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Monte Carlo Study of Shower Production

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Electron- and photon-initiated showers in lead have been calculated in a simple Monte Carlo manner for energies from 20 to 500 Mev. The results, exhibited in a series of transition curves, show considerable differences from the results of conventional cascade theory in that the number of electrons at the maximum are fewer and the shower is more penetrating.

The effects of multiple Coulomb scattering of the electrons have been included and depend markedly on the measurement considered. Thus, ionization currents are only slightly changed (the transition curves are foreshortened by about one-half radiation length) while the number of electrons counted behind lead plates in a cloud chamber can be reduced by as much as 50 percent by effects of multiple scattering. This result is nearly independent of the incident energy.

The final curves obtained are compared with the ionization measurement of Blocker, Kenney, and Panofsky and the cloud-chamber measurements of A. M. Shapiro. The agreement is satisfactory.

A simple quasi-analytic cascade theory is developed in which known low energy solutions are used to obtain successively higher energy solutions.

I. INTRODUCTION

THE cascade theory of shower production in light elements such as air has been worked out rather completely for very high energy initiating electrons or photons.[1] For energies of less than one Bev in lead, however, the cascade theory cannot be expected to apply very accurately: ionization losses are important, the cross sections for radiation and pair production change rapidly with energy, and not many generations occur. None of the calculations include effects of multiple scattering except for lateral spreading. The shower problem is inherently a stochastic one and lends itself naturally to a straightforward treatment by the Monte Carlo method. The calculations described here are all for lead and were done in two steps: first neglecting multiple scattering so that the results could be compared to conventional theory, and then including multiple scattering for the final results.

[1] H. J. Bhabha and W. Heitler, Proc. Roy. Soc. (London) **159**, 432 (1937); J. F. Carlson and J. R. Oppenheimer, Phys. Rev. **51**, 220 (1937); B. Rossi and K. Greisen, Revs. Modern Phys. **13**, 24 (1941); H. S. Snyder, Phys. Rev. **76**, 1563 (1949); I. B. Bernstein, Phys. Rev. **80**, 995 (1950).

II. THE MONTE CARLO MACHINE

The procedure used was a simple graphical and mechanical one. The distance into the lead was broken into intervals of one-fifth of a radiation length (about one mm). The electrons or photons were followed through successive intervals and their fate in passing through a given interval was decided by spinning a wheel of chance; the fate being read from one of a family of curves drawn on a cylinder, the unwrapped surface of which is shown in Fig. 1. The energy ordinate is parallel to the axis of the cylinder. For electrons, curves were drawn in such a way that the ratio of the distance between two adjacent curves to the peripheral distance of the cylinder was equal to the probability of radiation of a quantum, the energy of which was marked on each curve. For photons, dotted lines were drawn in a similar way to give the probability of electron pair production, the energies of the resulting electrons being read from the curves. Compton scattering was also included as is indicated. Accurate values of the probabilities, determined by the calculations of Bethe and Heitler[2] ad-

[2] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **146**, 84 (1934).

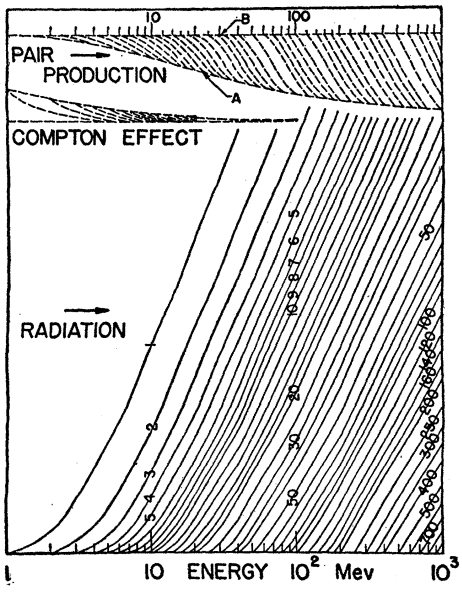

FIG. 1. The unwrapped surface of the cylinder of the wheel of chance.

justed to fit experiments,[3] were used in determining the curves.

A word about the wheel of chance: The cylinder, 4 in. outside diameter by 12 in. long, is driven by a high speed motor geared down by a ratio of about 20 to 1. The motor armature is heavier than the cylinder and determines where the cylinder stops. The motor was observed to stop at random and, in so far as the cylinder is concerned, its randomness is multiplied by the gear ratio. It appeared adequate for the operator simply to turn the motor on and then, after many revolutions of the cylinder, turn it off. As an improvement a simple switch was constructed which the operator turned on but which a Geiger counter, activated by cosmic-ray particles, turned off after a few seconds. It was also possible to set the position of the cylinder using a graduated dial and a table of random numbers. This was, of course, the most satisfactory procedure for obtaining true randomness but the operators found it considerably more tedious than using the motor. Many obvious tests of randomness have been made which indicate that the machine is more than adequately random for the accuracy aimed at in this treatment.

III. CALCULATIONS NEGLECTING MULTIPLE SCATTERING

Figure 2 exhibits a typical photon shower calculation for an initiating photon of 100 Mev. The energy of the electrons or photons of the cascade is plotted as ordinate, and distance into the Pb is the abscissa, each division corresponding to one interval (0.2 radiation length). The horizontal dotted lines are photons; the black sloping lines are electrons. The procedure of calculation is to follow the initial photon through successive intervals, spinning the cylinder at each interval until the stationary index falls between the dotted curves A and B, at the energy of the photon. The nearest curve to the index at that point indicates the energy of one of the electrons formed in pair production. The energy of the other electron is obtained by subtraction from the initial photon energy since energy is conserved. A further subtraction of one Mev for the rest mass of the electrons was made. The photon track disappears and one of the pair electrons is now followed. At each interval, the wheel is spun and the energy radiated by the electron is indicated on the black radiation curve nearest the index line at the electron energy. The newly created photon energy is plotted on the graph at that point. The energy of the radiating electron has now been reduced by the emitted photon energy plus a constant ionization loss of 1.5 Mev per interval. In this way the electron is followed until it has lost all its energy and so stops. The operator then goes back and follows the other electron and photons in turn until the whole shower is dead. The graph is thus a record of the history of that particular shower. The procedure is then successively repeated, until the statistics are adequate. Variations in form and complexity of each shower from that shown in Fig. 2 are indeed striking.

In practice photons are indicated but are not followed once their energy falls below 10 Mev because their mean free path by them is too long. The mean free path is nearly constant between 10 Mev and 1 Mev, and there is little difficulty in calculating the propagation of such radiation. The number of electrons n_2 at a

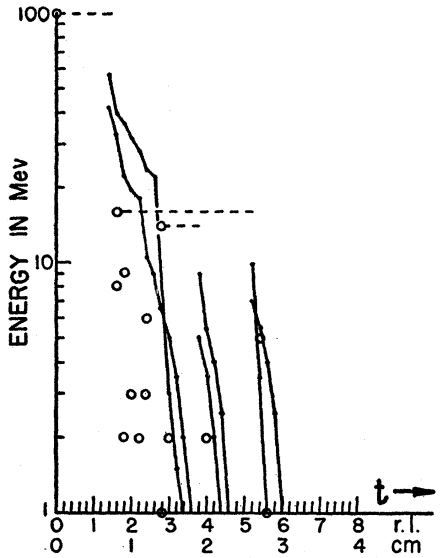

FIG. 2. A typical shower resulting from a 100-Mev photon. The dotted lines indicate photon paths and the solid lines indicate electron paths, the ordinate in each case giving the energy. The open circles are photons that were not followed. Distance in the Pb is measured in radiation lengths.

[3] DeWire, Beach, and Ashkind, Phys. Rev. **83**, 505 (1951); J. L. Lawson, Phys. Rev. **75**, 433 (1949); R. L. Walker, Phys. Rev. **76**, 1440 (1949).

distance t due to this radiation is given by

$$n_2(t)=\int_0^t [w(t')/\beta]e^{-(t-t')\sigma_m}dt', \qquad (1)$$

where $w(t')dt'$ is the average energy per shower starting as gamma-rays below 10 Mev in the interval dt' at t'. It is obtained from the shower charts by adding up the energy of all the indicated photons of energy less than 10 Mev which occur in intervals of distance that were chosen to be one radiation length and then dividing by the number of charts to get the average. The assumption is made that this energy propagates in the forward direction as $e^{-(t-t')\sigma_m}$ where σ_m is the average cross section per radiation length of the photon radiation below 10 Mev and above 1 Mev. The value of 0.24 was chosen for σ_m in lead. In obtaining (1), the assumption is also made that after the energy is absorbed the resulting electrons give up all of their energy into ionization in a region small compared to the mean free path of the radiation. The track length contributed in that region by these electrons is $w(t')/\beta$, where β is the ionization per radiation length or the critical energy; β is assumed to be 7.5 Mev for lead.

It is customary to present shower calculations in the form of curves giving the average number of electrons as a function of distance into the lead. This is done in Fig. 3 for incident photons. The number of electrons n, which is the ordinate, is the sum of n_2, the electrons as t due to low energy photons, and n_1, the electrons at t due

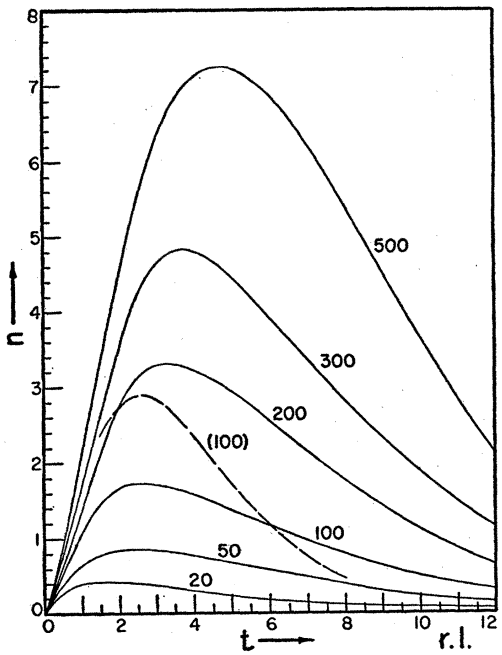

FIG. 3. Photon initiated shower curves in lead without consideration of multiple scattering. The solid curves give the Monte Carlo results and the dashed curve is calculated by conventional cascade theory. The energy in Mev is indicated on the curves. Depth units are in radiation units.

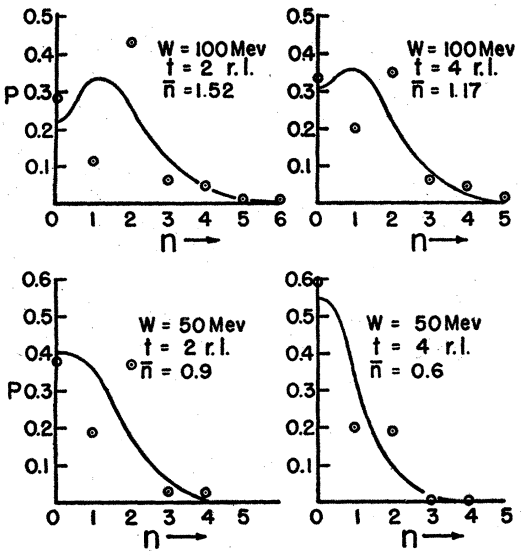

FIG. 4. Fluctuations in the number of electrons seen at a given distance. The circles are given by the Monte Carlo method and the curve is a Poisson distribution for the average number indicated. Only electrons resulting from photons above 10 Mev are shown here.

to photons of energy greater than 10 Mev. The number n_1 is obtained directly from the shower charts by adding up all the electrons seen at a given distance and then dividing by the number of charts used. In general one or two hundred shower histories were made for each curve; thus the statistical accuracy of each point is roughly 10 percent. Actually the curves have been normalized by a few percent as follows: the initial slope of the curves for gamma-rays must be equal to $2\sigma_\pi+\sigma_C$, where the cross sections are for pair production and Compton scattering, respectively; the curves must all finally fall off as $e^{-\sigma_m t}$, and the integral, $\int_0^\infty \beta n dt$, must be equal to the initial energy. The purpose of the Monte Carlo method is thus to determine the shape of the curves in the intermediate region.

The labor becomes excessive for initial energies above 300 Mev, and for these energies the photons and electrons were followed down to 50 Mev where solutions already obtained by the Monte Carlo method were fed in numerically. For purposes of comparison, the results of the theory as given in Rossi and Greisen[1] including ionization loss are shown as a dotted curve for 100-Mev initiating photons.[4] There is no agreement in shape or absolute values for reasons already given in the introduction. The comparison is equally poor with the numerical calculations of Arley.[5]

In Fig. 4 are plotted distributions of the empirical expectancy of observing various numbers of electrons at given depths in the lead (electrons below 10 Mev are

[4] I am indebted to Dr. L. Brown for calculating this curve and the one shown in Fig. 10.

[5] Niels Arley, *On the Theory of Stochastic Processes and their Application to the Theory of Cosmic Radiation* (G. E. C. Gad, Copenhagen, 1943).

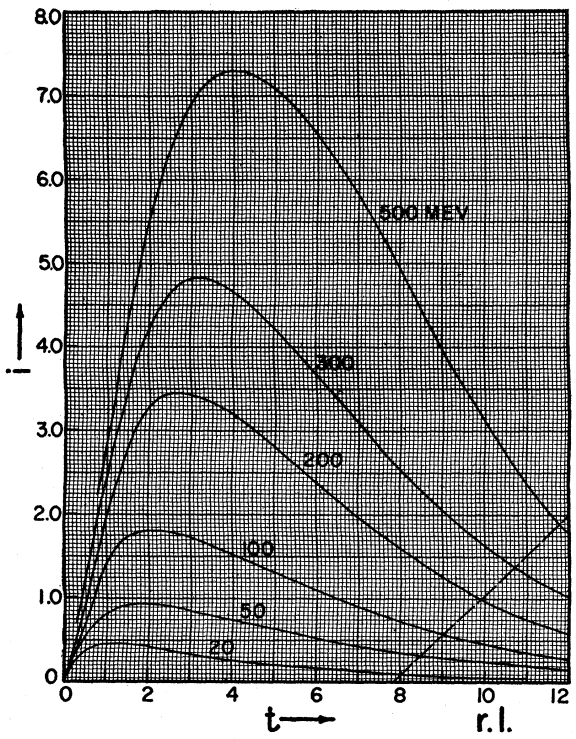

FIG. 5. Transition curves that would obtain in ionization measurements with incident photons whose energy in Mev is marked on each curve. Multiple scattering effects are included. The quantity i is defined in the text and would be proportional to the current measured in a thin ionization chamber. Beyond the oblique dashed line the curves are exponentials decaying with an absorption coefficient of 0.24. Depth units are in radiation lengths.

neglected here). Poisson distributions adjusted to the mean numbers are also drawn. It is seen that the empirical points have quite different values from those predicted by the Poisson distribution. The differences are largely inherent in the mechanism of pair production and in the fairly long range of electrons. Thus, if one member of a pair of electrons is observed, it is likely that the other electron will also be observed. Single electrons are infrequently seen by themselves and double electrons occur nearly twice as frequently as predicted by the Poisson curve. It is interesting that the expectancy of zero electrons agrees with the Poisson prediction well within the statistical accuracy.

For the experimentalist a sheaf of the history graphs is about as good a representation as any. He can leaf through the set and get a feeling for the fluctuations that are likely to occur. For a given set of experimental conditions, such as an arrangement that measures the ionization between the second and fourth radiation lengths, the graphs can be consulted directly to get an approximate answer. If one attempts a calculation, a cursory examination of the graphs gives valuable clues as to what is significant and as to what can be neglected.

IV. CALCULATIONS INCLUDING MULTIPLE SCATTERING

In principle one could include multiple scattering at the same time that the shower is being calculated by spinning an auxiliary wheel to find the multiple scattering angle of an electron as it passed through each interval of length. This would be far too laborious. Instead the effects of multiple scattering on the range of individual electrons has been worked out theoretically in a separate paper,[6] in which the results were checked with the Monte Carlo method. A two-group theory was assumed, in which the electron paths were considered as straight until the energy had been degraded to a critical value E_r after which the motion was considered to be completely random. It was found that the random energy E_r was nearly independent of the initial energy of the electron and is given in Mev by the approximate formula[7]

$$E_r \approx 20/\beta^{\frac{1}{2}}. \qquad (2)$$

Once the electron reaches this energy further displacement is negligible because the residual range is spent in a tight random motion. This means that one of the

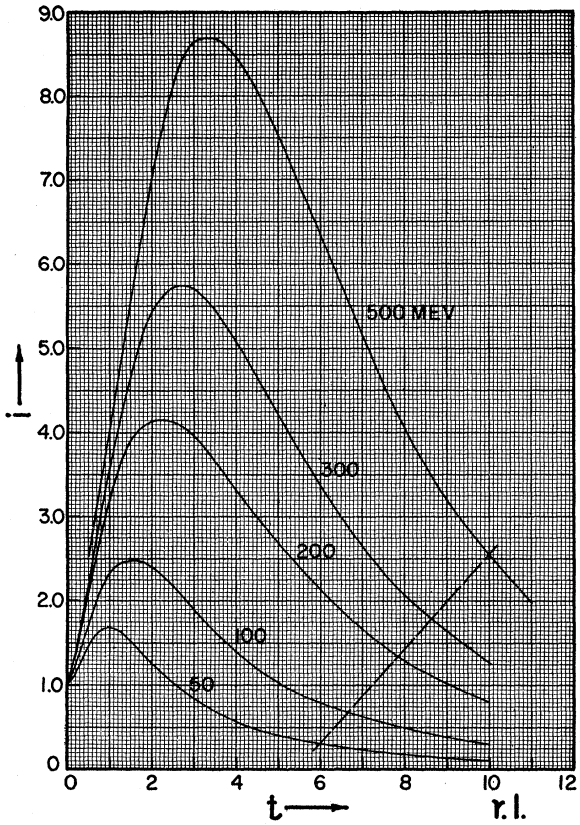

FIG. 6. Same as Fig. 5 but for incident electrons.

[6] R. R. Wilson, Phys. Rev. 84, 100 (1951).

[7] The formula appears different than in reference 6 because different units are used. More exact values are given by Eq. (14) of the same paper.

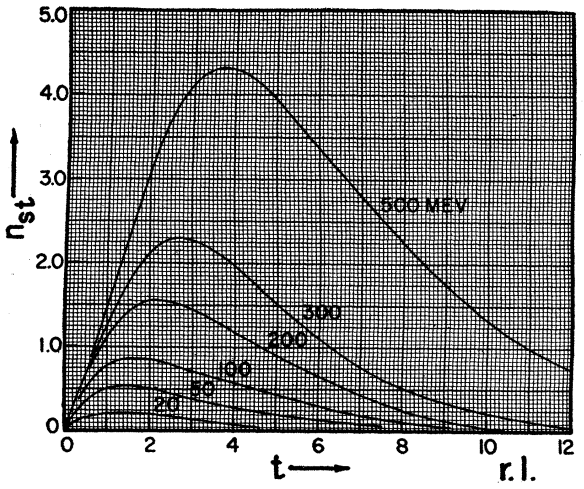

FIG. 7. The number of electrons that would be counted in the core of a shower and with paths in the direction of the shower is plotted against t, in radiation lengths, for incident photons. The energy is marked on the curves in Mev.

effects of multiple scattering is to shorten the paths of all electrons in a shower by an amount r, the range of an electron with energy E_r. This range was shown[6] to be given in radiation lengths by

$$r/\ln 2 = \ln(1+E_r/\beta \ln 2), \tag{3}$$

where β is the critical energy in the same units as E_r.

In presenting the effects of multiple scattering on curves such as those in Fig. 3 showing n against t, it is necessary to delineate the problem a little more closely in terms of measurement. One might try to measure such a curve with a thin ionization chamber inserted in the material, or the experimenter might endeavor to count the number of electrons at a given depth using a Geiger counter, or using a cloud chamber in which many plates were inserted. Quite different results would be obtained in lead in each case.

In the case of an ionization measurement, electrons moving with random motion contribute considerably to the ionization, especially those electrons moving nearly parallel to the plane of the chamber. Let us call the measured quantity i, defined as the ionization charge collected per incident electron or photon divided by the charge that would be collected were one straight electron path to cross the chamber normally. Thus if multiple scattering were negligible, i would equal n. As an approximation, we obtain curves of i as a function of t from the curves of Fig. 3 by simply shortening all the abscissas of the curves by an amount r to correspond to the decrease in electron range. Clearly this procedure breaks down for small values of t, in which case the abscissas are shortened by an amount,

$$\Delta t = r(1-e^{-t}). \tag{4}$$

This expression was chosen arbitrarily. It expresses the fact that most of the electrons in a shower have their energy near the critical energy and hence have a range of roughly one radiation length. In a more accurate theory it would be necessary to make a better determination of Δt. The fore-shortened curve should now be renormalized because the integral $\int_0^\infty i\beta dt$ must still be equal to the initial energy—this renormalization takes care of the oblique electron traversals of the ion chamber. Curves of i as a function of t for various energies of initiating electrons and photons are shown in Figs. 5 and 6.

Now for the second case in which the electrons are counted, let us discuss the example of the cloud chamber: the case of the Geiger counter will follow from this discussion. There are two possibilities in counting electrons in the cloud chamber: all the electrons appearing in the chamber between two plates can be counted, n_c; or just those electron tracks in the core and with a forward direction, say within 30°, can be counted, n_{st}. In the latter case the shower is just cut off at the energy E_r, for nearly all electrons below this energy will be either off the shower axis or if on the axis will be oriented in the wrong direction. This is a good procedure in counting because it discriminates against background electrons produced throughout the chamber. Curves of n_{st} against distance obtained by cutting showers off at E_r (8 Mev for lead) are shown in Figs. 7 and 8.

Finally we treat the case where all electron paths between successive plates are counted. Now we must add to n_{st}, just calculated for straight paths, the electrons which are moving randomly. This can be done most easily by taking the difference Δn between i given in Figs. 5 or 6 and n_{st} given in Figs. 7 or 8. The difference is the track length due to randomly moving electrons. Only the component of track length along the initial direction of the electron or photon is of interest now, and this is $\cos\theta$ times the track length, where θ is the direction of the electron with respect to the initial direction. The average of $\cos\theta$ over a hemisphere is $\frac{1}{2}$, hence we add one-half Δn to n_{st} to obtain

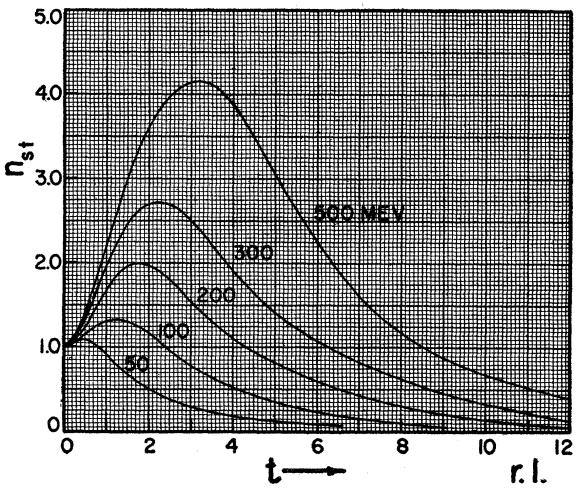

FIG. 8. Same as Fig. 7 but for incident electrons.

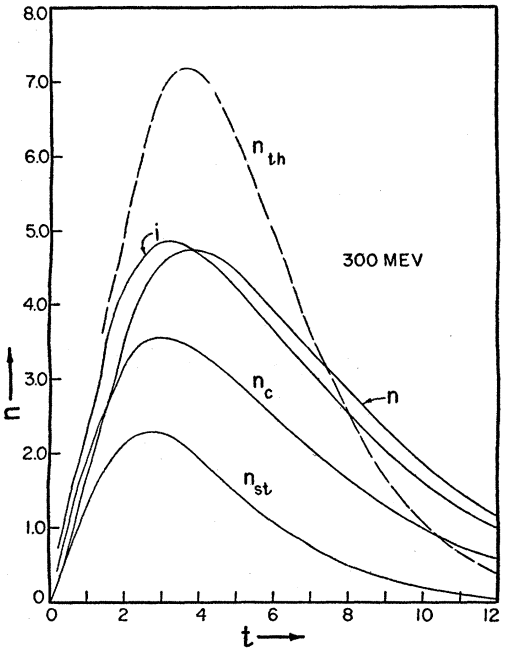

FIG. 9. The effects of multiple scattering on the transition curve of a 300-Mev incident photon. The ordinate plotted corresponds to the quantity labeled on the curve and these quantities are defined in the text for various conditions of measurement . . . the dashed curve is the result of conventional theory. (See reference 4.)

our result n_c. Curves are not given for this quantity because it can be so easily obtained from the curves of i and n_{st}.

In general one will count those electrons having an angle with respect to the initial direction less than say θ_1. In that case a fraction $\langle\cos\theta\rangle_{Av}=(1-\cos^2\theta_1)/4$ of Δn should be added to n_{st}. One must not forget the crudeness of this approximation.

Comparisons between the four above quantities n, i, n_c, and n_{st} for 300-Mev initiating electrons and photons are given in Fig. 9. The effects of multiple scattering were also calculated at 300 Mev by Mrs. Elizabeth Baranger[8] by inserting the multiple scattering in detail into the Monte Carlo calculations and her work will be published separately. Her results are statistically indistinguishable from those obtained by the simplified treatment given here.

V. EXTENSION OF CALCULATIONS TO OTHER MATERIALS

The above calculations apply to lead. For neighboring elements the results will be similar but the ordinates of the curves must be multiplied by the ratio of critical energy of the new substance to that of lead. For light elements conventional theory should apply for incident particles of high energy: at low energy or for elements of intermediate Z, the method of Bernstein[1] or the simple theory outlined in the appendix should be used.

Multiple scattering effects become relatively smaller at low Z for two reasons: (a) The cascade tends to cut off at an energy β which varies roughly as $1/Z$, and (b) E_r, the energy below which random motion sets in, varies as $1/\beta$ and hence as $Z^{\frac{1}{2}}$. Thus the relative number of electrons in a shower with energy below E_r decreases rapidly with Z.

Conventional theory can be easily corrected, if curves of i *vs* t are desired, by simply fore-shortening values of t by the amount given by Eqs. (2) and (3), as was done in this paper for the Monte Carlo results in lead, together with a renormalization of the curve. To obtain curves of n_{st} and n_c *vs* t, we must find the total track length below E_r. The computations of Richards and Nordheim[9] show that the integral track length for low energy electrons is nearly independent of Z. Their results can be expressed, at low energy and very roughly (about 10 percent), by the empirical formula

$$F(E)=W(1-e^{-2E/\beta})/\beta, \tag{5}$$

where $F(E)$ is the integral track length of electrons *up to* energy E and W is the initial particle energy. Hence we compute n_{st} by subtracting $F(E_r)$ from i as obtained above: n_c is obtained by subtracting $F(E_r)/2$. This procedure cannot be expected to work near the beginning of the shower when the calculations of track length do not apply. On the other hand it may be fairly

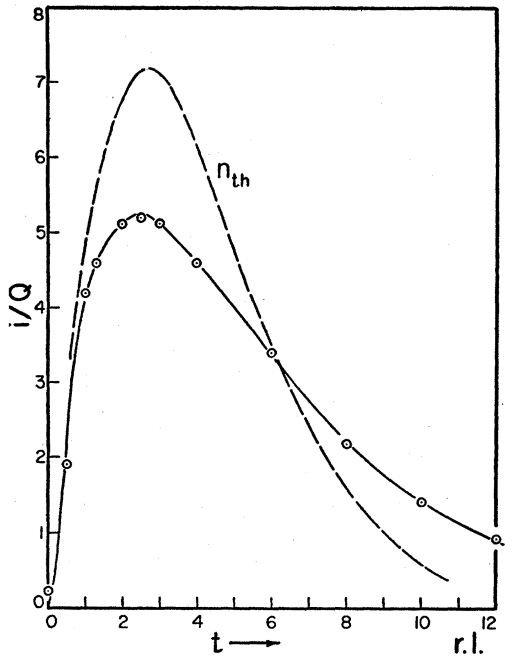

FIG. 10. A comparison of theory and experiment. The circles are the measurements in lead of ionization resulting from incident 330-Mev bremsstrahlung radiation. The solid curve which fits the points is the Monte Carlo theory and the dashed curve is the result of conventional shower theory (see reference 4). The areas of the curves are all adjusted to the same value.

[8] E. Baranger, Master's thesis, Cornell University (1951).

[9] J. Richards and L. Nordheim, Phys. Rev. 74, 1106 (1948).

reliable near the shower maximum. Incidentally, this calculation shows that the effects of multiple scattering are essentially independent of initial energy.

VI. COMPARISON WITH EXPERIMENT

Ionization curves have been measured for lead by Blocker *et al.*[10] for incident bremsstrahlung from 330-Mev electrons. Assuming the photon spectrum given by theory,[2] Monte Carlo values taken from Fig. 5 were added numerically to give the theoretical ionization curves of Fig. 10. The experimental points are also shown and the excellent agreement must be considered as partly fortuitous. The curve marked n_{th} is that given by conventional theory.[4]

Shapiro[11] has used a cloud chamber in which were placed a number of lead plates. With 200-Mev electrons incident, obtained from the Cornell synchrotron, he counted the number of electron paths in the core of the shower between the plates. His average numbers as well as the fluctuations from the averages are in very good agreement with the Monte Carlo results. An extensive comparison will be published by Shapiro.

There is an experimental check on the variation of multiple scattering effects with Z. Blocker *et al.*[10] measured the ionization current essentially at shower maximum with and without material behind the ion chamber. The difference is presumably due to multiple scattering effects. The increase in ionization resulting from back scatter was 0.41 of the total ionization in lead, 0.20 in copper, 0.09 in aluminum, and 0.04 in carbon. Our theory requires the backscattering to be proportional to the number of electrons below E_r. Using values of E_r given by Eq. (2) and the corresponding integral track lengths for lead and air given by Richards and Nordheim,[9] and assuming that half the source of randomly moving electrons is removed upon taking away the material behind the chamber, we calculate 0.38 for the backscattering in lead and 0.03 for carbon. Using Eq. (3) to interpolate track length for Cu and Al we get 0.20 and 0.07 for the backscattering respectively, instead of 0.20 and 0.08 as measured. The effects of reflections will tend to destroy this rather fortuitous agreement.

It is a pleasure to acknowledge the cooperation of Miss Leonilda Altman and Miss Ruth Seiwatz who did all of the tedious running of the machine and who carried out miscellaneous numerical calculations. Without their help the work would not have been done. I am also indebted to Mrs. Elizabeth Baranger for her work at 300 Mev on the inclusion of effects of multiple scattering.

APPENDIX. A SIMPLE SHOWER THEORY

The Monte Carlo method as used here has suggested the following simple analytic method of computing cascade phenomena. It is presented because it can be useful in extending the present results to higher energy. It is applicable to all substances, retains the use of actual cross sections for photon absorption, and should give considerably more accurate results than does the Monte Carlo method.

The theory will be described first for incident photons and then extended later on to incident electrons. The central idea is to feed in known low energy shower curves and to get out high energy curves, which can be used in turn to get ever higher energy curves. Basic for the calculation is the theory of the propagation of individual electrons, and this has been developed separately in reference 6. There it was shown that the average range R of an electron of energy E_0 is, in shower units,[12]

$$R=\ln(E_0+1), \tag{6}$$

and that the mean square straggling of the range is

$$y=(1-R/E_0)^2R. \tag{7}$$

It was also shown that the average range of the electrons of a pair produced by a photon of energy W is

$$R=(1-1/W)\ln(W+1)-1, \tag{8}$$

and that the spacial distribution of a pair of electrons produced at $t=0$, i.e., the number of electrons reaching a distance t or more, is given approximately by[13]

$$n=2\exp(-t/R_\pi). \tag{9}$$

A two-group theory is used for the shower calculation, in which the first group of electrons will be the pair made directly on the initial absorption of the incident gamma-ray. The second group will then correspond to all of the electrons resulting from the photons radiated by the initial pair electrons. We will call the electron groups primary and secondary, respectively.

The primary electron distribution $n_1(t)$ can be obtained readily from (9), i.e.,

$$\begin{aligned} n_1(t)&=2\sigma\int_0^t e^{-\sigma x}e^{-(t-x)/R_\pi}dx \\ &=2\sigma(e^{-t/R_\pi}-e^{-\sigma t})/(\sigma-1/R_\pi), \end{aligned} \tag{10}$$

where σ is the absorption cross section of the initial photon.

The secondary electrons will be calculated by first obtaining the average spectrum and source strength of the secondary photons, i.e., those emitted by the primary pair electrons. These photons will be of much

[10] Blocker, Kenney, and Panofsky, Phys. Rev. **79**, 419 (1950).
[11] A. M. Shapiro, Phys. Rev. **82**, 307 (1951).

[12] Shower units of length and energy are used in this appendix as in reference 6. Shower lengths are radiation lengths times ln2, and energies are here measured in units equal to ln2 times the critical energy, i.e., $E=E_{\text{Mev}}/\beta\ln 2$.

[13] The exponential expression applies for energies of the order of magnitude of the critical energy. For much higher energies, the integral Gaussian form (21) should be used, substituting the pair range, R_π, for R. Actually at such high energies the primary electrons will be completely masked by the secondaries and the approximation (9) will introduce little if any error.

lower energy in general than the initial photon, and a shower curve can then be calculated for this spectrum either by feeding in known shower curves such as the Monte Carlo results given in this paper or by going down to such a low energy compared to the critical energy that the shower curves will be given directly by (10), i.e., at low enough energy that the secondary group is negligible compared to the primary group. The spacial distribution, on the average, of the source strength of the secondary photons must also be computed, and then the distribution of secondary electrons will be obtained numerically through the knowledge of how such photons propagate, namely, by knowing the averaged secondary photon shower curve.

Now for the actual calculation of the secondary electrons: it is assumed that an electron in going dt emits a spectrum of photons given by

$$w(k)dkdt=dkdt/k. \tag{11}$$

which is cut off at $k=E_0$. We wish to find the average spectrum of photons from an electron of energy E_0 going a distance corresponding to its average range. The integral number of photons of energy k emitted by an electron will be obtained by integrating over that part of the range where the energy of the electron is greater than k. Assuming that the energy of the electron is related to the residual range as given by (6), then the interval of integration must be from zero to $R-t_k$, where $t_k=\ln(k+1)$ corresponds to the residual range of an electron of energy k and in this range no photons of energy k can be emitted. Accordingly, the average distribution of photons from an electron will be given by

$$w'(k)dk=(dk/k)\ln[(E_0+1)/(k+1)]. \tag{12}$$

We notice that this distribution cuts out the high energy photons. In fact a fair approximation is to assume a dk/k distribution which cuts off at $k_1=E_0-R$, the cutoff being chosen here to give the correct amount of total energy radiated compared to ionization loss which is, of course, just R in these units. This may be more accurate than using (12), for it automatically takes into account the effects of large fluctuations—an important consideration in the derivation of (6).

The spectrum of photons (12) must now be averaged over the energies of the pair electrons. Assuming that the energy of the pair is split between electron and positron such that it is equally probable for the electron to receive any fraction of the photon energy W between 0 and 1, then the average photon spectrum from the pair is

$$\begin{aligned} w''(k)dk&=\int_k^W (2dk/k)\ln[(E+1)/(k+1)]dE/W \\ &=2(dk/kW)\{(W+1)\ln[(W+1)/(k+1)] \\ &\qquad -(W-k)\}. \end{aligned} \tag{13}$$

Here the photons are degraded even farther in comparison to the initial photon energy W, and again we can approximate (but less accurately) with a simple $2dk/k$ spectrum with cutoff at $k_1=W-2R_\pi$.

To calculate $n_2(t)$, the distribution of electrons due to secondary photons, we must now know the distribution in space of the sources of secondary photons represented by (13). The total energy radiated in photons from an electron as it goes dt is given by[14]

$$-dE/dt=E,$$

and solving (6) for the average energy of an electron E as a function of the residual range r gives $E=e^r-1$; hence

$$-dE/dt=e^r-1. \tag{14}$$

An electron which starts at $t=0$ will have a photon source S_1 which is proportional to $-dE/dt$ and which varies as the energy of the electron varies. It will be given by

$$S_1(t)=e^{R-t}-1. \tag{15}$$

Assume now that the members of the initial pair of electrons both have just the average pair range given by (8). Then the source distribution from the pair produced by the initial photon will be given by

$$S_2(t)=2\sigma\int_{t-R_\pi}^{t}(e^{R_\pi-t+t'}-1)e^{-\sigma t'}dt',$$

where the lower limit is zero for $t<R_\pi$. After integration we get, for $t<R_\pi$,

$$S_2(t)=2\sigma(e^{-R_\pi-\sigma t}-e^{R_\pi-t})/(1-\sigma)-2(e^{-\sigma t}-1) \tag{16}$$

and for $t>R_\pi$,

$$\begin{aligned} S_2(t)&=2\sigma(e^{-R_\pi-\sigma t}-e^{-(t-R_\pi)\sigma})/(1-\sigma) \\ &\qquad -2(e^{-\sigma t}-e^{-(t-R_\pi)\sigma}). \end{aligned} \tag{17}$$

At this point we must calculate $n'(t)$, the shower curve due to a unit source at $t=0$ for the spectrum of photons given by (13) or by the approximate $2dk/k$ spectrum with cutoff at $k=W-2R_\pi$. Then, knowing the distribution of the sources as given above, we get at once the secondary electron distribution $n_2(t)$ by numerical integration, namely,

$$n_2(t)=\int_0^t S_2(t')\cdot n'(t-t')dt'. \tag{18}$$

The normalization of $n_2(t)$ can be checked by noticing that all the energy in the shower must be given up to ionization energy; thus

$$\int_0^\infty n_1 dt=2R_\pi, \tag{19}$$

[14] The ln2 term does not appear because of the effects of fluctuations, see reference 6.

and hence

$$\int_0^\infty n_2 dt = W - 2R_\pi. \quad (20)$$

The sum of n_1 and n_2 gives the resulting shower curve, and then one proceeds to progressively higher energies by feeding this solution back into the method.

For the case of incident electrons the procedure is exactly the same, except that the primary distribution $n_1(t)$ is due to the incident electron itself and, as shown in reference 6, is given by the integral Gaussian form,

$$n_1(t) = \tfrac{1}{2}\{1 - \mathrm{erf}[(x-R)/(2y)^{\frac{1}{2}}]\}, \quad (21)$$

R and y being given by (6) and (7). The photon spectrum is given by (12) or by its simple approximation dk/k with cutoff at $k = E_0 - R$. The spacial source distribution is given by (15).

Multiple scattering effects are included in much the same manner as in the Monte Carlo work. We calculate the number of electrons below the random energy, given in shower units of energy, approximately by

$$E_r = (10/\beta)^{\frac{1}{2}}, \quad (22)$$

or more accurately by Eq. (14) of reference 6. (Notice that β is in Mev in these formulas.) The fraction of the primary electrons, n_1, below E_r is just r/R for an incident electron or r/R_π for an incident photon, where $r = \ln(E_r+1)$. The fraction of the secondary electrons n_2 below E_r is just $r/\bar{R}$, where $\bar{R}$ is the pair range given by (7) averaged over the spectrum (12) or (13).

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Comparative Models in Nuclear Scattering*

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The "optical" and "compound" methods for treating nucleon-nucleus encounters are compared on the basis of a very simple model. It is concluded that the optical procedure is valid for incident energies $E > \sim 80$ Mev, the compound for $E < \sim 30$ Mev. The statistical model of the excited nucleus can be applied only when the excitation energies of individual nucleons may all be considered to be within the lower limit. For π-mesons incident on nuclei similar considerations show that the optical method must be used under all circumstances.

1. INTRODUCTION

ELASTIC scattering and absorption of nucleons have been treated by two different methods, which may be designated as the optical[1] and compound[2] procedures. The present note compares these approaches and attempts to define their respective regions of validity, with a view toward determining which approach is suitable for π-meson scattering and absorption by nuclei; it appears that the optical model is preferred under all circumstances.

In the compound procedure the logarithmic derivative f at the nuclear surface is represented by a suitable phenomenological form that reflects the complex situation inside the compound nucleus; namely,

$$f = -K_0 \tan\{\pi/D(E - E_0 + i\Gamma_a/2)\}$$

where $K_0 \approx 10^{13}$ cm^{-1} is a wave number appropriate to the interior of the nucleus and corresponds to a potential well about 30 Mev deep. The energy of the system is E, a resonance energy is E_0, the average spacing of successive resonance levels is D, and Γ_a is the half-width for absorption. This approach yields the Breit-Wigner form for isolated levels at low energy and has been extended[3] to incident nucleon energies as high as 25 Mev, where the resonances are completely smeared out.

The optical method has been used for incident nucleon energies on the order of 90 Mev or more and consists in integrating the phase difference over all paths through the nucleus. The wave number external to the nucleus is k, internally is $k' = k + k_1 + iK/2$, where $|k' - k| \ll k$. The beam may be analyzed into partial waves of angular momentum l, for which the appropriate W.K.B. path length extends radially from R to r_l, the classical turning point radius. A characteristic feature of this treatment is the complete neglect of reflection at the nuclear surface, which implies an infinitely diffuse boundary.

A superficial difference between the models is the question of "sharp" *vs* "diffuse" boundaries: to argue that it is unimportant, consider the error introduced by assuming a sharp boundary when a diffuse boundary is correct. The parameter measuring the sharpness of the boundary is $1/(k\Delta R)$, where the uncertainty in nuclear radius ΔR happens roughly to equal $1/K_0$ in magnitude. In the simplest case of no external potential the phase shift is given by

$$k \cot(kR + \delta) = k \cot K'R + (K' - k) \cot K'R, \quad (1)$$

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[1] Fernbach, Serber, and Taylor, Phys. Rev. 75, 1352 (1949).

[2] See, for example, Feshbach, Peaslee, and Weisskopf, Phys. Rev. 71, 145 (1947).

[3] H. Feshbach and V. F. Weisskopf, Phys. Rev. 76. 1550 (1949).