in the literature since the present paper was submitted. The results quoted lead to values of c/b in agreement with the calculations of Marshall and Guth,[9] while the values reported for a/b suggest that this coefficient rises about 0.06 at 6 Mev to 0.14 at 15 Mev. These results are reasonably consistent with those reported here.

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Covariant Theory of Radiation Damping

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Schwinger's expression of the S matrix in the Cayley form, in terms of a Hermitian operator K, is shown to be identical with the previous noncovariant expression used in Heitler's theory of radiation damping. The comparison of the two formalisms leads, furthermore, to a clear understanding of mass renormalization which is necessary for internal consistency, quite independently of the eventual removal of divergences. For the computation of $\bar{K}$ in a covariant way, new formulas generalizing and connecting Gupta's and Fukuda and Miyazima's results are presented. The nth order approximations of $\bar{K}$ and S are closely related, and $\bar{K}_n$ may be expressed in terms of the S_p of order $p\leqq n$ or in terms of their anti-Hermitian parts only.

1. THE TWO FORMS OF THE S MATRIX

THE solution of the Schrödinger equation in the interaction representation

$$i\delta\Psi[\sigma]/\delta\sigma(x)=H(x)\Psi[\sigma] \tag{1}$$

by means of the usual perturbation method leads to the collision S matrix

$$S=1+\sum_{n=1}^{\infty}S_n, \tag{2}$$

with[1]

$$S_n=(-i)^n\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}H(x_1)\theta^+(\sigma_1,\sigma_2)H(x_2)\times\theta^+(\sigma_2,\sigma_3)\cdots H(x_n)dx_1\cdots dx_n, \tag{3}$$

where

$$\theta^+(\sigma_1,\sigma_2)=\begin{cases}1 \text{ if } \sigma_1 \text{ is after } \sigma_2\\ 0 \text{ if } \sigma_1 \text{ is before } \sigma_2.\end{cases} \tag{4}$$

This expression of S has been extensively used because the presence of θ^+ functions alone, which are closely related to the principle of causality, leads very simply to the causal D^c functions of Stueckelberg and Feynman enabling a simple computation of (3) to be made by means of the Feynman rules.

However, even if all the S_n have been made convergent by a suitable regularization, it is not known whether the series (2) is always convergent, although it has a certain similarity with the development of an exponential, as was pointed out by Heisenberg.[2]

In any case, for large values of the coupling constant as in meson theories the convergence is presumably slow, and it is more indicative to write the unitary S matrix in the Cayley form

$$S=(1-\tfrac{1}{2}i\bar{K})/(1+\tfrac{1}{2}i\bar{K}), \tag{5}$$

which is closely connected with the effect of radiation damping. For the computation of transition probabilities one usually makes use of the alternative form

$$S=1-i\bar{R}, \tag{6}$$

which is equivalent to (5), provided $\bar{R}$ is deduced from the Heitler integral equation[3]

$$\bar{R}=\bar{K}-\tfrac{1}{2}i\bar{K}\cdot\bar{R}. \tag{7}$$

The scattering cross sections are then proportional to the square of the modulus of the corresponding matrix elements of $\bar{R}$.

The Hermitian operator $\bar{K}$ can be easily obtained as a series

$$\bar{K}=\sum_{n=1}^{\infty}\bar{K}_n \tag{8}$$

by a suitable perturbation method. According to Schwinger,[4]

$$\bar{K}_n=\left(-\frac{i}{2}\right)^{n-1}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}H(x_1)\epsilon(\sigma_1,\sigma_2)H(x_2)\times\epsilon(\sigma_2,\sigma_3)\cdots H(x_n)dx_1\cdots dx_n, \tag{9}$$

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[1] F. J. Dyson, Phys. Rev. 75, 486 (1949); D. Rivier, Helv Phys. Acta 22, 965 (1949); A. Houriet and A. Kind, Helv. Phys. Acta 22, 319 (1949).

[2] W. Heisenberg, Z. Naturforsch. A5, 251 (1950).

[3] W. Heitler, Proc. Cambridge Phil. Soc. 37, 291 (1941).

[4] J. Schwinger, Phys. Rev. 74, 439 (1948).

with

$$\epsilon(\sigma_1, \sigma_2) = \begin{cases} +1 \text{ if } \sigma_1 \text{ is after } \sigma_2 \\ -1 \text{ if } \sigma_1 \text{ is before } \sigma_2. \end{cases} \tag{10}$$

Before the development of the present covariant formalism Pauli[5] gave the formula

$$(k|\bar{K}_n|0) = 2\pi\delta(E_0 - E_k) \times \sum_P \frac{(k|H|l)(l|H|m)\cdots(s|H|0)}{(E_0-E_l)\cdots(E_0-E_s)}, \tag{11}$$

where P indicates that the principal value must be used in integrating over the energy.

2. EQUIVALENCE OF THE OLD AND NEW FORMALISMS

The identity of Eqs. (9) and (11), which is not generally explicitly recognized, is immediately checked by means of the relation

$$\int_{-\infty}^{+\infty} \epsilon(t) e^{-i\omega t} dt = \frac{2}{i}\frac{P}{\omega}, \tag{12}$$

where P means that the principal value must be taken in integrating over ω.

However, a more detailed comparison of the old and new formalism will be useful for the discussion of the mass renormalization. To derive the expression (11) of $\bar{K}_n$ in the old formalism Pauli wrote the wave function as the sum of an incident wave and an outgoing wave as follows:

$$(k|\Psi|0) = \delta_{k0} - 2\pi i (k|R|0)\delta_+(E_0 - E_k) \tag{13a}$$

with

$$\delta_+(\omega) = \frac{1}{2\pi}\int_0^\infty e^{i\omega t} dt = \frac{1}{2}\left\{\delta(\omega) - \frac{1}{i\pi\omega}\right\}. \tag{13b}$$

Inserting this wave function in the Schrödinger equation

$$(E-E_k)(k|\Psi|0) = \sum_l (k|H|l)(l|\Psi|0) \tag{14}$$

and assuming the energy E to be equal to the energy E_0 of the unperturbed incoming wave, one gets

$$(k|R|0) = (k|H|0) - i\pi \sum_l (k|H|l)(l|R|0)\delta(E_l - E_0) + \sum_l \frac{(k|H|l)(l|R|0)}{E_0 - E_l}. \tag{15}$$

The solution of this equation by repeated substitution in the term containing the δ-function of the value of R given by the first member of (15) leads to the Heitler integral equation in the form

$$(k|R|0) = (k|K|0) - i\pi \sum_l (k|K|l)(l|R|0)\delta(E_0 - E_l), \tag{16a}$$

where

$$(k|K|0) = (k|H|0) + \sum_{n=1}^{\infty} \sum_P \frac{(k|H|l)(l|H|m)\cdots(s|H|0)}{(E_0-E_l)\cdots(E_0-E_s)}. \tag{16b}$$

One finally obtains the Heitler integral equation in the form (7) by putting

$$\begin{aligned} (k|\bar{R}|0) &= 2\pi\delta(E_0-E_k)(k|R|0), \\ (k|\bar{K}|0) &= 2\pi\delta(E_0-E_k)(k|K|0). \end{aligned} \tag{17}$$

This straightforward derivation is easily put in covariant form by writing

$$\Psi[\sigma] = \left\{1 - i\int_{-\infty}^{\sigma} R(x) dx\right\}\Psi[-\infty]. \tag{18}$$

Inserting this in the Schrödinger equation (1) we obtain

$$R(x) = H(x) - iH(x)\int_{-\infty}^{\sigma} R(x')dx', \tag{19a}$$

or by introducing the function ϵ given by (10)

$$R(x) = H(x) - \frac{i}{2}H(x)\int_{-\infty}^{+\infty} R(x')dx' - \frac{i}{2}H(x)\int_{-\infty}^{+\infty}\epsilon(\sigma,\sigma')R(x')dx'. \tag{19b}$$

Putting in the last term the value of $R(x')$ given by the first member of the same formula, we get

$$\begin{aligned} R(x) = {} & H(x) - \frac{i}{2}H(x)\cdot\int_{-\infty}^{+\infty} R(x')dx' \\ & - \frac{i}{2}\int_{-\infty}^{+\infty} H(x)\epsilon(\sigma,\sigma')H(x')dx' \\ & - \frac{i}{2}\int_{-\infty}^{+\infty} H(x)\epsilon(\sigma,\sigma')H(x')dx'\cdot\int_{-\infty}^{+\infty} R(x')dx' \\ & + \left(\frac{i}{2}\right)^2 \int_{-\infty}^{\infty} H(x)\epsilon(\sigma,\sigma')H(x') \times \epsilon(\sigma',\sigma'')R(x'')dx'dx''. \end{aligned}$$

Inserting again in the last term the value of $R(x'')$ given by the first member of (19b) and repeating this operation indefinitely, we get the Heitler integral equation

$$R(x) = K(x) - \frac{i}{2}K(x)\int_{-\infty}^{+\infty} R(x')dx' \tag{20}$$

[5] W. Pauli, Phys. Soc. Cambridge Conference Report **5** (1947); W. Pauli, *Meson Theory* (Interscience Publishers, Inc., New York, 1948).

with

$$K(x)=H(x)+\sum\left(-\frac{i}{2}\right)^{n-1}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}H(x)\times\epsilon(\sigma,\sigma')H(x')\epsilon(\sigma',\sigma'')\cdots H(x^{(n-1)})\times dx'\cdots dx^{(n-1)}. \quad (20b)$$

This expression can be considered also as the solution, by iteration, of the integral equation

$$K(x)=H(x)-\frac{i}{2}\int_{-\infty}^{+\infty}H(x)\epsilon(\sigma,\sigma')K(x')dx'. \quad (20c)$$

Finally (20a) reduces to (7) by putting

$$\bar{R}=\int_{-\infty}^{+\infty}R(x)dx;\quad \bar{K}=\int_{-\infty}^{+\infty}K(x)dx. \quad (21)$$

The convergence condition of the series (8), representing $\bar{K}$, is no better known than that of the series (3), representing S. The two conditions might be different. If they are both fulfilled, the two developments will lead, of course, to the same unitary matrix S. However, for practical computations we have to break up one of them somewhere. In the case of series (8) this operation will not affect the hermiticity of $\bar{K}$ because each term $\bar{K}_n$ of the development is separately Hermitian, as is immediately verified by looking at formulas (9) or (11). Then the Cayley form might be more advantageous, because it is always unitary to any degree of approximation. This is no more true for the exponential-like development (3).

3. MASS RENORMALIZATION

The consistent subtraction of the self-energy in the construction of the S matrix in Cayley form, has already been treated[6,7] in the old noncovariant formulation. We remarked that the form (15) of the wave function is only consistent[8] with the picture of an incoming wave plus an outgoing wave if in $\delta_+(E_0-E_k)$ the energies of the system of "bare" particles are replaced by the actual total energies E_0' and E_k' with

$$E_k'=E_k+\Delta E_k, \quad (22)$$

ΔE_k being the contribution of the self-energies of the particles.

Moreover, in the Schrödinger equation the energy E of the system must be put equal to the perturbed energy E_0' of the incident wave. The Schrödinger equation can therefore be written in the form

$$(E_0'-E_k')(k|\Psi|0)=\sum_l(k|H'|l)(l|\Psi|0),$$

similar to the usual one, except that the perturbed energies have been substituted for the unperturbed energies and that $(k|H|l)$ has been replaced by

$$(k|H'|l)=(K|H|l)-\delta_{kl}\Delta E_k. \quad (23)$$

It was also shown that this substitution is necessary and sufficient to make $(k|R|0)$ free of singularities on the energy shell $E_k'=E_0'$, a condition which is indeed implied in the assumption that the second term of the wave function (13) describes an outgoing wave only, all the singularities being necessarily contained in the factor $\delta_+(E_0'-E_k')$. Finally, it was found that this condition of continuity is sufficient to determine completely the self-energies.

Now the substitution (23) is equivalent to the present covariant one

$$H'(x)=H(x)-\bar{\Psi}(x)\Psi(x)\delta mc^2. \quad (24)$$

If properly carried out, both should lead to the same results.

We see also now that the mass renormalization is not merely introduced, as is often believed, in order to subtract infinities and to obtain in this way finite physical results. In fact, it should still be performed, if the self-energies were made finite, but different from zero, as would be the case, for instance, in the nonrelativistic theory of extended particles. The actual mass renormalization is, therefore, only unsatisfactory because the self-energies are infinite and are nevertheless treated as small.

4. COVARIANT CALCULATION OF K

As S_n can be easily evaluated in a covariant way by means of the Feynman rules, it will be sufficient to show how to derive $\bar{K}_n$ from the $S_{n'}$.

To do this let us insert

$$\epsilon(x)=2\theta^+(x)-1 \quad (25)$$

in the expression (9) of $\bar{K}_n$. We get

$$-i\bar{K}_n=(-i)^n\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}H(x_1)\{\theta^+(\sigma_1,\sigma_2)-\tfrac{1}{2}\}\times H(x_2)\{\theta^+(\sigma_2,\sigma_3)-\tfrac{1}{2}\}\cdots H(x_n)dx_1\cdots dx_n. \quad (26)$$

The term containing all the θ^+ is obviously identical with the expression (3) of S_n. Another term, containing r factors $\frac{1}{2}$, is easily identified by means of the same Eq. (3) with a product of r factors S_p or lower order, $S_{p_1}\cdots S_{p_r}$, with $p_1+p_2+\cdots+p_r=n$. We so obtain the general formula

$$-i\bar{K}_n=S_n-\tfrac{1}{2}\sum_{p=1}^{n-1}S_{n-p}S_p+\tfrac{1}{4}\sum_{p=1}^{n-1}\sum_{q=1}^{p-1}S_{n-p}S_{p-q}S_q-\cdots, \quad (27)$$

which is for $n=1, 2, 3, 4$, identical with the particular expressions given by Gupta.[9]

[6] J. Pirenne, Helv. Phys. Acta 21, 226 (1948).
[7] W. Heitler and S. T. Ma, Phil. Mag. 40, 651 (1949).
[8] Heitler and Ma (see reference 7) start from analogous arguments of self-consistency but their treatment is somewhat different.
[9] S. N. Gupta, Proc. Cambridge Phil. Soc. 47, 454 (1951).

Another useful general formula can be obtained by a suitable grouping of terms

$$-i\bar{K}_n=S_n-\tfrac{1}{2}\sum_{p=1}^{n-1}S_{n-p}\{S_p-\tfrac{1}{2}\sum_{q=1}^{p-1}S_{p-q}S_q+\cdots\}. \quad (28)$$

By (9) the bracket is equal to $-i\bar{K}_p$. Therefore,

$$-i\bar{K}_n=S_n+\frac{i}{2}\sum_{p=1}^{n-1}S_{n-p}\bar{K}_p. \quad (29)$$

But by a simple change of notation in the summations contained in (27) and a different grouping we can also obtain

$$-i\bar{K}_n=S_n+\frac{i}{2}\sum_{p=1}^{n-1}\bar{K}_pS_{n-p}. \quad (30)$$

This is not surprising as $[S,K]=0$ and therefore

$$\sum_{p=1}^{n-1}[S_{n-p},K_p]=0. \quad (31)$$

Taking the Hermitian conjugate of (30) and using the fact that $\bar{K}_p$ is Hermitian, we get

$$+i\bar{K}_n=S_n^+-\frac{i}{2}\sum_{p=1}^{n-1}S_{n-p}^+\bar{K}_p. \quad (30b)$$

Now let us introduce the Hermitian and anti-Hermitian parts of S_n:

$$S_n=H_n+iA_n. \quad (32)$$

By adding and subtracting (29) and (30) we find

$$-\bar{K}_n=A_n+\tfrac{1}{2}\sum_{p=1}^{n-1}H_{n-p}\bar{K}_p, \quad (33a)$$

$$0=H_n-\tfrac{1}{2}\sum_{p=1}^{n-1}A_{n-p}\bar{K}_p. \quad (33b)$$

Using now in (33a) the value of H_{n-p} given by (33b), we obtain the general formula

$$\bar{K}_n=-A_n-\tfrac{1}{2}\sum_{\substack{p>0,\,q>0\\ p+q<n-1}}A_{n-p-q}\bar{K}_p\bar{K}_q, \quad (34)$$

in which the order of the factors in the last term is, in fact, irrelevant because of (31).

This formula simplifies a great deal for low values of n. As

$$A_1=\int_{-\infty}^{+\infty}H(x)dx=0, \quad (35)$$

it follows that $\bar{K}_1=0$. Therefore, the second term of (34) will only bring a contribution if all the three indices are at least equal to 2. In other words

$$\bar{K}_n=-A_n \text{ for } n\leqq 5. \quad (36)$$

This simple rule was given without proof by Fukuda and Miyazima.[10] (34) is its generalization. For instance, for $n=6$ we have

$$\bar{K}_6=-A_6+\tfrac{1}{4}A_2^3. \quad (37)$$

In writing (35) we have implicity assumed that there is no external (static) field; if such a field exists, the Fukuda-Miyazima rule only applies for $n\leq 2$.

We have thus a rather simple way to compute $\bar{K}$ up to a certain degree of approximation p from the development of S limited to terms of the same order (which are themselves easily written down by means of the Feynman rules).

But, finally, for practical purposes we have to introduce that $\bar{K}$ into the Cayley formula (or into the Heitler integral equation) in order to obtain a certain S matrix, because it is S that we really need to compute transition probabilities.

It might therefore seem at first sight that we have not gained anything new. However, it must be realized that the S we have now obtained is unitary and constitutes, therefore, a quite different approximation from the initial nonunitary approximation of S from which $\bar{K}$ was derived. In addition to the limited number of terms considered in the initial development of S the new S will contain terms of higher order than p, arising from the successive powers of $\bar{K}$ in the development of S in terms of $\bar{K}$. These higher order terms result from the product of lower order process for each one of which the conservation laws of energy and momentum are both satisfied, as is the case for real processes. The appearance of these new terms corresponds to renormalization of the probability amplitudes of these low order real processes, which are really the only ones considered in the chosen approximation. This renormalization is an improvement which is the more necessary, the larger the transition probabilities.

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[10] N. Fukuda and T. Miyazima, Prog. Theoret. Phys. 5, 849 (1950).