First-Forbidden Beta-Decay Matrix Elements*

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The general theory of first-forbidden beta-transitions involves seven nuclear matrix elements in the nonrelativistic approximation. In principle the number of independent parameters can be reduced to four with the help of the relation

$$(W_f-W_i)(f|X|i)=(f|[H_0,X]|i)+(f|[H_c,X]|i)+(f|[H_\nu,X]|i).$$

Here W is an energy eigenvalue, X a coordinate type first-forbidden operator, H_0 the free particle Hamiltonian, H_c the Coulomb interaction, and H_ν the specifically nuclear interaction. The commutators of H_0 and H_c with X are easily evaluated, but $[H_\nu, X]$ presents difficulties. However, the matrix elements of $[H_\nu, X]$ can be estimated by general physical arguments based on the semi-empirical energy surface and the validity of shell model considerations. The explicit form of H_ν is not required a fortunate circumstance since H_ν for complex nuclei is at present essentially unknown. These calculations determine the common proportionality factor Λ in the relations

$$\tfrac{1}{2}\Lambda\alpha Z\int \mathbf{r}/R=-i\int \boldsymbol{\alpha},$$
$$\tfrac{1}{2}\Lambda\alpha Z\int \boldsymbol{\sigma}\cdot\mathbf{r}/R=-i\int \gamma_5,$$
$$\tfrac{1}{2}\Lambda\alpha Z\int \boldsymbol{\sigma}\times\mathbf{r}/R=-\int \beta\boldsymbol{\alpha}.$$

1. INTRODUCTION

THE matrix elements[1]

$$\int \boldsymbol{\alpha},\quad \int \beta\boldsymbol{\alpha},\quad \int \gamma_5, \tag{1a}$$

$$\tfrac{1}{2}\alpha Z\int \mathbf{r}/R,\quad \tfrac{1}{2}\alpha Z\int \boldsymbol{\sigma}\cdot\mathbf{r}/R,\quad \tfrac{1}{2}\alpha Z\int \boldsymbol{\sigma}\times\mathbf{r}/R \tag{1b}$$

occur in the theory of first-forbidden transitions characterized by the selection rule $\Delta I=0, \pm 1$ (yes).[2–4] Nonrelativistic approximations to the first three matrix elements and the association with the different covariant formulations of the beta-decay theory are shown in Table I. The designations momentum type (1a) and coordinate type (1b) are convenient to distinguish the matrix elements in the two lines of Eq. (1).

Upper limits on the squares of the matrix elements can be derived from the completeness theorem. We use the estimates

$$\langle p^2\rangle_{\text{Av}}/2M\sim 20 \text{ Mev} \tag{2}$$

for the average value of the kinetic energy of the particle making the transition, and

$$\langle r^2\rangle_{\text{Av}}\sim 0.6R^2 \tag{3}$$

for the corresponding average square of the radial distance. Equation (2) is based on the single particle picture of a nucleon having binding energy of 6–8 Mev in a potential well of depth 25–30 Mev. Equation (3) follows from the assumption of uniform particle density. A small correction for a possible nonuniform particle density is unimportant in the present context. Results for the upper limits are collected in Table II.

TABLE I. Properties of momentum type matrix elements.

Covariant formulation	Matrix element	Nonrelativistic approximation
Polar vector	$\int\boldsymbol{\alpha}$	$-\int\mathbf{p}/Mc$
Tensor	$\int\beta\boldsymbol{\alpha}$	$-i\int\boldsymbol{\sigma}\times\mathbf{p}/Mc$
Axial vector	$\int\gamma_5$	$-\int\boldsymbol{\sigma}\cdot\mathbf{p}/Mc$

Table II suggests that the momentum type matrix elements are dominant for small Z, while leaving open the question of which type is most important for the very largest values of Z.

It must be admitted, however, that these upper limits are too high to support strong conclusions on the relative importance of momentum and coordinate type matrix elements. Allowed favored transitions, for which $\sum_{m_f}|M|^2\sim 1$, have $ft\sim 3\times 10^3$; first-forbidden transitions with $ft\sim 10^{6-7}$ have $\sum_{m_f}|M|^2\sim 10^{-3}$. Thus a factor of about 50 separates the upper limits of Table II from the mean experimental values.

2. THEORETICAL RELATIONS BETWEEN COORDINATE AND MOMENTUM TYPE MATRIX ELEMENTS

Information on the ratios of momentum to coordinate type matrix elements is needed for an unambiguous interpretation of the experimental material. These ratios are accessible to theoretical study following a procedure suggested by a remark in reference 2. The analysis starts from the nuclear Hamiltonian[5]

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[1] α = fine structure constant, Z = atomic number of the product nucleus, R = nuclear radius with the approximate value $(e^2/2mc^2)A^{\frac{1}{3}}$. The first-forbidden pseudoscalar matrix element is not included in the present discussion. The usual statement that it is $\int\beta\gamma_5$ is misleading because this matrix element is extremely small, of the order $(p/Mc)^3$, in the nonrelativistic approximation, if the lepton covariant $\psi_e^*\beta\gamma_5\varphi_n$ is removed from under the integration.

[2] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).

[3] R. Marshak, Phys. Rev. **61**, 431 (1942).

[4] E. J. Konopinski, Revs. Modern Phys. **15**, 209 (1943).

[5] The isotopic spin formalism [L. Rosenfeld, *Nuclear Forces* (Interscience Publications, New York, 1948), p. 43] is used in the following calculations; i.e., the two charged states of a nucleon are described by a charge variable τ with the values +1 (neutron

$$H=H_0+H_c+H_\nu, \tag{4}$$

$$H_0=-c\sum \boldsymbol{\alpha}_k\cdot \mathbf{p}_k-Mc^2\sum\beta_k, \tag{5}$$

$$H_c=\frac{e^2}{8}\sum_{l\neq k}\frac{1}{r_{lk}}(1-\tau_{kz})(1-\tau_{lz}), \tag{6}$$

H_ν—specifically nuclear interactions.

The equation

$$(W_f-W_i)(f|X|i)=(f|HX-XH|i) \tag{7}$$

relates the matrix elements of $X=\sum X_kQ_k$ and the commutator of H and X.[6]

Explicitly,

$$[H_0,\sum\mathbf{r}_kQ_k]=ic\hbar\sum_k \boldsymbol{\alpha}_kQ_k$$
$$\cong(\hbar/iM)\sum_k \mathbf{p}_kQ_k, \tag{8a}$$

$$[H_0,\sum\boldsymbol{\sigma}_k\cdot\mathbf{r}_kQ_k]=-(c\hbar/i)\sum_k\{3\gamma_{5k}-2i\boldsymbol{\alpha}\cdot(\mathbf{r}\times\nabla)\}Q_k$$
$$\cong(\hbar/iM)\sum_k\boldsymbol{\sigma}_k\cdot\mathbf{p}_kQ_k$$
$$\cong-(c\hbar/i)\sum_k\gamma_{5k}Q_k, \tag{8b}$$

$$[H_0,\sum\boldsymbol{\sigma}_k\times\mathbf{r}_kQ_k]=2c\hbar\sum_k(\boldsymbol{\alpha}_k-\boldsymbol{\alpha}_k\cdot\mathbf{r}_k\nabla_k+\mathbf{r}_k\cdot\nabla_k\boldsymbol{\alpha}_k)Q_k$$
$$\cong(\hbar/iM)\sum_k\boldsymbol{\sigma}_k\times\mathbf{p}_kQ_k$$
$$\cong\hbar c\sum_k\beta_k\boldsymbol{\alpha}_kQ_k, \tag{8c}$$

$$[H_c,\sum\boldsymbol{\sigma}_k\cdot\mathbf{r}_kQ_k]=\tfrac{1}{2}e^2\sum_{l\neq k}\frac{\boldsymbol{\sigma}_l\cdot\mathbf{r}_l}{r_{lk}}(1-\tau_{kz})Q_l, \tag{9a}$$

$$[H_c,\sum\mathbf{r}_kQ_k]=\tfrac{1}{2}e^2\sum_{l\neq k}\frac{\mathbf{r}_l}{r_{lk}}(1-\tau_{kz})Q_l, \tag{9b}$$

$$[H_c,\sum\boldsymbol{\sigma}_k\times\mathbf{r}_kQ_k]=\tfrac{1}{2}e^2\sum_{l\neq k}\frac{\boldsymbol{\sigma}_l\times\mathbf{r}_l}{r_{lk}}(1-\tau_{kz})Q_l. \tag{9c}$$

The expansion theorem

$$(f|[H_c,X]|i)=\sum_{f'}(f|H_c|f')(f'|X|i)-\sum_{i'}(f|X|i')(i'|H_c|i) \tag{10}$$

suggests the approximate relation

$$(f|[H_c,X]|i)\cong[(f|H_c|f)-(i|H_c|i)](f|X|i). \tag{11}$$

state) and -1 (proton state). The functions,

$$\delta_\pm(\tau)=1,\quad \tau=\pm1$$
$$=0,\quad \tau=\mp1,$$

describe definite charge states. These functions are eigenfunctions of the isotopic spin operator τ_z: $\tau_z\delta_\pm(\tau)=\pm\delta_\pm(\tau)$.

The conventional symbol $\int X$ of Eq. (1) denotes the matrix element of the operator

$$\sum_{k=1}^{A}X_kQ_k \quad\text{or}\quad \sum_{k=1}^{A}X_kQ_k^*,$$

in which Q and Q^* are displacement operators defined by

$$Q\delta_+=\delta_-\quad Q^*\delta_-=\delta_+$$
$$Q\delta_-=0\quad Q^*\delta_+=0.$$

[6] The symbols f and i denote final and initial states of the β-decay process while the W's are the corresponding energy eigenvalues.

TABLE II. Upper limits on the squares of the first-forbidden matrix elements.[a]

Covariant formulation	Matrix element M	Upper limit on $\Sigma_{m_f}\lvert M\rvert^2$
Scalar and polar vector	$\frac{1}{2}\alpha Z\int\mathbf{r}/R$	$0.15(\alpha Z)^2$
Polar vector	$\int\boldsymbol{\alpha}$	0.044
Tensor and axial vector	$\frac{1}{2}\alpha Z\int\boldsymbol{\sigma}\cdot\mathbf{r}/R$	$0.15(\alpha Z)^2$
Tensor and axial vector	$\frac{1}{2}\alpha Z\int\boldsymbol{\sigma}\times\mathbf{r}/R$	$0.30(\alpha Z)^2$
Tensor	$\int\beta\boldsymbol{\alpha}$	0.088
Axial vector	$\int\gamma_5$	0.044

[a] In the last column the sum is over all values of the magnetic quantum number m_f of the final state.

Equation (11) is not likely to be greatly in error considering that the Coulomb energy difference of two neighboring isobars is large compared with nondiagonal Coulomb matrix elements; destructive interference among the neglected terms in Eq. (10) may also be a helpful factor.[7]

It is also possible that the neglected terms in Eq. (10),

$$\sum_{f'\neq f}(f|H_c|f')(f'|X|i)-\sum_{i'\neq i}(f|X|i')(i'|H_c|i), \tag{12}$$

change in a fairly random manner as regards sign and magnitude, as N and Z are varied, suggesting the possibility that Eq. (11) is accurately equivalent to Eq. (10) on the average when fluctuations are smoothed out over a range of N, Z values.

These statements are supported by a test calculation, with determinental wave functions, in which the left and right-hand members of Eq. (11) differ by less than ten percent (Appendix A). By inserting the Coulomb energies of uniformly charged spheres into the right-hand member of Eq. (11), we obtain

$$(f|[H_c,X]|i)\cong1.2e^2(Z-1)R^{-1}(f|X|i). \tag{13}$$

The matrix elements of the commutator $[H_\nu,X]$ can be estimated under special assumptions on H_ν. Conditions under which the commutator vanishes include ordinary forces (nonexchange) and zero range exchange forces. Generally speaking, such estimates have value as sample calculations, but throw no light on what actually occurs in nature.

We attempt here to evaluate the commutator without introducing an explicit form for H_ν; the calculation is based on general arguments and the semi-empirical formula for the nuclear energy surface. We use the

[7] Equations (11) and (14) involve an implicit assumption on the relative magnitudes of $(f|X|i)$, $(f'|X|i)$, and $(f|X|i')$; the situation is most favorable when

$$|(f|X|i)|\gtrsim|(f'|X|i)|$$
$$|(f|X|i')|.$$

We note also that the two-particle character of the Coulomb interaction fixes f' in Eq. (10) to the extent that it need be considered only when $\psi_{f'}$ contains a substantial component derived from configurations each differing by not more than two orbitals from corresponding configurations in ψ_f (a similar remark applies to $\psi_{i'}$ and ψ_i).

expansion theorem once more in the form expressed by Eq. (10) with H_c replaced by H_ν. If nondiagonal matrix elements of H_ν can be neglected the exact equation reduces to

$$(f|[H_\nu, X]|i)\cong[(f|H_\nu|f)-(i|H_\nu|i)](f|X|i). \quad (14)$$

The remarks in the discussion of Eq. (11) on the relative magnitude of diagonal and nondiagonal energy matrix elements, the possibility of destructive interference among the neglected terms, and a possible smoothing effect produced by averaging over a range of N, Z values all apply here in support of Eq. (14) as a useful approximation. An additional argument follows.

The terms neglected in the derivation of Eq. (14) have the form of Eq. (12) with H_ν replacing H_c and require discussion only for values of f' and i' such that $(f'|X|i)$ and $(f|X|i')$ are not extremely small compared to $(f|X|i)$. Paraphrasing an earlier remark, this condition fixes f' to the extent that it enters the problem only when $\psi_{f'}$ contains a substantial component derived from configurations each differing in just one orbital from corresponding configurations in ψ_i (a similar remark applies to $\psi_{i'}$ and ψ_f). These restrictions on f' and i' hold equally well for the matrix elements of X occurring in Eq. (10).

When ψ_f is restricted as above, one can argue on physical grounds that $(f'|H_\nu|f)$ must be small. The argument is based on the success of shell model considerations in ordering and correlating a wide range of nuclear properties. This fact can be understood if nondiagonal matrix elements of the specifically nuclear interaction are small between states derived from configurations differing in only a few orbitals.[8] Otherwise the irregular, but strong, incidence of configuration interaction would be expected to result in many anomalous nuclear properties; anomalous, that is, from the point of view of a logically coherent shell model.

The semi-empirical energy formula contains Coulomb and symmetry terms in the form[9]

$$\tfrac{3}{5}Z(Z-1)e^2/R+u_\tau(N-Z)^2/A \\ -(\delta/4A)\{(-1)^N+(-1)^Z\} \quad (15)$$

producing a parabolic variation of energy with $N-Z$ along an isobaric series. At the bottom of the parabola

$$\begin{aligned}\tfrac{3}{5}(e^2/R)\{Z(Z-1)-(Z-1)(Z-2)\} &= (6/5)(e^2/R)(Z-1) \\ &\cong (u_\tau/A)\{(N-Z+2)^2-(N-Z)^2\}. \quad (16)\end{aligned}$$

It is found in the treatment of the nucleus as a degenerate gas of free particles that very little of the δ-term and about 40 percent of the term in $(N-Z)^2$ comes from the kinetic energy. Accepting this result we obtain

$$\begin{aligned}(f|H_\nu|f)-(i|H_\nu|i) &\cong -0.6(u_\tau/A)\{(N-Z+2)^2-(N-Z)^2\} \\ &\quad -(\delta/2A)\{(-1)^N+(-1)^Z\} \\ &\cong -0.72(e^2/R)(Z-1) \\ &\quad -(\delta/2A)\{(-1)^N+(-1)^Z\}. \quad (17)\end{aligned}$$

Equation (7) now becomes

$$\begin{aligned}[W_i-W_f+0.48(e^2/R)(Z-1) \\ -(\delta/2A)\{(-1)^N+(-1)^Z\}](f|X|i) \\ \cong -(f|H_0X-XH_0|i). \quad (18)\end{aligned}$$

The presence of δ in Eq. (18) suggests a difference in the relation between the coordinate and momentum type matrix elements for odd and even values of A. However, the δ term simply cancels out the contribution to the energy difference from the spacing of the odd-odd and even-even parabola's and thus tends to make the general relation expressed by Eq. (18) independent of whether A is odd or even and, in the latter case, independent of whether the product nucleus is odd-odd or even-even.

Results from Eqs. (8) and (18) are summarized in the relations

$$\begin{aligned}\tfrac{1}{2}\Lambda\alpha Z\int \mathbf{r}/R &= -i\int \boldsymbol{\alpha} \\ \tfrac{1}{2}\Lambda\alpha Z\int \boldsymbol{\sigma}\cdot\mathbf{r}/R &= -i\int \gamma_5 \quad (19) \\ \tfrac{1}{2}\Lambda\alpha Z\int \boldsymbol{\sigma}\times\mathbf{r}/R &= -\int \beta\boldsymbol{\alpha},\end{aligned}$$

in which, for odd A,

$$\Lambda = 1+\frac{W_i-W_f}{mc^2}\frac{A^{\frac{1}{3}}}{Z}. \quad (20)$$

For positron emission Z is replaced by $-Z$ in Eqs. (19) and (20). This follows from the anti-Hermitian property of $i\boldsymbol{\alpha}$, $i\gamma_5$, and $\beta\boldsymbol{\alpha}$ and the fact that the roles of ψ_i and ψ_f are interchanged if the direction of the transition is reversed.

To complete the theory for applications one needs also

$$\begin{aligned}W_i-W_f &= W_0-2.5mc^2 \quad \text{(negatron emission)} \\ &= W_0+2.5mc^2 \quad \text{(positron emission)} \\ &= \Delta M+1.5mc^2 \quad (K\text{-capture}). \quad (21)\end{aligned}$$

3. DISCUSSION

In a recent publication,[10] Pursey derives relations similar to those developed in the preceding section. The procedure differs greatly in detail from that

[8] Equation (14) holds accurately for the artificial example of long-range exchange forces. In this example the solutions are eigenfunctions of H_ν, all nondiagonal matrix elements of H_ν vanish, and configuration interaction is completely absent. As mentioned under Eq. (13), at the opposite extreme of zero range forces $[H_\nu, X]$ vanishes. In this case Eq. (19) still holds, but Λ is somewhat larger than in Eq. (20).

[9] E. Feenberg, Revs. Modern Phys. 19, 239 (1947).

[10] D. L. Pursey, Phil. Mag. 42, 1193 (1951).

followed here since complete reliance is placed on the accuracy of single particle wave functions[11] and on the correctness of an explicit Hamiltonian operator. The assumed form of H_ν is a linear combination of short-range two-particle ordinary, charge exchange, and spin-orbit coupling interactions. The contribution to Λ from the matrix element of $(f|[H_\nu, X]|i)$ is found to be small, in agreement with our conclusion for short-range two-particle interactions stated in the footnote following our Eq. (14). In our notation Pursey's results are equivalent to Eq. (19) with a modified value for Λ,

$$\Lambda \sim 2 + \frac{W_i - W_f}{mc^2} \frac{A^{\frac{1}{3}}}{Z}. \tag{22}$$

Our procedure with neglect of $[H_\nu, X]$ would yield

$$\Lambda \sim 2.4 + \frac{W_i - W_f}{mc^2} \frac{A^{\frac{1}{3}}}{Z}. \tag{23}$$

At present it is perhaps not possible to distinguish experimentally between Eq. (20) and Eqs. (22)–(23). Eventually, when the correct formulation of beta-decay theory is known down to the last coupling parameter, an experimental test should be possible.

Pursey also obtains a theoretical relation between the matrix elements of $\mathbf{r}$ and $\boldsymbol{\sigma}\times\mathbf{r}$ in the single particle approximation. Since experiment may eventually make possible a test of this relation, we give a simple nonrelativistic derivation,

$$[\boldsymbol{\sigma}\cdot\mathbf{L}, \mathbf{r}] = (\hbar/i)\boldsymbol{\sigma}\times\mathbf{r} \tag{24}$$

$$(f|[\boldsymbol{\sigma}\cdot\mathbf{L}, \mathbf{r}]|i) = \hbar\{I_f(I_f+1) - I_i(I_i+1) - L_f(L_f+1) + L_i(L_i+1)\}(f|\mathbf{r}|i); \tag{25}$$

for $I_f = I_i \pm 1$, $L_f = L_i \pm 1$

$$2(I_f - L_f)(f|\mathbf{r}|i) = \mp i(f|\boldsymbol{\sigma}\times\mathbf{r}|i). \tag{26}$$

The symmetry between $\mathbf{r}$ and $\mathbf{p}$ in the definition of $\mathbf{L}$ ensures that the same analysis goes through with $\mathbf{r}$ replaced by $\mathbf{p}$ in Eqs. (24)–(26).

APPENDIX A. ESTIMATES OF THE COULOMB AND COORDINATE MATRIX ELEMENTS

We compute the coordinate and Coulomb matrix elements in the approximation of single particle orbitals. These orbitals are combined to form antisymmetrical wave functions

$$\begin{aligned} \psi_i &= (A!)^{-\frac{1}{2}}\textstyle\sum_\nu(-1)^{n_\nu}P_\nu u_1(1)u_2(2)\cdots u_A(A), \\ \psi_f &= (A!)^{-\frac{1}{2}}\textstyle\sum_\nu(-1)^{n_\nu}P_\nu v(1)u_2(2)\cdots u_A(A). \end{aligned} \tag{A1}$$

The orbitals u_k are functions of space, spin, and charge coordinates of a single nucleon. In evaluating the matrix elements we use the notation: ω_n—space and spin component of u_1; ω_p—space and spin component of v; $(1|\rho_n|2)$—neutron density matrix derived from ψ_i; the indices 1 and 2 refer to both spin and space coordinates. Also $(1|\rho_p|2)$—as above for protons; $\rho_n(\mathbf{r})$—total initial neutron density; $\rho_p(\mathbf{r})$—total initial proton density.

The general formula

$$\int\cdots\int \psi_f^*\psi_i d\tau_{34\cdots A} = \frac{1}{A(A-1)}\sum_{k=1}^{A}\{v(1)u_\kappa(2) - v(2)u_\kappa(1)\}^* \cdot \{u_1(1)u_\kappa(2) - u_1(2)u_\kappa(1)\}, \tag{A2}$$

now yields

$$A(A-1)\int\cdots\int \psi_f^* Q_1\psi_i d\tau_{34\cdots A} = \omega_p^*(1)\omega_n(1)(\rho_p(2)+\rho_n(2)) - \omega_p^*(1)\omega_n(2)(2|\rho_n|1) - \tfrac{1}{2}\omega_p^*(2)\omega_n(1)(1|\rho_p|2), \tag{A3}$$

$$A(A-1)\int\cdots\int \psi_f^*\tfrac{1}{2}(1-\tau_{2z})Q_1\psi_i d\tau_{34\cdots A} = \omega_p^*(1)\omega_n(1)\rho_p(2) - \omega_p^*(2)\omega_n(1)(1|\rho_p|2). \tag{A4}$$

Here the integration symbols in Eqs. (A3) and (A4) include a summation over the charge variables of particles 1 and 2.

In the single particle approximation the transition may be described as the transfer of a nucleon from an occupied neutron orbital u_1 to a vacant proton orbital v. This implies

$$\begin{aligned} \textstyle\sum_{m_2}\int\int\int \omega_n(2)(2|\rho_n|1)dv_2 &= \omega_n(1) \\ \textstyle\sum_{m_2}\int\int\int \omega_p^*(2)(1|\rho_p|2)dv_2 &= 0. \end{aligned} \tag{A5}$$

With the aid of Eqs. (A3) and (A5) the coordinate matrix element $(f|X|i)$ reduces to a simple integral

$$(f|\textstyle\sum X_l Q_l|i) = A(f|X_1Q_1|i) = \displaystyle\sum_{m_1}\int\int\int \omega_p^*(1)X_1\omega_n(1)dv_1. \tag{A6}$$

In Eq. (A4) the second term of the right-hand member is generally unimportant, particularly so when no destructive interference occurs in the first term. Consequently, we drop the second term and also, for simplicity, treat ρ_p as a constant in evaluating the

[11] Note that this involves much more than the general semi-quantitative validity of the spin-orbit shell model.

Coulomb integral. The result is

$$(f|[H_c, X]|i)=\left(f\left|\frac{e^2}{2}\sum_{l\neq k}\frac{X_l}{r_{lk}}(1-\tau_{kz})Q_l\right|i\right)=\left(f\left|A(A-1)\frac{X_1e^2}{r_{12}}\frac{1}{2}(1-\tau_{2z})Q_1\right|i\right)$$

$$=\frac{3(Z-1)e^2}{4\pi R^3}\sum_{m_1}\int\cdots\int\frac{\omega_p{}^*(1)X_1\omega_n(1)}{r_{12}}dv_1dv_2=\frac{3(Z-1)e^2}{2R^3}\sum_{m_1}\int\int\int\omega_p{}^*(1)X_1\omega_n(1)(R^2-\tfrac{1}{3}r_1{}^2)dv_1. \quad \text{(A7)}$$

If now $(f|X|i)$ is comparatively large, $R^2-\frac{1}{3}r_1{}^2$ may be replaced safely by an approximate average value, $R^2-(1/5)R^2=(4/5)R^2$; Eq. (A7) then reduces to

$$(f|[H_c, X]|i)\cong\{6e^2(Z-1)/5R\}(f|X|i) \quad \text{(A8)}$$

in agreement with Eqs. (11) and (13).

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Nonlinear Meson Theory for Heavy Nuclei*

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A potential well somewhat between a three-dimensional isotropic oscillator and a square well is determined by using a nonlinear meson theory for heavy nuclei in their ground states. The introduction of a phenomenological spin-orbit coupling is shown to group the energy levels obtained according to the magic numbers. Finally, it is shown that the existence of nuclear shell structure seems to imply a modification in the concept of constant nuclear density for heavy nuclei.

1. INTRODUCTION

THE failure of existing two-body interactions to account adequately for nuclear saturation and the apparent incompatibility of the observed relatively strong, short-range, two-body interaction between nucleons with the independent particle model of the nucleus, strongly suggest the possibility of many-body forces among nucleons. A detailed investigation of many-body interactions would be somewhat complicated for an exploratory calculation. However, an alternative and much simpler procedure suggests itself in the possibility of interpreting the independent particle model for a heavy nucleus in its ground state in terms of a phenomenological nonlinear meson theory.

In this paper, a sort of classical effective meson field will be determined for a heavy nucleus from a phenomenological nonlinear wave equation. In conjunction with this investigation, we will also discuss the compatibility of nuclear shells with some of our other concepts of nuclear structure.

2. AN EFFECTIVE MESON FIELD FOR A HEAVY NUCLEUS

We assume that an effective meson field for a heavy nucleus in its ground state can be characterized by a static nonlinear wave equation for a classical neutral scalar field with a time independent nucleon source density. As suggested in a previous note,[1] we assume the static wave equation to be

$$-\nabla^2\phi+\mu^2\phi+g^4(\hbar c)^{-3}\lambda\phi^3=g\rho, \quad (2.1)$$

where ρ is the nucleon density, μ the inverse meson Compton wavelength (1.4×10^{-13} cm for the π-meson) and g and λ are constants which are to be determined. Equation (2.1) is essentially the same form as that independently investigated by Schiff.[2] However, we solve this equation solely from the point of view of determining an effective meson field compatible with the independent particle model of the nucleus. As noted by Schiff,[3] the total energy associated with the meson field is

$$H=\int[\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}\mu^2\phi^2+\tfrac{1}{4}g^4(\hbar c)^{-3}\lambda\phi^4-g\rho\phi]d\tau, \quad (2.2)$$

so that from Eq. (2.1)

$$H=\int[-\tfrac{1}{4}g^4(\hbar c)^{-3}\lambda\phi^4-\tfrac{1}{2}g\rho\phi]d\tau. \quad (2.3)$$

* Based on part of a thesis presented in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Harvard University, November, 1951.

[1] B. J. Malenka, Phys. Rev. 85, 686 (1952). While we make use of the form of the wave equation that was calculated in this note, we do not wish to imply that the nonlinear meson theory considered in the present paper is a consequence of vacuum polarization.

[2] L. I. Schiff, Phys. Rev. 84, 1 (1951).

[3] See reference 2, Eqs. (12) and (14).