ is interpreted as the lowest state for a meson bound to the nucleon and the assumption is that all the mesons in the cloud go independently into this single state. [Note that the wave function is assumed to be independent of the spin direction of the nucleon. However, for a given set of k^r the nucleon momentum is fixed by (3.2). The factor $2^{-\frac{1}{2}}$ is a normalization caused by the two possible spin states of the nucleon for a given set k^r.] Substituting (3.8) into (3.7), we have

$$W=\sum_n^\infty\{n\alpha_n c_n^2+2n^{\frac{1}{2}}\beta_n c_{n-1}c_n\}, \qquad (3.10)$$

where

$$\alpha_n=\int(d\mathbf{k})^n(n^{-1}p_0+k_0^n)\prod_r^n f^2(k^r), \qquad (3.11)$$

and

$$\beta_n=f'\int(d\mathbf{k})^n\sum_{s,t}2^{-1}U(\mathbf{p}_s^{n-1},\mathbf{p}_t^n)f(k^n)\prod_r^{n-1}f^2(k^r). \qquad (3.12)$$

It is now our main purpose to find $f(k)$ and c_n which make W a minimum, subject to the condition (3.9). The momentum distribution of each meson is given by $f(k)$ and c_n^2 gives directly the probability distribution of the numbers of mesons in the cloud. The first step is to reduce the expressions α_n and β_n to integrals over a single variable.

We have (introducing the cutoff at κ)

$$\alpha_n=n^{-1}\int^\kappa(|\mathbf{k}^1+\cdots+\mathbf{k}^n|^2+\kappa^2)^{\frac{1}{2}}\prod_r^n f^2(k^r)(d\mathbf{k})^n$$
$$+\int^\kappa(k^2+\mu^2)^{\frac{1}{2}}f^2(k)d\mathbf{k} \qquad (3.13)$$

where κ and μ are the nucleon and meson masses, respectively. Make the assumption that

$$|\mathbf{k}^1+\cdots+\mathbf{k}^n|<\kappa, \qquad (3.14)$$

and expand in terms of the ratio. Then the first term of α_n is approximately

$$n^{-1}\int^\kappa\{\kappa+[(\mathbf{k}^1)^2+\cdots+(\mathbf{k}^n)^2]/(2\kappa)\}\prod_r^n f^2(k^r)(d\mathbf{k})^n;$$

and we thus have

$$\alpha_n=n^{-1}\kappa+\alpha, \qquad (3.15)$$

$$\alpha=\int^\kappa k_0'f^2(k)d\mathbf{k}, \qquad (3.16)$$

where

$$k_0'=k_0+(k^2/2\kappa). \qquad (3.17)$$

By inserting the explicit values[11] for $u(p)$ we find

$$\sum_{s,t}\tfrac{1}{2}U(\mathbf{p}_s^{n-1},\mathbf{p}_t^n)$$
$$=i\left[\frac{p_x^{n-1}}{\kappa+p_0^{n-1}}-\frac{p_x^n}{\kappa+p_n^0}\right]\mathcal{G}(p^n)\mathcal{G}(p^{n-1})(k_0^n)^{-\frac{1}{2}}, \qquad (3.18)$$

where

$$\mathcal{G}(p)=[1+p^2/(\kappa+p_0)^2]^{-\frac{1}{2}}. \qquad (3.19)$$

By expanding the integrand of β_n in the same way as was done for α_n and dropping odd terms which will not contribute,

$$\beta_n=f'\int^\kappa(d\mathbf{k})^n ik_x(\kappa+p_0^n)^{-1}(k_0^n)^{-\frac{1}{2}}\mathcal{G}(p^n)$$
$$\times\mathcal{G}(p^{n-1})f(k^n)\sum_r^{n-1}f^2(k^r). \qquad (3.20)$$

Putting $n=1$ (and dropping the suffix on β_1),

$$\beta=f'\int^\kappa\sum_{s,t}U(\mathbf{p}_s^0,\mathbf{p}_t^1)f(k)d\mathbf{k}, \qquad (3.21a)$$

$$=f'\int^\kappa ik_x(\kappa+p_0^1)^{-1}k_0^{-\frac{1}{2}}\mathcal{G}(p^1)f(k)d\mathbf{k},$$

$$=f'\int^\kappa g(k)f(k)d\mathbf{k}, \qquad (3.21)$$

where $g(k)$ is defined by the final equality. Returning to β_n we replace $(p^n)^2$ by $n\bar{k}^2$ (not $n^2\bar{k}^2$), where $\bar{k}$ is defined as the average value of k,

$$\bar{k}^2=\int^\kappa f^2(k)k^2d\mathbf{k}. \qquad (3.22)$$

This is reasonable since the directions of the vectors $\mathbf{k}^r$ are assumed to be at random. The k^n dependence of the integrand is made to be the same as in β by the insertion of the required factors. In the compensating factors k^n is replaced by $\bar{k}$. Then,

$$\beta_n=\beta[\kappa+(\bar{k}^2+\kappa^2)^{\frac{1}{2}}][\kappa+(n\bar{k}^2+\kappa^2)^{\frac{1}{2}}]^{-1}$$
$$\times\mathcal{G}(\bar{k})\mathcal{G}(n^{\frac{1}{2}}\bar{k})\mathcal{G}[(n-1)^{\frac{1}{2}}\bar{k}]$$
$$=\beta[1-(n-1)(\bar{k}^2/2\kappa^2)];\quad n=2,3,\cdots, \qquad (3.23)$$

the last equation following from an expansion in $(\bar{k}^2/\kappa^2)$.

(b) Evaluation of C_n

Let

$$W_0=\sum_n^\infty\{n\alpha c_n^2+2n^{\frac{1}{2}}\beta c_{n-1}c_n\}, \qquad (3.24)$$

and

$$W'=\sum_n^\infty 2n^{\frac{1}{2}}\beta_n'c_{n-1}c_n, \qquad (3.25)$$

[11] W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, London, 1944), Chapter III, Sec. 9.

where

$$\beta_n'=\beta_n-\beta, \tag{3.26}$$

so that

$$W-\kappa=W_0+W'. \tag{3.27}$$

The significance of this splitting of W is that α and β are independent of n. This allows us to employ methods previously applied to the no recoil approximation to derive a complete set of exact solutions for the minimizing of W_0. W' will then be treated as a perturbation using this set of solutions as a basis for the expansion.

The minimal equations for W_0 subject to the condition (3.9) are

$$(n\alpha-E)c_n+\beta(n^{\frac{1}{2}}c_{n-1}+(n+1)^{\frac{1}{2}}c_{n+1})=0, \tag{3.28}$$

where E is the eigenvalue of the energy. This can be written

$$(n-\epsilon)c_n+b(n^{\frac{1}{2}}c_{n-1}+(n+1)^{\frac{1}{2}}c_{n+1})=0, \tag{3.29}$$

where

$$\epsilon=E/\alpha \quad\text{and}\quad b=\beta/\alpha. \tag{3.30}$$

We now introduce the notation

$$c_n=(n|c), \tag{3.31}$$

where the bras $(n$ can be interpreted as the eigenbras of a family of bosons with only one available energy level (or, equivalently, to the eigenbras of a single harmonic oscillator). Now,

$$c)=\textstyle\sum_n n)(n|c). \tag{3.32}$$

so that, if q and q^* are the annihilation and creation operators associated with $n)$, the Eq. (3.29) can be written

$$|q^*q+b(q+q^*)|c)=\epsilon c). \tag{3.33}$$

A complete set of solutions to this equation has been given[12] by Glauber and Luttinger, namely,

$$\epsilon^m=m-b^2, \tag{3.34}$$

$$(n|c^m)=\exp[-\tfrac{1}{2}b^2]\sum_{l=0}^{m}\frac{(m!)^{\frac{1}{2}}(n!)^{\frac{1}{2}}(-1)^{n-m+l}}{l!(m-l)!(n-m+l)!}b^{n-m+l}. \tag{3.35}$$

The solution belonging to the lowest level is thus

$$(n|c^0)=\exp[-\tfrac{1}{2}b^2](-b)^n(n!)^{-\frac{1}{2}}, \tag{3.36}$$

with corresponding energy

$$E^0=-\beta^2/\alpha. \tag{3.37}$$

Both $(n|c^0)$ and E^0 are here given as functions of $f(k)$ which has now to be chosen to make E^0 a minimum. This can also be done exactly using Schwarz' inequality[13] and gives

$$f(k)=L^{-\frac{1}{2}}g(k)(k_0')^{-1}, \tag{3.38}$$

where $g(k)$ and k_0' are defined in (3.21) and (3.17) and L is a normalizing factor.

$$L=4\pi\int^{\kappa}f^2(k)k^2dk=4\pi A. \tag{3.39}$$

Substituting for $f(k)$ into (3.17) and (3.21), we obtain from (3.30)

$$b=f''(A/\pi)^{\frac{1}{2}}, \tag{3.40}$$

where

$$f''=(f/2\pi^{\frac{1}{2}}). \tag{3.41}$$

A has been evaluated graphically (with cutoff at κ) and leads to

$$b=f''(2/25). \tag{3.42}$$

The plot of $f(k)$ is given in Fig. 1. $(n|c^0)=c_n{}^0$ can readily be evaluated for particular values of f''. For $f^2/4\pi=10$ we have

$$c_0{}^2=0.95,\quad c_1{}^2=0.05,\quad c_2{}^2=0.00. \tag{3.43}$$

For $f^2/4\pi=40$ we have

$$c_0{}^2=0.79,\quad c_1{}^2=0.18,\quad c_2{}^2=0.02. \tag{3.44}$$

By substituting (3.40), (3.38), (3.36), [with $g(k)$ defined by (3.21a)] into (3.8), it will be found that, apart from the normalizing factor $\exp[-\frac{1}{2}b^2]$, the intermediate coupling gives exactly the same expression for $\sum_t(\psi|p_t,k)$ as small coupling theory. Thus one goes into the other as the coupling becomes small.

---

12 On (3.33) make the unitary transformation

$$c)=\exp[-b(q^*-q)]c').$$

The equation reduces to $|q^*q-b^2-\epsilon|c')=0$. But this is the standard oscillator equation so that we have a set of solutions

$$c')=m),\quad m=1,2,\cdots$$

giving the spectrum for ϵ, $\epsilon^m=m-b^2$. From (5.14) and (5.16)

$$c^m)=\exp[-b(q^*-q)]m).$$

Since $[q^*,q](=1)$ is a c-number, [see R. J. Glauber, Phys. Rev. **84**, 395 (1951)]

$$\exp[-b(q^*-q)]=\exp[-\tfrac{1}{2}b^2]\exp[-bq^*]\exp[bq].$$

After substitution in (5.18) we have the equation for the ket

$$c^m)=\exp[-\tfrac{1}{2}b^2]\sum_{j=0}^{\infty}\sum_{l=0}^{m}(-1)^jb^{j+l}\frac{q^{*j}}{j!}\frac{q^l}{l!}\bigg|m\bigg)$$

$$=\exp[-\tfrac{1}{2}b^2]\sum_{j=0}^{\infty}\sum_{l=0}^{m}(-1)^jb^{j+l}\frac{(m!)^{\frac{1}{2}}(m-l+j)!^{\frac{1}{2}}}{j!l!m-l!}|m-l+j),$$

which follows from repeated use of (1.7) and (1.8). Equation (3.35) is direct consequence of this relation.

13 This is also due to Dr. Glauber and Dr. Luttinger. The lowest energy $E^0=-\beta^2/\alpha$. By (5.1) and (5.2)

$$E^0\sim-\left(\int f(k)g(k)d\mathbf{k}\right)^2\left(\int f^2(k)k_0'd\mathbf{k}\right)^{-1}.$$

Let

$$\bar{f}(k)=k(k_0')^{\frac{1}{2}}f(k),\quad \bar{g}(k)=k(k_0')^{-\frac{1}{2}}g(k).$$

Then

$$E^0\sim-\left(\int\bar{f}(k)\bar{g}(k)dk\right)^2\left(\int\bar{f}^2(k)dk\right)^{-1},$$

which, by Schwarz' inequality, is a minimum if $\bar{f}(k)\sim\bar{g}(k)$. We thus have (3.38).

(c) The Perturbation

With the notation introduced in (3.31), (3.24) can be written as

$$W_0=(c|w|c)=\sum_{n,n'}(c|n)(n|w|n')(n'|c), \tag{3.45}$$

where

$$(n|w|n)=n\alpha, \tag{3.46}$$

$$(n-1|w|n)=(n|w|n-1)=n^{\frac{1}{2}}\beta. \tag{3.47}$$

The operator w has the eigenkets and eigenvalues $c^m)$ and E^m derived in the last section;

$$w|c^m)=E^mc^m). \tag{3.48}$$

Similarly,

$$W'=\sum_{n,n'}(c|n)(n|w'|n')(n'|c), \tag{3.49}$$

where

$$(n-1|w'|n)=(n|w'|n-1)=n^{\frac{1}{2}}\beta_n' \quad (n=2,3,\cdots). \tag{3.50}$$

Suppose that the eigenkets for $w+w'$ are $c^m)+d^m)$ where $d^m)$ is of the same order as w'. Then by general perturbation theory

$$(c^m|d^0)=(c^m|w'|c^0)/(E^m-E^0). \tag{3.51}$$

The modification of $(n|c^0)$ due to w' is thus

$$\begin{aligned}(n|d^0)&=\sum_{m=1}^{\infty}(n|c^m)(c^m|d^0),\\ &=\sum_{m=1}^{\infty}\sum_{l,j}(n|c^m)(c^m|l)(l|w'|j)(j|c^0)(\alpha m)^{-1},\\ &=\sum_{m=1}^{m}\sum_{j=2}^{\infty}-(n|c^m)(c^m|j)(j-1|c^0)j^{\frac{1}{2}}\\ &\qquad\times(j-1)(\bar{k}^2/2\kappa^2)(\beta/\alpha m).\end{aligned} \tag{3.52}$$

The final equality follows from (3.23). The only term which is nonzero is $j=2$, since $(j-1|c^0)$ is negligible for $j\geqslant 3$. Thus

$$(n|d^0)=\sum_{m=1}^{\infty}-(n|c^m)(c^m|2)(1|c^0)(\bar{k}^2/2\kappa^2)(\beta/\alpha m). \tag{3.53}$$

Now, by (5.21),

$$(n|c^m)=(-1)^{n-m}(m|c^n), \tag{3.54}$$

so that the largest component of $d^0)$ is

$$(0|d^0)=-(0|c^1)(c^1|2)(1|c^0)(\bar{k}^2/2\kappa^2)(\beta/\alpha). \tag{3.55}$$

This is readily evaluated and leads to a negligible reduction (less than 0.01 for $f^2/4\pi=10$) in the probability for the single bare nucleon state.

4. ONE PAIR APPROXIMATION

We now attempt to solve the problem under the less restrictive assumption that one nucleon pair may be present in the cloud, but that the representatives which

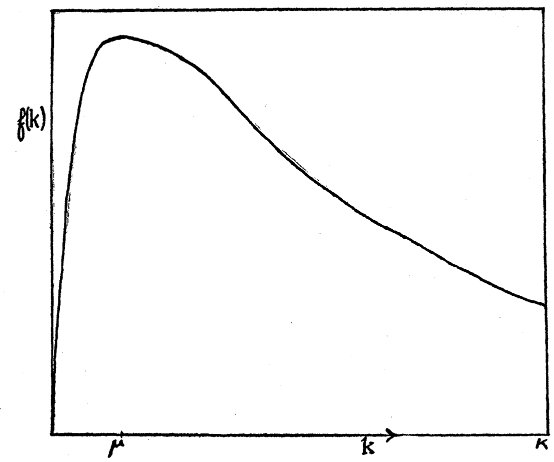

FIG. 1. Plot of $f(k)$.

involve more than one pair are negligible. (The energy contained in the rest mass alone of these neglected components is so large that it is unlikely that they are appreciable, unless the average energy of the mesons in the cloud is comparable with that of a pair.) The typical representatives are

$$(\psi|\mathbf{p}_t^n, \mathbf{k}^1\cdots\mathbf{k}^n)=(\psi|1,n), \tag{4.1}$$

and

$$(\psi|\mathbf{p}_s', \mathbf{p}_t''; \mathbf{q}_r^n; \mathbf{k}^1\cdots\mathbf{k}^n)=(\psi|3,n), \tag{4.2}$$

where

$$\mathbf{q}^n=-(\mathbf{p}'+\mathbf{p}''+\textstyle\sum^n\mathbf{k}^n). \tag{4.3}$$

The representatives satisfy the equations

$$(\psi|H-E|3,n)=0, \tag{4.4}$$

and the energy is given by[14]

$$\begin{aligned}(\psi|H|\psi)&=(\psi|H|1,n)(1,n|\psi)+(\psi|H|3,n)(3,n|\psi)\\ &=(\psi|H|1,n)(1,n|\psi)+E(\psi|3,n)^2,\end{aligned} \tag{4.5}$$

where

$$H=H_0+H_1.$$

Also

$$1=(\psi|\psi)=(\psi|1,n)^2+(\psi|3,n)^2. \tag{4.6}$$

Now

$$\delta((\psi|H|\psi)-E(\psi|\psi))=0. \tag{4.7}$$

Substituting from (4.5) and (4.6),

$$\delta X=\delta(Y-E(\psi|1,n)^2)=0, \tag{4.8}$$

where (4.8) defines X and

$$Y=(\psi|H|1,n)(1,n|\psi). \tag{4.9}$$

We will minimize X, subject to the Tomonaga approximation for $(\psi|1,n)$. Note that Y depends on $(\psi|3,n)$ through the operation of H_1 on $|1,n)$. This can be determined in terms of $(\psi|1,n)$ through the Schrödinger

[14] $1,n)$ $(1,n$ implies summation over all such states. $1,n)$ $(1,n+3,n)$ $(3,n$ is the unit operator, if representatives with more than one pair can be neglected.

Eq. (4.4) provided we approximate by taking only large components (that is, for pseudoscalar interaction, the pair creation or annihilation parts) of the interaction. This equation, derived just like (3.3) with the help of (1.11–13) is

$$(p_0'+p_0''+q_0^n+\sum^n k_0^n-E)(\psi|\mathbf{p}_s',\mathbf{p}_t'';\mathbf{q}_v^n;\mathbf{k}^1\cdots\mathbf{k}^n)$$
$$=-f'\Big[(n+1)^{\frac{1}{2}}2^{-\frac{1}{2}}\int\{V^*(\mathbf{p}_s',\mathbf{q}_v^n,\mathbf{k}^{n+1})$$
$$\times(\psi|\mathbf{p}_t'',\mathbf{k}^1\cdots\mathbf{k}^{n+1})\delta(\mathbf{p}''-\mathbf{p}^{n+1})$$
$$-V^*(\mathbf{p}_t'',\mathbf{q}_v^n,\mathbf{k}^{n+1})(\psi|\mathbf{p}_s',\mathbf{k}^1\cdots\mathbf{k}^{n+1})$$
$$\times\delta(\mathbf{p}'-\mathbf{p}^{n+1})\}d\mathbf{k}^{n+1}+(2n)^{-\frac{1}{2}}$$
$$\times\sum_r^n\{V^*(\mathbf{p}_s',\mathbf{q}_v^n,\mathbf{k}^r)(\psi|\mathbf{p}_t'',\mathbf{k}^1\cdots\mathbf{k}^{r-1},\mathbf{k}^{r+1}\cdots k^n)$$
$$\times\delta(\mathbf{p}''+\sum_{r'\neq r}^n\mathbf{k}^{r'})-V^*(\mathbf{p}_t'',\mathbf{q}_v^n,\mathbf{k}^r)$$
$$\times(\psi|\mathbf{p}_s',\mathbf{k}^1\cdots\mathbf{k}^{r-1},\mathbf{k}^{r+1}\cdots\mathbf{k}^n)$$
$$\times\delta\Big(\mathbf{p}'+\sum_{r'\neq r}^n\mathbf{k}^{r'}\Big)\Big\}\Big],\quad(4.10)$$

where

$$V(\mathbf{p}_s,\mathbf{q}_v,\mathbf{k})=iu_t^*(p)\beta\gamma_5u_v(\mathbf{q})k_0^{-\frac{1}{2}}.\quad(4.11)$$

[$u(q)$ is a negative energy state]. This expression for the representative (wave function) $(\psi|\mathbf{p}',\mathbf{p}'';\mathbf{q}^n;\mathbf{k}'\cdots\mathbf{k}^n)$ is antisymmetrical in $\mathbf{p}'$, $\mathbf{p}''$, as it should be for Fermi statistics.

Y can also be evaluated with the use of the equations of Sec. 1. We have

$$Y=\sum_n\int(d\mathbf{k})^n\sum_t\Big\{(p_0^n+nk_0^n)(\psi|\mathbf{p}_t^n,\mathbf{k}^1\cdots\mathbf{k}^n)^2$$
$$+f'\Big[\sum_s n^{\frac{1}{2}}U(\mathbf{p}_s^{n-1},\mathbf{p}_t^n)(\psi|\mathbf{p}_s^{n-1},\mathbf{k}^1\cdots\mathbf{k}^{n-1})$$
$$\times(\mathbf{p}_t^n,\mathbf{k}^1\cdots\mathbf{k}^n|\psi)+\text{c.c.}+2^{\frac{1}{2}}(n+1)^{\frac{1}{2}}$$
$$+\int d\mathbf{p}'d\mathbf{k}^{n+1}\sum_{s,v}V(\mathbf{p}_s',\mathbf{q}_v''^{n+1},\mathbf{k}^{n+1})\quad(4.12)$$
$$\times(\psi|\mathbf{p}_s',\mathbf{p}_t^n;\mathbf{q}_v''^{n+1};\mathbf{k}'\cdots\mathbf{k}^{n+1})(\mathbf{p}_t^n,\mathbf{k}^1\cdots\mathbf{k}^n|\psi)$$
$$+2^{\frac{1}{2}}n^{\frac{1}{2}}\int d\mathbf{p}'\sum_{s,v}V(\mathbf{p}_s',\mathbf{q}_v'^{n-1},\mathbf{k}^n)$$
$$\times(\psi|\mathbf{p}_s',\mathbf{p}_t^n;\mathbf{q}'^{n-1};\mathbf{k}^1\cdots\mathbf{k}^{n-1})(\mathbf{p}_t^n,\mathbf{k}^1\cdots\mathbf{k}^n|\psi)\Big]\Big\},$$

where

$$\mathbf{q}''^n=\mathbf{q}^n(\mathbf{p}''=\mathbf{p}^{n-1})=-(\mathbf{p}'+\mathbf{k}^n),\quad(4.13)$$
$$\mathbf{q}'^n=\mathbf{q}^n(\mathbf{p}''=\mathbf{p}^{n-1})=-(\mathbf{p}'-\mathbf{k}^{n+1}).\quad(4.14)$$

The four terms of (4.10) are referred to as (*a*), (*b*), (*c*), and (*d*). These are substituted into the fourth term of (4.12) to give A, B, C, and D, and into the third term to give A^1, B^1, C^1, and D^1.

When these substitutions are made the terms appear consistently in a certain pattern. Some terms (D, A, C^1, and A^1) come from unrestricted pair creation, as though no other Fermi particles were present. The other terms D, B, D^1, and B^1 represent the restrictions on these terms, because of the presence of a Fermi particle, through the operation of the Pauli principle. As is done in perturbation theory (see (2.12) and following discussion), we treat the first type of term as a vacuum effect and consider only the second (or restricting) terms as genuine contributions to the energy of the free particle under consideration. In this way one obtains an expression for Y, similar to that for W, with the addition of an extra term coming from D and B^1, which represent either the annihilation or creation of two mesons in the cloud. These two terms are the complex conjugates of each other. There are also the terms D^1 and B, which modify α, the coefficient of c_n^2. Of these, D^1 corresponds to the positron part of the nucleon self energy in which a meson is created and annihilated again with virtual pair creation; B is the absorption of one meson and the creation of another, with, again, pair creation in the intermediate state. These two terms are small compared with α and will not be considered further. We thus have, finally, the expression,

$$Y=\sum_n[n\alpha_nc_n^2+2n^{\frac{1}{2}}\beta_nc_{n-1}c_n+2n^{\frac{1}{2}}(n-1)^{\frac{1}{2}}\gamma_nc_{n-2}c_n],\quad(4.15)$$

where α_n and β_n are the same as in the no-pair expression, (3.11 and 3.12), and γ_n, after putting in the Tomonaga assumption for $(\psi|p^n,k^1\cdots k^n)$, (3.8), is

$$\gamma_n=-(f')^22^{-1}\int^\kappa(d\mathbf{k})^n\sum_{v,s,t}V(\mathbf{p}_s^{n-2},\mathbf{q}_v,\mathbf{k}^n)$$
$$\times V^*(\mathbf{p}_t^n,\mathbf{q}_v,\mathbf{k}^{n-1})(p_0^{n-2}+p_0^n+q_0$$
$$+\sum^{n-1}k_0^r-E)f(k^{n-1})f(k^n)\prod_r^{n-2}f^2(k^r),\quad(4.16)$$

where

$$\mathbf{q}=\mathbf{p}^n+\mathbf{k}^n=\mathbf{p}^{n-2}-\mathbf{k}^{n-1}.$$

γ_n can be reduced by the methods applied to β_n to the approximate expression

$$\gamma_n=(f')^24^{-1}\kappa^{-3}\Big[\int^\kappa k_xk_0^{-\frac{1}{2}}f(k)d\mathbf{k}\Big]^2.\quad(4.17)$$

To this approximation γ_n is independent of n and the suffix will be dropped.

We can proceed by the same method as before.

$$Y-\kappa=Y_0+Y',\quad(4.18)$$

where

$$Y_0=\sum_n\{n\alpha c_n{}^2+2n^{\frac{1}{2}}\beta c_{n-1}c_n+2n^{\frac{1}{2}}(n-1)^{\frac{1}{2}}c_nc_{n-2}\}, \quad (4.19)$$

and

$$Y'=\sum_n 2\beta_n' n^{\frac{1}{2}}c_{n-1}c_n. \quad (4.20)$$

The minimal Eq. (4.8) for Y_0 can again be solved exactly, giving a complete spectrum for E^m and the corresponding solutions $c_n{}^m$), which again form the basis for treating Y' as a perturbation. We proceed to the exact solution.

The minimal Eq. (4.8) for Y_0 is

$$(n-\epsilon)c_n+b(n^{\frac{1}{2}}c_{n-1}+(n+1)^{\frac{1}{2}}c_{n+1})+g(n^{\frac{1}{2}}(n-1)^{\frac{1}{2}}c_{n-2} +(n+2)^{\frac{1}{2}}(n+1)^{\frac{1}{2}}c_{n+2})=0, \quad (4.21)$$

where

$$\epsilon=E/\alpha, \quad b=\beta/\alpha, \quad g=\gamma/\alpha. \quad (4.22)$$

Introducing the notation of (3.31) this can be written

$$|q^*q-\epsilon+b(q+q^*)+g(qq+q^*q^*)|c)=0. \quad (4.23)$$

Instead of solving this equation in operator form, we use the more general method of transforming to the Schrödinger representation of the q-operators.[15]

Multiplying (4.23) by $(x$, we have

$$q^*=-i2^{-\frac{1}{2}}(d/dx-x), \quad q=-i2^{-\frac{1}{2}}(d/dx+x), \quad (4.24)$$

and the equation becomes

$$\{\tfrac{1}{2}[(1+2g)d^2/dx^2-x^2(1-2g)+1] +i2^{\frac{1}{2}}bd/dx+\epsilon\}(x|c)=0. \quad (4.25)$$

To eliminate the term in d/dx, make the transformation

$$(x|c)=\exp[-i2^{\frac{1}{2}}\delta](x|c'), \quad (4.26)$$

where

$$\delta=b(1+2g)^{-1}, \quad (4.27)$$

then

$$\{(1+2g)(d^2/dx^2-2\epsilon)+r^2(1-2g)+1+2\delta^2\}(x|c')=0. \quad (4.28)$$

Define r, y, and ϕ) by the equations,

$$r^2=(1-2g)(1+2g)^{-1}, \quad (4.29)$$

and

$$r^{\frac{1}{2}}x=y, \quad (4.30)$$

$$(x|c')=(y|\phi). \quad (4.31)$$

Then,

$$\{d^2/dy^2+\lambda-y^2\}(y|\phi)=0, \quad (4.32)$$

where

$$\lambda=(2m+1)=(1-4g^2)^{-\frac{1}{2}}[1+2\delta^2(1+2g)+2\epsilon]. \quad (4.33)$$

But (4.32) is the representative form of the standard oscillator equation.[16] This has solutions of the form

$$(y|\phi^m)=H_m{}^*(y)\exp[-\tfrac{1}{2}y^2], \quad (4.34)$$

where $H_m{}^*$ is the Hermite polynomials, satisfying

$$H_m{}^{*\prime\prime}-2yH_m{}^{*\prime}+2mH_m{}^*=0. \quad (4.35)$$

Thus, by (4.26) and (4.30)

$$(x|c^m)=H_m{}^*(r^{\frac{1}{2}}x)\exp[-\tfrac{1}{2}rx^2-i2^{\frac{1}{2}}\delta x]. \quad (4.36)$$

The components

$$(n|c^m)=\int_{-\infty}^{\infty}(n|x)(x|c^m)dx, \quad (4.37)$$

where $(n|x)$ are the solutions of the standard oscillator equation, (4.32), with m replaced by n. (We are here repeating in representative form the argument of reference 12.) Thus

$$(n|c^m)=K^m\int_{-\infty}^{\infty}H_n{}^*(x)H_m{}^*(r^{\frac{1}{2}}x) \times\exp[-\tfrac{1}{2}(1+r)x^2-i\delta 2^{\frac{1}{2}}x]dx, \quad (4.38)$$

where K^m is a normalizing factor. This is a complete set of exact solutions to the Eq. (4.32).

The corresponding spectrum E^m is given by (4.33). We are primarily interested in the lowest level

$$E^0=-\beta^2/[\alpha(1+2g)]-\tfrac{1}{2}\alpha[1-(1-4g^2)^{\frac{1}{2}}]. \quad (4.39)$$

This is a function of $f(k)$ and $f(k)$ must be chosen to make a minimum. Expanding (4.39) in powers of g,

$$E^0=-(\beta^2/\alpha)(1-2g)+O(g^2). \quad (4.40)$$

If g is small we can take $f(k)$ to be the same as in the no-pair approximation, (3.38).

The values of $(n|c^0)$ or $c_n{}^0$ are determined by

$$c_n{}^0=K\int_{-\infty}^{\infty}H_n{}^*(x)\exp[-\tfrac{1}{2}(1+r)x^2-i\delta 2^{\frac{1}{2}}x]dx, \quad (4.41)$$

which can be evaluated exactly by substituting the explicit expressions for $H_n{}^*(x)$,

$$H_0{}^*(x)=1, \qquad H_1{}^*(x)=2x,$$
$$H_2{}^*(x)=4x^2-2, \qquad H_3{}^*(x)=8x^3-12x. \quad (4.42)$$

One thus finds after some elementary integration

$$c_0{}^0=K,$$

$$c_1{}^0=-K\left(\frac{2\delta}{1+r}\right),$$

$$c_2{}^0=K\left(\frac{1}{2}\right)^{\frac{1}{2}}\left[\left(\frac{2\delta}{1+r}\right)^2+\frac{r-1}{r+1}\right]$$

$$c_3{}^0=K\left(\frac{1}{3}\right)^{\frac{1}{2}}\left[\left(\frac{2\delta}{1+r}\right)^3+\frac{3\delta(r-1)}{(r+1)^2}\right] \quad (4.43)$$

where K is determined by the normalization of the state function.

[15] P. A. M. Dirac, (see reference 7) Sec. 34 and 60. Our q is Dirac's $\bar{\eta}$.

[16] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), Chapter IV.

Since we are using the same form for $f(k)$, α and β are the same functions as before. The values of α and γ are obtained by substituting for $f(k)$ in (3.16) and (8.1) and by graphical integration. Then

$$g=(f^2/4\pi)(1/350). \tag{4.44}$$

$c_n{}^0$ can now be evaluated for different values of f. To determine K we must satisfy the equation

$$(\psi|1,n)^2+(\psi|3,n)^2=1,$$

or

$$\sum_n^\infty\bigg[(c_n{}^0)^2+\sum_{a,b,c}\int d\mathbf{p}'d\mathbf{p}''(d\mathbf{k})^n \times(\psi|\mathbf{p}_a{}',\mathbf{p}_b{}'';\mathbf{q}_c{}^n;\mathbf{k}^1,\cdots,\mathbf{k}^n)\bigg]=1. \tag{4.45}$$

The second term of the left-hand side has to be evaluated using (4.10) with the Tomonaga assumption for $(\psi|1,n)$ and $f(k)$ given by (3.38). In general this is a calculation very similar to that for Y which involves the separation of vacuum terms. For the values of f we are considering only $(\psi|3,1)^2$ is significant, which reduces to just the small coupling expression (2.13) multiplied by $(c_0{}^0)^2=K^2$. Expanding in (k/κ)

$$(\psi|3,1)^2=d_1{}^2\text{ (say)}=K^2(f')^2\int^\kappa(1-k^2/4\kappa^2)k_0{}^{-1} \times(2\kappa+k^2/2\kappa+k_0)^{-2}d\mathbf{k}. \tag{4.46}$$

This has been evaluated graphically, giving

$$d_1{}^2=K^2(f^2/4\pi)(1/60). \tag{4.47}$$

Combining this with the values for $c_n{}^0$ for particular values of f, K can be chosen to satisfy (4.45). The results are, for $f^2/4\pi=10$,

$$c_0{}^2=0.83,\quad c_1{}^2=0.04,\quad d_1{}^2=0.13. \tag{4.48}$$

For $f^2/4\pi=40$,

$$c_0{}^2=0.57,\quad c_1{}^2=0.05,\quad d_1{}^2=0.38. \tag{4.49}$$

These are exact solutions for the expression Y_0 for the lowest energy level. The expression (4.38) for the complete set of solutions for Y_0 could now be used to treat Y' as a perturbation, as was done in Sec. 3(c). These are negligible to the degree of accuracy to which we are working and will not be treated here.

Using (4.40) one can also evaluate E^0. The result is

$$E^0=-(f^2/4\pi)(1/500). \tag{4.50}$$

Thus for $f^2/4\pi=10$,

$$E^0=-(\kappa/50). \tag{4.51}$$

5. DISCUSSION

The approximations made in the above calculations are certainly crude. (To the order of magnitude considered electromagnetic effects are entirely negligible.) However, if the reasonableness of the cutoff at κ is accepted provisionally, they are at least self consistent. The technique would have to be considerably refined before it could be made the basis for a reliable calculation (of the nuclear magnetic moments for instance), but even in its present form there are certain interesting indications.

One striking feature is the way in which the representatives get smaller with increasing numbers of mesons, and the fact that for a coupling constant $(f^2/4\pi)$ as large as ten, the representative $(\psi|\mathbf{p})^2$ is still the largest by a factor of six. Even for a coupling constant of forty, the representatives fall off in the same order, but not nearly as rapidly as is assumed in small coupling theory. A quite different result is obtained for scalar mesons, where already for a coupling constant of two the free nucleon is just as likely to be in the bare nucleon plus bare meson state as in the single bare nucleon state, and the small coupling approximation breaks down altogether.[4] This is because the significant parameter in intermediate coupling theory (and to a rough approximation in weak coupling theory also) is not f^2 but f^2A [see (3.40)], and A is particularly small for pseudoscalar mesons. (It is this which leads one to consider large values of the coupling constant in the first place!) In fact the only difference between the values (4.49) quoted for $f^2/4\pi=10$ and those given by weak coupling theory lies in the wave function renormalization, which is treated as negligible to lowest order in a genuine expansion in f. This is certainly not correct and is particularly important in magnetic moment calculations, where a large part of the effect comes from the $(\psi|p)$ term. (The concept of *anomalous* magnetic moments ceases to have much meaning as soon as the normalization factor is appreciably different from one.)

The rate at which successive terms in the weak coupling expansion will fall off is very roughly indicated by b. For $f^2/4\pi=10$ we have $b=\frac{1}{4}$. Judging by particular calculations which have been made,[17] this is a very optimistic estimate and it is more probable that successive terms in the S-matrix will diminish in the ratio of about one-half. The great advantage of a renormalized weak coupling theory is that it deals exactly with the fundamental problems of divergences, virtual pair creation, and the separation of vacuum effects. The calculations presented here indicate that, although the weak coupling method yields a very slow convergence for the size of coupling which is required, the implicit assumption of the relative importance of the different

[17] E. Corinaldesi and G. Field, Phil. Mag. **40**, 1159 (1949); K. A. Brueckner, Phys. Rev. **79**, 641 (1950); K. Nakabayasi and I. Sato, Prog. Theoret. Phys. **6**, 252 (1951). These general remarks, based as they are on the consideration of the self-energy problem only, should not be taken too seriously. It may well be that the weak coupling expansion for a particular S-matrix element starts to diverge at about the third term. If this is so, the only line of progress seems to lie in a refinement of some intermediate coupling theory.

representatives is correct, and that the first two terms of the expansion should give correctly the general features of the theory. More reliance could be placed on an equation such as that of Bethe and Salpeter[18] for the deuteron in which the important terms are iterated to all orders in the coupling constant.

All the quantitative conclusions given above depend on the cutoff at κ. This was inserted because it was felt that the problem of the divergences was solved in principal for pseudoscalar coupling by renormalization, which would, in general, cut down the integrals at about this value. The qualitative conclusions just reached are much less sensitive to the cutoff, since to alter them the estimate of $(\psi|3,1)^2$ would have to be wrong by a factor of three, which we consider unlikely.

A mathematically rigorous method for incorporating renormalization into such a calculation has been given by Dyson.[19] In his theory a modified Schrödinger equation is obtained in the Schrödinger representation [IV, Eq. (99)] in which the Hamiltonian itself is an expansion in the charge. Each term in this expansion is finite after renormalization and also contains a convergence factor [IV, Eq. (102)] which may be made powerful enough to render finite any integral in which it occurs (for example the expressions for α, β, and γ above). The idea is (III) that only the first few terms in the Hamiltonian are important. The point at which the convergence factors cut down the integrands depends on certain arbitrary constants, Γ. The great advantage of Dyson's theory over a simple cutoff is that the convergence factors are introduced by a unitary transformation. Thus, if one approximates by considering only a certain number of terms in the Hamiltonian, it is always possible to write down an expression for what has been neglected, and, in principle at least, the validity of the approximation for particular values of Γ can be tested.

The other main source of error in the above calculations is in the reduction of the multiple integrals for α_n, β_n, and γ_n, [Sec. 3(a) and (4.15)]. These were complicated by the fact that any number of mesons was allowed for. On the basis of the above conclusion, that only representatives with a limited number of mesons are significant, a more accurate evaluation should be possible. If only two representatives were considered the method would reduce to that suggested by Tamm and Dancoff for the deuteron.[20] Refinements along these and other lines are being considered.

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Note added in proof:—The calculations of Koba, Kotani, and Nakai [Prog. Theoret. Phys. 6, 849 (1951)], which have appeared recently, of the higher order c corrections to photomeson production, indicate that the optimistic view of renormalized perturbation theory for nuclear phenomena taken above is not justified, but the assumption in reference 17 is much more likely to be correct.

APPENDIX

The Charged No-Pair Model

If we consider a charged meson field, then the Hamiltonian of the Schrödinger equation is the same as (2.3) except that the factor $(q^*(\mathbf{k})+q(\mathbf{k}))$ is replaced by

$$(q_+(\mathbf{k})+q_-^*(\mathbf{k}))\tau+(q_+^*(\mathbf{k})+q_-(\mathbf{k}))\tau^*, \qquad \text{(A.1)}$$

where the suffixes + and − denote operators associated with positive and negative mesons, respectively. If P and N denote protons and neutrons and P_A and N_A antiprotons and antineutrons, then

$$\begin{array}{llll} \tau N=P, & \tau P=0; & \tau^* N=0 & \tau^* P=N; \\ \tau P_A=N_A, & \tau N_A=0; & \tau^* P_A=0 & \tau^* N_A=P_A; \end{array} \qquad \text{(A.2)}$$

which simply states that τ creates charge and τ^* annihilates.

If we consider a state which represents a single free stationary proton in the no-pair approximation, the various representatives all have zero momentum and unit positive charge, and are thus of the typical forms, (i),

$$(\psi|\mathbf{p}_t^{2n};\mathbf{l}^1,\cdots,\mathbf{l}^n;\mathbf{m}^1,\cdots,\mathbf{m}^n)=(\psi|1,n,n), \qquad \text{(A.3)}$$

where l and m refer to positive and negative mesons, respectively, and

$$\mathbf{p}^{2n}=-\left(\sum_r^n \mathbf{l}^r+\sum_s^n \mathbf{m}^s\right) \qquad \text{(A.4)}$$

represents a bare proton; or (ii),

$$(\psi|\mathbf{p}_t^{2n+1};\mathbf{l}^1,\cdots,\mathbf{l}^{n+1};\mathbf{m}^1,\cdots,\mathbf{m}^n) =(\psi|1,n+1,n), \qquad \text{(A.5)}$$

where $\mathbf{p}_t^{2n+1}$ is a neutron.

[18] H. A. Bethe and E. E. Salpeter, Phys. Rev. 82, 309 (1951); F. Low and M. Gell-Mann, Phys. Rev. 84, 350 (1951).

[19] F. J. Dyson, Phys. Rev. 82, 428 (1951); 83, 608 (1951); Proc. Roy. Soc. (London) A207, 395 (1951); Phys. Rev. 83, 1207 (1951); to be referred to as I, II, III, and IV, respectively. See in particular the first and last sections of IV for the modified Schrödinger equation. In I methods are developed for obtaining the terms of the Hamiltonian from generalized Feynman graphs. In II the machinery for renormalizing such a term is set up.

[20] I. G. Tamm, J. Phys. (U.S.S.R.) 9, 449 (1945); S. M. Dancoff, Phys. Rev. 78, 382 (1950).

The energy of the state ψ is

$$W=\sum_n^\infty\int(d\mathbf{l})^n(d\mathbf{m})^n\sum_t\bigg\{(\psi|H|1,n,n)(1,n,n|\psi)+\int d\mathbf{l}^{n+1}(\psi|H|1,n+1,n)(1,n+1,n|\psi)\bigg\}. \quad (A.6)$$

This can be evaluated just as in Sec. 3, and we make the Tomonaga approximation,

$$(\psi|1,n,n)=2^{-\frac{1}{2}}c_{2n}\prod_r^n f_+(l^r)\prod_s^n f_-(m^s). \quad (A.7)$$

We thus obtain

$$W=\sum_n^\infty\int(d\mathbf{l})^n(d\mathbf{m})^n\bigg\{(p_0^{2n}+nl_0^n+nm_0^n)c_{2n}^2\times\prod_r^n f_+^2(l^r)\prod_s^n f_-^2(m^s)+\int(p_0^{2n+1}+(n+1)l_0^n+nm_0^n)c_{2n+1}^2\prod_r^{n+1}f_+^2(l^r)\prod_s^n f_-^2(m^s)d\mathbf{l}^{n+1}$$
$$+f'2^{-1}\sum_{s,t}\bigg[n^{\frac{1}{2}}U(p_s^{2n+1},p_t^{2n})c_{2n-1}{}^*c_{2n}\times\prod_r^n f_+^2(l^r)\prod_s^{n-1}f_-^2(m^s)f_-(m^n)+\text{c.c.}$$
$$+(n+1)^{\frac{1}{2}}\int\bigg(U(p_s^{2n},p_t^{2n+1})c_{2n}c_{2n+1}{}^*f_+(l^{n+1})\times\prod_r^n f_+^2(l^r)\prod_s^n f_-^2(m^s)+\text{c.c.}\bigg)d\mathbf{l}^{n+1}\bigg]\bigg\}. \quad (A.8)$$

This expression can be considerably simplified if we introduce the notation

$$\mathbf{l}^n=\mathbf{k}^{2n-1},\quad \mathbf{m}^n=\mathbf{k}^{2n}, \quad (A.9)$$

in which case

$$W=\sum_n[n\alpha_n c_n^2+2n^{\frac{1}{2}}(\beta_{2n-1}c_{2n-2}c_{2n-1}+\beta_{2n}c_{2n-1}c_{2n})], \quad (A.10)$$

where

$$\alpha_n=n^{-1}p_0^n+k_0^n, \quad (A.11)$$

$$\beta_n=f'2^{-1}\sum_{s,t}\int(dk)^nU(\mathbf{p}_s^{n-1},\mathbf{p}_t^n)f(k^n)\prod_r^{n-1}f^2(k^r), \quad (A.12)$$

where $f(k^r)$ is $f_+(k)$ or $f_-(k)$ according to whether r is odd or even. If we make the further assumption that

$$f_+(k)=f_-(k)=f(k), \quad (A.13)$$

thus neglecting the correlation due to charge,[21] we can write

$$W=\sum_n[nc_n^2\alpha_n+2\{(n-\tfrac{1}{2})^{\frac{1}{2}}c_{2n-2}c_{2n-1}\beta_{2n-1}+n^{\frac{1}{2}}c_{2n-1}c_{2n}\beta_{2n}\}+2(n^{\frac{1}{2}}-(n-\tfrac{1}{2})^{\frac{1}{2}})\times c_{2n-2}c_{2n-1}\beta_{2n-1}]$$
$$=\sum_n[n\alpha_n c_n^2+2n^{\frac{1}{2}}c_{n-1}c_n(\beta_n2^{-\frac{1}{2}})+2(n^{\frac{1}{2}}-(n-\tfrac{1}{2})^{\frac{1}{2}})c_{2n-2}c_{2n-1}\beta_{2n-1}]. \quad (A.14)$$

The final expression is obtained by summing over $2n=n'$ in the bracket $\{\cdots\}$, in which case the two terms can be combined into a single sum over all values of n'. We can obtain an exact solution for the n independent parts of the first two terms and treat the remainder as a perturbation.

[21] Investigations of R. H. Dalitz and D. G. Ravenhall show that this is not a bad approximation. Phil. Mag. 42, 1378 (1951).