The mean value of the coordinate of the electron taken over the Wannier localized function $\varphi_{ln}(x)$ is

$$(\varphi_{ln}(x),\ x^\mu\varphi_{ln}(x))=x_l{}^\mu+X_n{}^\mu. \tag{1}$$

Thus $X_n{}^\mu$ is the mean displacement of the center of charge from the origin in unit cell.

It was shown in (I) that for any choice of phase of the Bloch functions ψ_{kn} so that they are periodic in p space, there exists a function $\eta_n(p)$ such that

$$\xi_{nn}{}^\mu=\frac{\partial}{\partial p^\mu}\eta_n(p). \tag{2}$$

The error made in (I) was to treat η_n as a periodic function of p_μ whereas it had been proven only that $\xi_{nn}{}^\mu$ was a periodic function. Since $\xi_{nn}{}^\mu$ is periodic, however, the most general form of η_n is

$$\eta_n(p)=\eta_n'(p)+X_n{}^\mu p^\mu. \tag{3}$$

In a band n for which some of the $X_n{}^\mu$ are different from zero, the treatment of (I) requires that across some plane in the Brillouin zone the phase of the wave function ψ_{kn} jumps by the increment of $X_n{}^\mu p^\mu$ across the zone. The difficulty noted in the footnotes to (I) is that derivatives of the ψ_{kn} by k do not converge on this plane.

There is, of course, an arbitrariness in the definition of the Bloch functions ψ_{kn} on which the theory is based, since it has been assumed of them only that they are eigenfunctions of H_0 and are continuous periodic functions of k. An equally suitable set of basic functions $\bar{\psi}_{kn}$ are obtained from the ψ_{kn} by multiplying them by a factor $U(\hbar k, n)$ which is any continuous and periodic function of k with modulus unity. If $U(\hbar k, n)$ is chosen to be

$$U(\hbar k, n)=\exp[-i\eta_n'/\hbar], \tag{4}$$

and the $\bar{\psi}_{kn}$ used as the basic functions one gets

$$\xi_{nn}{}^\mu=X_n{}^\mu. \tag{5}$$

Further, by a suitable choice of $U(\hbar k, n)$, $X_n{}^\mu$ can always be made so small that if plotted from the center of unit cell it does not extend past the surface of the cell. The change of phase of basic functions is accomplished by transforming all operators in the Wannier theory with the unitary operator

$$\mathbf{U}=\Sigma_n\mathfrak{v}_{nn}U(p, n). \tag{6}$$

As a consequence of the argument above, one can now show that the treatment of the problems discussed in (I) is essentially correct. In particular, the chief modification in the equation of motion in a magnetic field is that Luttinger's Hamiltonian should involve $E_n(p+(e/c)\mathfrak{A}(x+X_n))$ rather than $E_n(p+(e/c)\mathfrak{A}(x))$. By proper choice of the origin in unit cell one could always arrange that for any one band $X^\mu=0$. Therefore, the conclusions originally drawn as to the validity of Luttinger's form of Wannier's one-band theory remain unchanged.

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The Saturation Property of Nuclear Forces

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IN the Fermi gas model of heavy nuclei all the lowest states of neutrons and protons are occupied up to those with a maximum momentum. On account of the Pauli principle the interaction potentials between nucleons in heavy nuclei are expected to be different from those between free nucleons.

The matrix element for the transition from an initial state in which two nucleons have momenta $\mathbf{p}_1$, $\mathbf{p}_2$ to a final state with the momenta $\mathbf{p}_1-\mathbf{k}$, $\mathbf{p}_2+\mathbf{k}$ *via* a neutral scalar meson field is given by

$$g^2f(p_1)f(p_2)[1-f(p_1-k)][1-f(p_2+k)]\left\{\frac{1}{E(p_1)-E(p_1-k)-\omega_k}+\frac{1}{E(p_2)-E(p_2+k)-\omega_k}\right\}\exp i(\mathbf{k},\ \mathbf{x}_2-\mathbf{x}_1)(\psi_1'{}^*\beta\psi_1)(\psi_2'{}^*\beta\psi_2).$$

Here $f(\mathbf{p})$ is the probability that the state $\mathbf{p}$ is occupied and E, ω are the energies of the nucleon and the meson, respectively. The interaction potential between two nucleons having momenta $\mathbf{p}_1$, $\mathbf{p}_2$ within the Fermi sphere is given by

$$V(1, 2)=g^2\beta^{(1)}\beta^{(2)}\int[1-f(p_1-k)][1-f(p_2+k)]\times\left\{\frac{1}{E(p_1)-E(p_1-k)-\omega_k}+\frac{1}{E(p_2)-E(p_2+k)-\omega_k}\right\}\times\exp i(\mathbf{k},\ \mathbf{x}_2-\mathbf{x}_1)\frac{d\mathbf{k}}{(2\pi)^3}.$$

If we apply the energy law, two nucleons within the Fermi sphere cannot go over to the states outside the fermi sphere. However, according to Araki and Huzinaga,[1] the energy law cannot be made use of in general when forces between fermions in bound states are to be derived. In nonrelativistic approximation we get the following potential

$$V(1, 2)=-g^2\int[1-f(p_1-k)][1-f(p_2+k)]\frac{\exp i(\mathbf{k},\ \mathbf{x}_2-\mathbf{x}_1)}{\omega_k{}^2}\frac{d\mathbf{k}}{(2\pi)^3}.$$

We now calculate the mean energy of $V=\frac{1}{2}\Sigma_{i,k}V(i, k)$ using the Fermi gas model.

$$\bar{V}=\bar{V}_{\rm ord}+\bar{V}_{\rm exc}$$

$$\bar{V}_{\rm ord}=\frac{1}{2}\Big\{\sum_{p_1,p_2}^{(+)}\int\psi_{p_1}{}^*(x_1)\psi_{p_2}{}^*(x_2)V(1, 2)\psi_{p_1}(x_1)\psi_{p_2}(x_2)dx_1dx_2+\sum_{p_1,p_2}^{(-)}\int\psi_{p_1}{}^*(x_1)\psi_{p_2}{}^*(x_2)V(1, 2)\psi_{p_1}(x_1)\psi_{p_2}(x_2)dx_1dx_2+2\sum_{p_1}^{(+)}\sum_{p_2}^{(-)}\int\psi_{p_1}{}^*(x_1)\psi_{p_2}{}^*(x_2)V(1, 2)\psi_{p_1}(x_1)\psi_{p_2}(x_2)dx_1dx_2\Big\},$$

$$\bar{V}_{\rm exc}=\frac{-1}{2}\Big\{\sum_{p_1,p_2}^{(+)}\int\psi_{p_1}{}^*(x_1)\psi_{p_2}{}^*(x_2)V(1, 2)\psi_{p_2}(x_1)\psi_{p_1}(x_2)dx_1dx_2+\sum_{p_1,p_2}^{(-)}\int\psi_{p_1}{}^*(x_1)\psi_{p_2}{}^*(x_2)V(1, 2)\psi_{p_2}(x_1)\psi_{p_1}(x_2)dx_1dx_2\Big\},$$

where $\psi_p(x)=\Omega^{-\frac{1}{2}}\exp i(\mathbf{p},\ \mathbf{x})$ and $\Sigma^{(+)}$, $\Sigma^{(-)}$ represent summations extending over the occupied neutron states and proton states, respectively. If we first perform the integration over x, we find that only $k=0$ contributes to $\bar{V}_{\rm ord}$, but $[1-f(p_1)][1-f(p_2)]$ vanishes because of $f(p_1)=f(p_2)=1$. So the nonsaturating $\bar{V}_{\rm ord}$ vanishes, but $\bar{V}_{\rm exc}$ does not.

The details will be published elsewhere.

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β-Decay of Mirror Nuclei and the Shell Model

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SPIN and parity predictions from the nuclear shell model are in good agreement with nearly all experimental β-decay data.[1] For mirror nuclei the ft-values agree within a factor two with the theoretical nuclear matrix elements for suitably chosen β-decay coupling constants.[2-3]

It is the intention in this note to base a determination of these coupling constants on those mirror nuclei which have closed shells (0, 2, 8, 20) in both neutrons and protons±one particle. For these

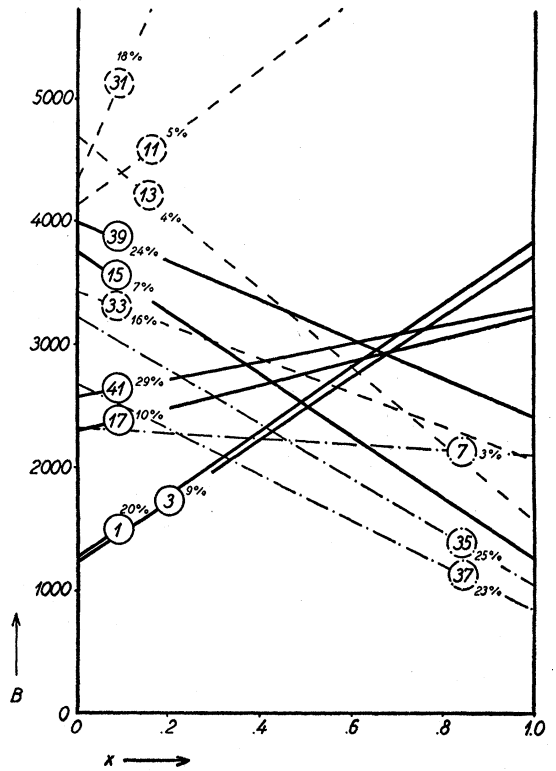

FIG. 1. B as a function of x for some mirror nuclei. The mass number is given in a circle at each curve and the experimental uncertainty in ft is indicated. ———— represents data in Table I. — — — — — represents data in Table II. — · — · — · — represents data in Table III.

nuclei one would expect the shell model to prescribe the wave functions with the least ambiguity. The view that these nuclei have a simple structure is also supported by the fact that their magnetic moments lie close to the Schmidt lines.

The nuclear wave functions are thus taken as single particle wave functions, and the normalization of the coupling is evident when it is noted that the square of the Fermi matrix element between $s_{1/2}$ states is given by $|F|^2=|\langle s_{1/2}|1|s_{1/2}\rangle|^2=1$ and that the corresponding Gamow-Teller (G—T) matrix element squared is $|G|^2=|\langle s_{1/2}|\sigma|s_{1/2}\rangle|^2=3$. The results are given in Table I.[4]

The decay constant is given by

$$\lambda=\frac{\ln 2}{t}=\frac{g^2m^5c^4}{2\pi^3\hbar^7}[(1-x)|F|^2+x|G|^2]f,$$

where x is the relative contribution from the G—T coupling; i.e., the partial coupling constant for G—T transitions equals $g[x]^{\frac{1}{2}}$, and correspondingly $g[1-x]^{\frac{1}{2}}$ is the partial Fermi coupling constant.

With

$$B=2\pi^3\hbar^7\ln 2/(g^2m^5c^4),$$

we get

$$B=ft[(1-x)|F|^2+x|G|^2].$$

In an x, B-diagram this represents a straight line connecting $B=ft|F|^2$ for $x=0$ and $B=ft|G|^2$ for $x=1$. For each of the transitions listed in Table I this straight line is given as a full drawn curve in Fig. 1. If the data are consistent with a single value of x and B all these lines ought to pass through a common point. This is actually the case within the limits of the experimental uncertainties. A least square fit of the data yields $B=2600\pm85$ and $x=0.50\pm0.05$, where the errors given are mean square deviations computed from internal consistency of the data. From this value of B we find $g=(2.19\pm0.03)\times10^{-49}$ erg cm^3.

These values differ somewhat from those given by Moszkowski[5] who based his determination on H^3, Be^7 (see below) and on He^6.

It should be emphasized that the above uncertainties of course do not contain the changes which might arise from refinements in the nuclear wave functions.

It is clear that more precise experimental investigations would be of great value in deciding whether the shell model is sufficiently accurate to provide a unique intersection point for the transitions listed in Table I. At the same time one would obtain an improved determination of B and x.

TABLE I. Mirror nuclei with closed shells 0, 2, 8, 20±one particle.

Transition	Spin	Configuration	Maximum kinetic energy, Mev	Half-life, t	ft[a]	$\|F\|^2$	$\|G\|^2$
$_0n^1-{}_1H^1$	1/2	$s_{1/2}$	0.782 ±0.001	12.8 ±2.5 min	1280±20%	1	3
$_1H^3-{}_2He^3$	1/2	$-s_{1/2}$[b]	0.0191±0.0005	12.45±0.20 yr	1240± 9%	1	3
$_8O^{15}-{}_7N^{15}$	1/2	$-p_{1/2}$	1.68 ±0.01	2.1 ±0.1 min	3750± 7%	1	1/3
$_9F^{17}-{}_8O^{17}$	5/2	$d_{5/2}$	1.72 ±0.03	70 ±3 sec	2300±10%	1	7/5
$_{20}Ca^{39}-{}_{19}K^{39}$	3/2	$-d_{3/2}$	5.13 ±0.15	1.1 ±0.2 sec	3990±24%	1	3/5
$_{21}Sc^{41}-{}_{20}Ca^{41}$		$f_{7/2}$	4.9 ±0.3	0.87±0.05 sec	2570±29%	1	9/7

[a] The Fermi integral f is determined from the formulas of Feenberg and Trigg (reference 4).
[b] The symbol $-s_{1/2}$ indicates that one $s_{1/2}$ particle will fill up the shell.

TABLE II. Mirror nuclei with closed shells 6, 16±one particle.

Transition	Spin	Configuration	Maximum kinetic energy, Mev	Half-life, t	ft	$\|F\|^2$	$\|G\|^2$
$_6C^{11}-{}_5B^{11}$	3/2	$-p_{3/2}$	0.975±0.010	20.5±0.2 min	4130± 5%	1	5/3
$_7N^{13}-{}_6C^{13}$	1/2	$p_{1/2}$	1.24 ±0.02	10.1±0.2 min	4700± 4%	1	1/3
$_{16}S^{31}-{}_{15}P^{31}$	1/2	$-s_{1/2}$	4.06 ±0.12	3.2±0.3 sec	4320±18%	1	3
$_{17}Cl^{33}-{}_{16}S^{33}$	3/2	$d_{3/2}$	4.43 ±0.13	1.8±0.1 sec	3440±16%	1	3/5

TABLE III. Unique $J-T$ multiplet mirror nuclei.

Transition	Spin	Configuration	Maximum kinetic energy, Mev	Half-life, t	ft	$\|F\|^2$	$\|G\|^2$
$_4Be^7-{}_3Li^7$	3/2	$(p^3_{3/2})_{3/2^{1/2}}$[a]	0.863±0.002[b]	54 ±1 day	2330± 3%	1	121/135
$_{18}A^{35}-{}_{17}Cl^{35}$	3/2	$(d^3_{3/2})_{3/2^{1/2}}$	4.4 ±0.2	1.90±0.05 sec	3220±25%	1	121/375
$_{19}K^{37}-{}_{18}A^{37}$		$(d^5_{5/2})_{3/2^{1/2}}$	4.57 ±0.13	1.2 ±0.2 sec	2700±23%	1	121/375

[a] The upper right index 1/2 means $T=1/2$.
[b] K-capture.

We have also calculated $|F|^2$ and $|G|^2$ for mirror nuclei having 6 protons and neutrons±one nucleon and 16 protons and neutrons±one nucleon, since the numbers 6 and 16 are assumed to represent closed configurations. The data are listed in Table II and shown in Fig. 1. Especially for the nuclei of mass numbers 11 and 31 the fit is bad. This can hardly be explained without the assumption of a rather strong distortion of the nuclear wave functions. Also the magnetic moments of these nuclei show that one does not have pure single particle states. Estimates indicate that perturbations of the order of magnitude required may arise from the coupling between the single particle motion and nuclear surface deformations.[6,7]

For the more complicated mirror nuclei the shell model wavefunctions are more ambiguous. However, in a few cases included in Table III the states are given uniquely by the $j-j$ coupling shell model together with the charge symmetry requirements. The results for these nuclei are plotted in Fig. 1, and it is seen that the matrix elements are slightly too small.

A more detailed account of β-matrix elements derived from nuclear shell models will appear in Dan. Mat. Fys. Medd.

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Tests of Charge Independence from Pion Production in Nuclear Collisions*

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NEW possibilities of testing the hypothesis of charge independence[1] arise in connection with studies of pion production in nuclear collisons. Formally, charge independence is most conveniently expressed with the help of the isotopic spin formalism which can be extended to include pions; it amounts to assuming that the total isotopic spin is a good quantum number in reactions involving nucleons and pions only.[2] If this assumption is true, it turns out that the cross sections for the production of charged and neutral pions in nuclear collisions must satisfy rather stringent relations.

Thus, consider a production process,

$$N_1+N_2\rightarrow N+\pi, \tag{1}$$

where N_1 and N_2 designate the colliding nuclei and N the outgoing nuclear fragments (bound or unbound). We designate by t_1, t_2, t, ϑ the isotopic spins of N_1, N_2, N, and π, respectively, ($\vartheta=1$) and by T the total isotopic spin ($|t_1-t_2|\leqslant T\leqslant t_1+t_2$).

The simplest case is $t_1=0$, $t_2=0$ (e.g., $d+\alpha$, $\alpha+\alpha$, $d+C^{12}$, $\cdots$); then $T=0$ and the cross sections σ_+, σ_0, σ_-, for the production of π^+, π^0, π^- at a given solid angle and with a given energy must be equal since there exists no preferred direction in charge space:

$$\sigma_+=\sigma_0=\sigma_-. \tag{2}$$

The equality of σ_+ and σ_- is merely a consequence of charge symmetry; therefore one must measure the π^0 yield if one wishes to use (2) as a test of charge independence.

When $t_1=0$, $t_2=\frac{1}{2}$ (e.g., $p+d$, $p+\alpha$, $p+C^{12}$, $\cdots$, $d+Be^9$, $\cdots$), $T=\frac{1}{2}$; therefore t may assume two values: $\frac{1}{2}$ or $\frac{3}{2}$. For convenience, we treat the particular case of p-d collisions; it will become apparent that the general case leads to identical results. The 3-nucleon system leads to one quartet and two doublets; the corresponding charge functions will be designated by the symbols ξ_q, ξ'_q, ξ''_q, respectively, where the subscript q stands for the total charge of the nuclear system (with the convention that the proton has isotopic spin $+\frac{1}{2}$, $q=t_z+\frac{3}{2}$) and ϑ_+, ϑ_0, ϑ_- will designate the charge functions of the 3 types of pions. Only 3 mutually orthogonal charge states can be formed with 3 nucleons and 1 pion, having the proper values $T=\frac{1}{2}$, $T_z=\frac{1}{2}$:

$$\begin{aligned}\chi&=\sqrt{\tfrac{1}{6}}\xi_1\theta_+-\sqrt{\tfrac{1}{3}}\xi_2\theta_0+\sqrt{\tfrac{1}{2}}\xi_3\theta_-\\ \chi'&=\sqrt{\tfrac{2}{3}}\xi_1'\theta_+-\sqrt{\tfrac{1}{3}}\xi_2'\theta_0\\ \chi''&=\sqrt{\tfrac{2}{3}}\xi_1''\theta_+-\sqrt{\tfrac{1}{3}}\xi_2''\theta_0.\end{aligned} \tag{3}$$

Therefore, the scattered wave function assumes the form $\Psi\chi+\Psi'\chi'+\Psi''\chi''$ where Ψ, Ψ', and Ψ'' depend on the nucleon spins, the relative coordinates of the nucleons and the coordinate $\mathbf{r}$ of the pion relative to the center of mass of the nucleons. Using Eq. (3), this scattered wave can be rewritten in order to separate the terms corresponding to mesons of different charge, namely

$$\begin{aligned}&[\sqrt{\tfrac{1}{6}}\Psi\xi_1+\sqrt{\tfrac{2}{3}}(\Psi'\xi_1'+\Psi''\xi_1'')]\theta_+\\ &\quad-[\sqrt{\tfrac{1}{3}}\Psi\xi_2+\sqrt{\tfrac{1}{3}}(\Psi'\xi_2'+\Psi''\xi_2'')]\theta_0+[\sqrt{\tfrac{1}{2}}\Psi\xi_3]\theta_-.\end{aligned} \tag{4}$$

Let us consider first the cross sections for producing π^++H^3 and π^0+He^3, respectively. In this case, $t=\frac{1}{2}$ and the nuclear wave functions in the final state assume the form

$$\begin{aligned}\Phi_{H^3}&=\phi'\xi_1'+\phi''\xi_1''\\ \Phi_{He^3}&=\phi'\xi_2'+\phi''\xi_2'',\end{aligned} \tag{5}$$

where ϕ', ϕ'' are functions of the space and spin coordinates of the nucleons. That the same ϕ', ϕ'' appear in both wave functions in the way indicated is a consequence of the assumption of charge independence of nuclear forces. To find the desired cross sections, we take the scalar products of the scattered wave (4) with $\Phi_{H^3}\theta_+$ and with $\Phi_{He^3}\theta_0$ obtaining,

$$\begin{aligned}F_+(\mathbf{r})&=\sqrt{\tfrac{2}{3}}[\langle\phi'|\Psi'\rangle+\langle\phi''|\Psi''\rangle]\\ F_0(\mathbf{r})&=-\sqrt{\tfrac{1}{3}}[\langle\phi'|\Psi'\rangle+\langle\phi''|\Psi''\rangle].\end{aligned} \tag{6}$$

Since, according to (6), $F_+(\mathbf{r})=-2F_0(\mathbf{r})$, the same relationship holds between the scattering amplitudes; taking then the absolute squares,

$$\sigma(p+d\rightarrow H^3+\pi^+)=2\sigma(p+d\rightarrow He^3+\pi_0). \tag{7}$$

It is clear that relation (7) would hold if one replaced the two final states considered above, H^3 and He^3, by any other charge doublet.[3] In the same way, one finds that the cross sections for the production of π^+, π^0, π^- leading to nuclei in charge quartet states go as 1:2:3, to be compared with the ratios 2:1:0 found for doublet transitions. It follows that the cross sections σ_+, σ_0, σ_- defined above satisfy the relations,

$$\sigma_+/(1+2\rho)=\sigma_0/(2+\rho)=\sigma_-/3, \tag{8}$$

where ρ is the ratio of the doublet to quartet contribution. Equation (8) leads to the (rather weak) inequality: $\sigma_-<3\sigma_+$ and, by elimination of ρ, to

$$\sigma_0=\tfrac{1}{2}[\sigma_-+\sigma_+]. \tag{9}$$

The type of derivation given above applies to the more complicated cases and relations similar to (9) can be derived for any value of t_1 and t_2. In particular, in nucleon-nucleon collisions one finds

$$\sigma_0{}^{np}+\sigma_0{}^{pp}=\tfrac{1}{2}\sigma_+{}^{pp}+\sigma_+{}^{np}, \tag{10}$$

where the superscripts indicate the charge of the colliding nucleons, the subscript the charge of the meson produced. Equation (10) is valid provided the initial beams (n or p) have the same energy, and the mesons are observed with the same energy and at the same angle.[4] An interesting particular case of Eq. (10) is obtained by considering π^0 and π^+ production with deuteron formation;[5] then

$$\sigma(n+p\rightarrow d+\pi^0)=\tfrac{1}{2}\sigma(p+p\rightarrow d+\pi^+). \tag{11}$$

Relations (2), (9), (10) and their particular exemplifications like (7) and (11) provide many possible tests of the charge independence hypothesis. In order to eliminate small uncertainties in