Application of the Fermi Model to Cosmic-Ray Events of Primary Energy Greater Than 10^{13} ev*

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The Fermi model for nucleon-nucleon collisions at high energies has been used to calculate the energy distribution, the energy dependence of the angular distribution, and the number of emitted pions as a function of primary energy and impact parameter. We calculate the effect on the pion distribution of nucleon-antinucleon production in the same collision. The results approach agreement with air-shower observations.

I. INTRODUCTION

EXPERIMENTAL observations of the density structure of air showers[1,2] seem to indicate that the density does not vary appreciably over distances of about one meter or less from the axis of the shower. This result might be explained either by a peculiar density distribution for a single shower core or by a multiplicity of cores with average separations somewhat less than one meter. The latter seems to be the more reasonable explanation.

The most promising model[3] is the customary one in which π^0 mesons are emitted in nuclear interactions of the primary cosmic rays and subsequently decay into photons. If one assumes that the emission is nearly isotropic in the center-of-mass system and merely exploits the relativistic contraction in explaining the observed separation of the air-shower cores, it is necessary to attribute events of shower energy $10^{13}-10^{14}$ ev to primaries of energy 10^{17} ev. The objection to this result is that less than one percent of the energy goes into π^0 mesons, if their number is kept small enough to agree with observed shower sizes. This is apparently in contradiction with estimates from observations at lower energies and with general arguments of equipartition of energy.

Fermi[4] has improved the situation by an order of magnitude in the energy. In this model, conservation of angular momentum already dictates (for a collision with median-impact parameter) a concentration of mesons near the collision axis in the center-of-mass system. The present discussion seeks to show that a more detailed analysis of the Fermi model of single nucleon-nucleon collisions results in an additional gain of nearly another order of magnitude. The angular distribution and number of π^0 mesons emitted "during" a collision will be calculated as a function of energy of the primary nucleons and of the mesons.

Another question that we shall seek to answer is whether or not a reasonable number of mesons of sufficient energy are emitted at the energy required by considerations of the angular separation.

The calculation of the angular distribution and spectral distribution of π-mesons and nucleons produced in a high energy collision of two nucleons follows and extends the work done by Fermi[4] on the statistical theory of multiple meson production for extremely high energies. Only a bare outline of the method is contained here; for a fuller discussion of the ideas and limitations, Fermi's paper should be consulted. The results contained in this paper are obtained mainly by an extension of the ideas presented in Fermi's paper.

We shall consider the question of the angular distribution and the number of particles produced as a function of energy of the primary nucleon and of the secondary particles. In the first part of the work, we assume that statistical equilibrium is attained only by the π-mesons and that the impact parameter is the median one. For extreme relativistic energies calculations have been made in which it is assumed that the incident nucleon energy is high enough to bring the mesons and the nucleon-antinucleon pairs into statistical equilibrium and also for cases with the impact parameter greater than median.

It is found that the angular distribution as a function of the energy of the secondaries is only very weakly dependent upon whether or not one assumes nucleon-antinucleon production in addition to meson production. Of course, the number of mesons produced is smaller if nucleons are produced. If the impact parameter is increased, the angular distribution is found to be more peaked, as Fermi has stated,[4] and the proportion of higher energy mesons is increased.

II. CALCULATIONS

We follow closely the work of Fermi, using as far as possible his notation. Unprimed quantities will refer to the center-of-mass system, primed ones to the laboratory system. The total energy W is deposited initially into a sphere of radius $R=\hbar/\mu c=1.4\times10^{-13}$ cm, which is Lorentz contracted because of the relative motion of the colliding nucleons. This volume we take as

$$V=(2Mc^2/W)(4\pi R^3/3),$$

where Mc^2 is the rest energy of a nucleon. Figure 1

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[1] R. W. Williams, Phys. Rev. 74, 1689 (1948); J. M. Blatt, Phys. Rev. 75, 1584 (1948).
[2] W. E. Hazen, Phys. Rev. 85, 455 (1952).
[3] Lewis, Oppenheimer, and Wouthuysen, Phys. Rev. 73, 127 (1948).
[4] E. Fermi, Phys. Rev. 81, 683 (1951).

shows the flattened sphere and the initial direction of motion of the two nucleons (along a and b) having only an angular momentum M_z, along the z axis, which is perpendicular to the plane of the figure and outwards. For the extreme relativistic case, in which we are interested here, the volume is very flattened and the y dimension may be neglected in computing the angular momentum, Z, of a particle produced at the point x, z. Then

$$Z = xp\cos\theta = x(w/c)\eta, \qquad (1)$$

where $\eta = \cos\theta$, p is the momentum, w the energy of the emitted particle (assumed relativistic), and θ is the angle between p and the y axis.

We use the thermodynamic approximation and require conservation of energy and of angular momentum. Then the average number of particles in a state of energy w and angular momentum Z is proportional to

$$1/(e^{\beta w - \lambda Z} \pm 1), \qquad (2)$$

with (+) for Fermi-Dirac statistics and (−) for Bose-Einstein statistics. We consider both statistics, for we shall consider the formation of pions and nucleon-antinucleon pairs. The parameters β and λ are determined by conservation of energy and angular momentum, respectively. We let

$$\begin{aligned} \gamma &= c\beta = c/kT, & \rho &= \lambda R/c\beta, \\ x/R &= \xi, & \zeta &= \gamma p(1-\rho\eta\xi). \end{aligned} \qquad (3)$$

Then using (1), (2), and (3) we may write for the number of particles in a volume element in phase space

$$dN = (AMc^2/W)[g_+F_+(\zeta) + g_-F_-(\zeta)](1-\xi^2)d\xi p^2 dp d\eta, \qquad (4)$$

where

$$F_\pm(\zeta) = 1/(e^\zeta \pm 1) \quad \text{and} \quad A = R^3/(2\pi\hbar^3). \qquad (5)$$

The statistical weights g_+ and g_- are taken as $g_+ = 8$ for the nucleon-antinucleon pairs and $g_- = 3$ for the pions.

The angular distribution, if one takes the momentum integral from 0 to ∞, is independent of the statistics and we have

$$N(\eta) = (AMc^2/\gamma^3 W)(g_+B_+ + g_-B_-)f_4(\rho\eta), \qquad (6)$$

where

$$f_4(\alpha) = \frac{2}{\alpha^2(1-\alpha^2)} - \frac{1}{\alpha^3}\ln\frac{1+\alpha}{1-\alpha}$$

and

$$B_\pm = \int_0^\infty \zeta^2 F_\pm(\zeta)d\zeta. \qquad (7)$$

If we integrate (6) over η from −1 to +1, we have for the total number of particles

$$N = (AMc^2/\gamma^3 W)(g_+B_+ + g_-B_-)f(\rho), \qquad (8)$$

where

$$f(\rho) = \frac{1+\rho^2}{\rho^3}\ln\frac{1+\rho}{1-\rho} - \frac{2}{\rho^2}. \qquad (9)$$

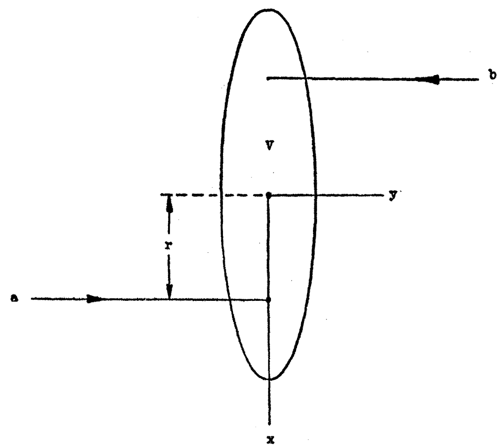

FIG. 1. Diagrammatic view of the interaction volume V of two colliding nucleons, a and b.

Multiplying (4) by cp and integrating over all variables, we have for the total energy

$$W = \tfrac{2}{3}(AMc^3/\gamma^4 W)(g_+b_+ + g_-b_-)f_2(\rho), \qquad (10)$$

where

$$f_2(\rho) = \frac{1}{\rho}\ln\frac{1+\rho}{1-\rho} + \frac{2}{1-\rho^2} \qquad (11)$$

and

$$b_\pm = \int_0^\infty \zeta^3 F_\pm(\zeta)d\zeta. \qquad (12)$$

Multiplying (4) by $xp\eta = R\xi p\eta$ and integrating, we find for the total angular momentum

$$M_Z = (ARMc^2/\gamma^4 W)(g_+b_+ + g_-b_-)f_1(\rho), \qquad (13)$$

where

$$f_1(\rho) = \frac{2}{\rho^3} + \frac{4/(3\rho)}{1-\rho^2} - \frac{1+\frac{1}{3}\rho^2}{\rho^4}\ln\frac{1+\rho}{1-\rho}. \qquad (14)$$

One then finds the parameter ρ, which measures the impact parameter, by putting the ratio M_z/W, i.e., the ratio of (13) to (10), equal to r/c. We have then

$$r/R = \tfrac{3}{2}f_1(\rho)/f_2(\rho). \qquad (15)$$

Having found ρ for a particular collision, we then find the parameter γ, which depends upon ρ and W, from (10), (11), and (12).

The constants $B_\pm$ and $b_\pm$ are given by

$$B_+ = 2\sum_{n=1}^\infty (-1)^{n+1}\frac{1}{n^3} = 1.803,$$

$$B_- = 2\sum_{n=1}^\infty \frac{1}{n^3} = 2.413,$$

$$b_+ = 6\sum_{n=1}^\infty (-1)^{n+1}\frac{1}{n^4} = 5.682,$$

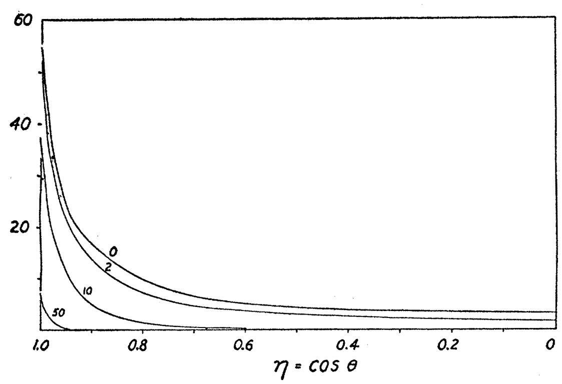

FIG. 2. Angular distribution of pions with no nucleon emission. The number of pions emitted with η between η and $\eta+d\eta$ and having an energy $cp_0\geqslant 0$ is $(AMc^2/\gamma^3W)g_-B_-f_4(0.959\eta)$, and having an energy $cp_0>0$ is $(AMc^2/\gamma^3W)g_-\sum_n h_n(\gamma p_0,\ 0.959\eta)$. We plot $B_-f_4(0.959\eta)$ vs η (upper curve) and $\sum_n h_n(\gamma p_0,\ 0.959\eta)$ vs η for $\gamma p_0=2$, 10, 50.

and

$$b_-=6\sum_{n=1}^{\infty}\frac{1}{n^4}=6.494. \tag{16}$$

One notices that the value of ρ for a given impact parameter, r, is independent of whether or not nucleon-antinucleon pairs are formed in addition to the pions. The influence of nucleon-antinucleon pairs is felt only in the effect on γ. This affects the total number of pions of all energies and the angular distribution of pions of momentum $p\geqslant p_0>0$.

To find the angular distribution of pions alone, we write

$$dN_\pi=\frac{AMc^2}{W}\left(g_-\sum_{n=1}^{\infty}e^{-n\zeta}\right)(1-\xi^2)d\xi p^2dpd\eta. \tag{17}$$

For the nucleons, we write

$$dN_N=\frac{AMc^2}{W}\left(g_+\sum_{n=1}^{\infty}(-1)^{n+1}e^{-n\zeta}\right)(1-\xi^2)d\xi p^2dpd\eta. \tag{18}$$

Integrating (17) and (18) over ξ from -1 to $+1$ and over p from p_0 to ∞, we get, respectively,

$$N_\pi(p_0,\eta)=\frac{A}{\gamma^3}\frac{Mc^2}{W}g_-\sum_{n=1}^{\infty}h_n(\gamma p_0,\rho\eta) \tag{19}$$

and

$$N_N(p_0,\eta)=\frac{A}{\gamma^3}\frac{Mc^2}{W}g_+\sum_{n=1}^{\infty}(-1)^n h_n(\gamma p_0,\rho\eta), \tag{20}$$

where

$$h_n(\gamma p_0,\rho\eta)=\frac{2\gamma p_0}{n^3\rho^2\eta^2}\left\{\frac{e^{-n\sigma}}{\sigma}+\frac{e^{-n\tau}}{\tau}\right\}-\frac{2}{n^3\rho^3\eta^3}\{-\mathrm{Ei}(-n\sigma)+\mathrm{Ei}(-n\tau)\},$$

$$\sigma=\gamma p_0(1-\rho\eta),\quad\text{and}\quad\tau=\gamma p_0(1+\rho\eta).$$

The convergence of (19) and (20) is quite rapid for $\gamma p_0=cp_0/kT\gtrsim 2$. We notice that $N_\pi(p_0,\eta)$ depends upon γ and γp_0. It is here that the influence of nucleon-antinucleon pair formation is felt, in its effect on γ.

We should like to describe a collision by taking for r some median value. We take for $P(r)$ the form

$$P(r)=r^2/R^2,$$

where $P(r)$ is the probability of a collision in which the impact parameter is less than r. For the median collision this probability is $\frac{1}{2}$, which gives

$$r=R/\sqrt{2}.$$

With this result and using (15), we find $\rho=0.959$. Later, in part B, we shall consider collisions for which $\rho=0.99$. This is not an exceptional case since this corresponds to $P(r)=0.77$, so that there are still 23 percent of the collisions whose impact parameters are larger.

A. Pion Production Only, Median Impact Parameter, $\rho=0.959$

Here we calculate the angular distribution and total number of pions above a certain minimum energy with the assumption that no available energy is used for nucleon-antinucleon pair formation. In this case (6), (8), (10), and (13) become, respectively,

$$N_\pi(\eta)=(AMc^2/\gamma^3W)g_-B_-f_4(\rho\eta), \tag{21}$$

$$N_\pi(\rho)=(AMc^2/\gamma^3W)g_-B_-f(\rho), \tag{22}$$

$$W=\tfrac{2}{3}(AMc^3/\gamma^4W)g_-b_-f_2(\rho), \tag{23}$$

and

$$M_Z=(ARMc^2/\gamma^4W)g_-b_-f_1(\rho). \tag{24}$$

We compute γ from (23) using $\rho=0.959$. We have then

$$\gamma=1.23\times10^{-8}(Mc^2/W)^{\frac{1}{4}}c/\mathrm{ev}. \tag{25}$$

Using this value of γ in (21), we find the total number of emitted pions, one-third of which are neutral pions. Their number is given by

$$N_{\pi^0}=0.46(W/Mc^2)^{\frac{3}{4}}. \tag{26}$$

One finds the angular distribution of pions with energy greater than a given cp_0 by using (19) if $cp_0>0$ or (21) if $cp_0=0$. In Fig. 2, we plot for $\gamma p_0=0$, the function

$$N_\pi(\eta)\Big/\left(\frac{AMc^2}{\gamma^3W}g_-\right)=B_-f_4(\rho\eta),$$

for $\rho=0.959$ and as a function of η. Also in Fig. 2, for $\gamma\rho_0=2$, 10, and 50, we plot the function

$$N_\pi(p_0,\eta)\Big/\left(\frac{AMc^2}{\gamma^3W}g_-\right)=\sum_{n=1}^{\infty}h_n(\gamma p_0,\rho\eta),$$

for $\rho=0.959$ as a function of η. The curves are sym-

TABLE I. The energies and numbers of emitted π^0 mesons corresponding to the parameter γp_0 for various primary energies and impact parameters.

Primary energy W/Mc^2	W' ev	$\rho=0.959$, γp_0 50	10	2	$\rho=0.990$, γp_0 50	10	2	Total number of π^0-mesons (forward and backward) $\rho=0.959$	$\rho=0.990$
Only π-mesons produced. Energy of π^0 mesons in units of 10^{13} ev (cp_0')									
50	1.2×10^{12}	0.14	0.03	0.006	0.10	0.02	0.004	3.3	1.7
200	1.9×10^{13}	1.15	0.23	0.05	0.83	0.17	0.034	6.5	3.5
500	1.2×10^{14}	4.6	0.91	0.18	3.3	0.66	0.13	10.3	5.5
1000	4.7×10^{14}	13	2.6	0.52	9.4	1.9	0.38	14.6	7.8
2000	1.9×10^{15}	36	7.3	1.45	26.4	5.3	1.06	21	11.0
5000	1.2×10^{16}	144	29	5.8	104	21	4.2	33	17.3
20,000	1.9×10^{17}	1150	230	46	833	167	3.3	65	35

Primary energy W/Mc^2	W' ev	$\rho=0.959$, γp_0 50	10	2	$\rho=0.990$, γp_0 50	10	2	Total number of π^0 meson (forward and backward) $\rho=0.959$	$\rho=0.990$
Both π-mesons and nucleons produced. Energy of π^0 mesons in units of 10^{13} ev (cp_0')									
50	1.2×10^{12}	0.104	0.021	0.004	0.053	0.011	0.0021	1.3	0.72
200	1.9×10^{13}	0.853	0.171	0.034	0.44	0.089	0.018	2.6	1.4
500	1.2×10^{14}	3.4	0.68	0.14	1.77	0.354	0.071	4.2	2.3
1000	4.7×10^{14}	9.6	1.92	0.38	5.03	1.0	0.201	5.9	3.2
2000	1.9×10^{15}	26.5	5.4	1.07	19.2	3.9	0.775	8.3	4.5
5000	1.2×10^{16}	107	21.5	4.3	77	15	3.1	13	7.2
20,000	1.9×10^{17}	853	170	34	615	123	24	26	14.3

metrical around $\eta=0$ (in the c.m. system) and we plot only the range $0\leq\eta\leq1$. These plotted curves are independent of the energy of the incoming nucleon, if we consider a given γp_0. The angular distribution for a given cp_0, however, depends on W through γ.

To compare the curves with experimental data, we must translate γp_0 into cp_0', the energy of the emitted pion in the laboratory system. We have (if the colliding nucleon is extreme relativistic)

$$cp_0'=\tfrac{1}{2}(W/Mc^2)(c/\gamma)(1+\eta)\gamma p_0. \tag{27}$$

The factor $(1+\eta)$ in (27) makes those pions emitted in the forward direction with $0.82<\eta<1$ higher in energy by an order of magnitude over those emitted with $-1<\eta<-0.82$. This is the justification for considering only the former, more energetic, group for the interpretation later in the paper.

From Eqs. (25) and (27) we get (using $\eta\approx1$)

$$cp_0'=0.82\times10^8\gamma p_0(W/Mc^2)^{\frac{3}{4}}\text{ ev}. \tag{28}$$

In Table I, (28) is tabulated for various values of γp_0 and W. Also given there is the total number of π^0's, (forward and backward) from (26). We plot in Fig. 3, the integral spectrum of π^0's for several values of W', the laboratory-system energy of the nucleon producing them.

In Table II we give $\eta_{\frac{1}{2}}$, the cosine of the angle within which is emitted one-half of the particles in the forward cone. We include there also the value of η for which the intensity has fallen to one-half the value for $\eta=1$.

B. Pion Production Only; Impact Parameter Largar than Median, $\rho=0.99$

If we increase ρ, there are two effects: (a) The angular distributions become concentrated around $\theta=0$ and $\theta=\pi$, (b) more energy goes into rotation, which lowers the temperature, thereby increasing γ. For $\rho=0.99$, we recompute:

$$\gamma=1.70\times10^{-8}(Mc^2/W)^{\frac{1}{4}}c/\text{ev}. \tag{29}$$

The total number of neutral mesons (forward and backward) becomes

$$N_{\pi^0}=0.25(W/Mc^2)^{\frac{1}{2}},$$

which is almost a factor two less than for the median impact parameter. We have for this case

$$cp_0'=0.59\times10^8\gamma p_0(W/Mc^2)^{\frac{3}{4}}\text{ ev}.$$

In Fig. 4, we plot the same function as Fig. 2, using

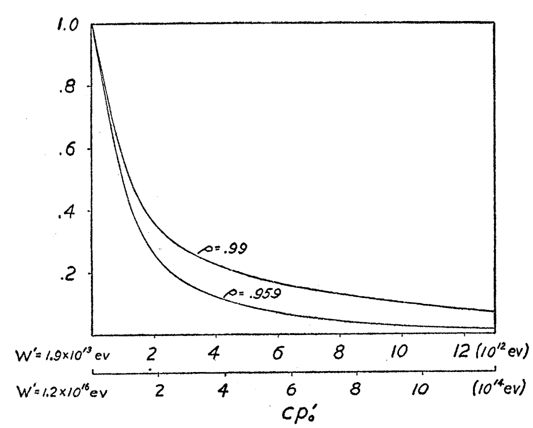

FIG. 3. Integral-energy spectra for emitted pions for two primary energies, considering pion emission only (upper scale $W'=1.9\times10^{13}$ ev) and pion and nucleon emission (lower scale $W'=1.2\times10^{16}$ ev). For other values of W' the cp_0' scale should be varied as $(W')^{3/4}$.

TABLE II. The cosine of the angle within which are emitted one-half of the particles in the forward cone, and the cosine of the angle at half-intensity, as functions of γp_0 and ρ.

γp_0	$\rho=0.959$ $\eta_{\frac{1}{2}}$	η at half intensity
0	0.82	0.958
2	0.865	0.964
10	0.960	0.974
50	0.989	0.989
	$\rho=0.99$	
0	0.89	0.991
50	0.991	0.994

$\rho=0.99$. The dotted line is taken from Fig. 2 with $\gamma p_0=10$ for comparison.

C. Effects of Nucleon-Antinucleon Pairs on Pion Distribution for $\rho=0.959$ and $\rho=0.99$

If the incoming nucleon energy is high enough ($W/Mc^2\geqslant 100$) so that nucleon-antinucleon pairs are brought into equilibrium, we must consider their production in addition to the pion production. The effect is to reduce the temperature and hence to reduce the number of π^0's emitted for a given W. In this case we use Eq. (10) to compute γ and Eq. (22) to compute the number of π's, one-third of which are π^0's. We find

$$\begin{aligned} &\rho=0.959\\ &\gamma=1.66\times10^{-8}(Mc^2/W)^{\frac{1}{4}}c/\text{ev}\\ &N_{\pi^0}=0.19(W/Mc^2)^{\frac{3}{4}}\\ &cp_0'=0.60\gamma p_0(W/Mc^2)^{\frac{3}{4}}\text{ ev.}\\ &\rho=0.99\\ &\gamma=2.29\times10^{-8}(Mc^2/W)^{\frac{1}{4}}c/\text{ev}\\ &N_{\pi^0}=0.099(W/Mc^2)^{\frac{3}{4}}\\ &cp_0'=0.44\times10^8\gamma p_0(W/Mc^2)^{\frac{3}{4}}\text{ ev.}\end{aligned} \tag{30}$$

One should note that the angular distribution as a function of γp_0 is not changed, Figs. 2 and 4, but again the energies of the mesons, cp_0', corresponding to a given γp_0 are not the same as any of the other cases. In Fig. 3 the integral energy spectrum for the π's is plotted for an energy $W'=1.2\times10^{16}$ ev.

III. DISCUSSION

A. Comparison of Shower Observations with Fermi's Calculations

Counter observations[5] of the lateral structure of air showers show that there is no multiplicity of singularities of comparable strength separated by distances from a few meters to 200 meters. Ionization chamber[1] measurements show no multiplicity for distances from one meter to about ten meters but there is evidence that either the Molière distribution is wrong or that there is a multiplicity of singularities within distances of about one meter. Cloud-chamber observations show no distinctly resolved singularities for separations less than one meter, but they do confirm the ion-chamber observation that there is a plateau region with very little variation in particle density near the shower axis.[2] Since the cloud-chamber observations should be able to resolve two Molière singularities separated by more than about 20 cm, the cloud-chamber observations of particle densities imply either that there are usually more than two shower cores with separations less than one meter or that the Molière singularities are too sharp.

There is some evidence of multiple cores in the cloud-chamber pictures[2] as evidenced by cases where there are two separate concentration areas for rays of energy $>10^{10}$ ev. Since from the theory of lateral spread of cascade showers the probability is about one-half for rays of energy $>10^{10}$ to lie within 20 cm of the shower axis,[6] concentrations of such rays can be used to identify cores with separations of the order of 50 cm or more.

As a first approximation we shall assume that the major contribution to the air showers observed in the lower atmosphere is made by the secondaries produced in the initial nuclear encounter of a primary. Thus the energy of the initiating rays will be determined with the ordinary cascade shower theory by assuming that the shower originated near the top of the atmosphere. The shower energies involved can be estimated as follows: Both the ion-chamber and cloud-chamber observations are for cases in which the minimum particle densities are $\sim$500 m^{-2} in a region of about 0.2 m^2 surrounding the shower axis. If we use the Molière distribution to obtain the total number of electrons at the observation level and the cascade theory for longitudinal development to obtain the minimum initiating energy therefrom, we obtain 3×10^{13} ev for a single ray or 10^{13} ev for each of four initiating rays.

A satisfactory model should therefore give a multiplicity of 10^{13}-ev rays with angular separations of

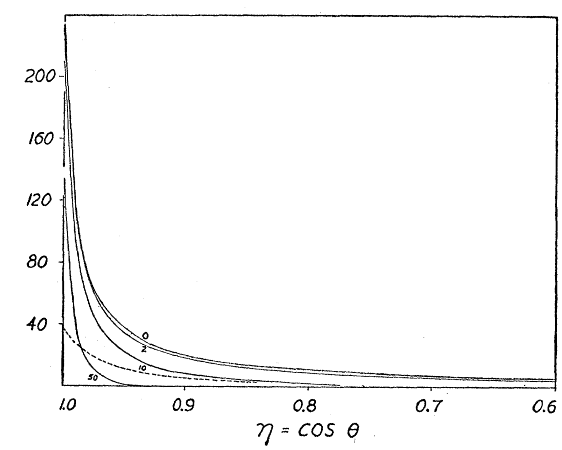

FIG. 4. $B_-f_4(0.99\eta)$ *vs* η (upper curve) and $\Sigma h_n(\gamma p_0, 0.99\eta)$ *vs* η for $\gamma p_0=2$, 10, and 50. The dotted curve is for $\gamma p_0=10$ from Fig. 2, for comparison.

[5] Cocconi, Tongiorgi, and Greisen, Phys. Rev. **76**, 1020 (1949).

[6] L. Eyges and S. Fernbach, Phys. Rev. **82**, 23 (1951); **82**, 287 (1951).

roughly 10^{-4} radian or less. The decay of π^0 mesons in flight generates two gamma-rays with an angular separation of about 1.5×10^{-5} radian for photon energies of 10^{13} ev. This mechanism alone would result in a saddle 20 cm long in the density distribution, but it fails to explain a plateau. In order to obtain a plateau, we require a multiplicity of π^0 mesons themselves with an angular spread less than 8×10^{-5} (one meter separation at the observation level of 3000 meters). The angle with the primary axis would be 4×10^{-5} radian.

The relativistic transformation from the rest system (θ) to the observation system (θ') is, for small values of the angles,

$$\theta'=\theta/(2W'/Mc^2)^{\frac{1}{2}}.$$

Thus we need a model that will provide values for θ and W'. As mentioned in the introduction, a model that assumes isotropic emission in the center-of-mass system is probably unsatisfactory.

If we turn to Fermi's calculations, we find $\theta_{\frac{1}{2}}=0.6$ for the angle that includes one-half of the forwardly-emitted mesons of all energies. We disregard the backwardly-emitted mesons because their energy in the laboratory system is an order of magnitude lower, as previously shown. A primary energy of 10^{17} ev is now required in order to effect a contraction of the angle to 4×10^{-5} radian.

Since we really should be considering most probable events, the angle at $\frac{1}{2}$ intensity is perhaps more appropriate than $\theta_{\frac{1}{2}}$. Fermi's angular distribution[4] for mesons of all energies ($\gamma p_0=0$ in Fig. 2) gives 0.28 for the angle at $\frac{1}{2}$ intensity and the corresponding primary proton energy is 2.4×10^{16} ev. Thus, as we go from an isotropic-emission model to the Fermi model, we have a factor of ten reduction (from 3×10^{17} to 2.4×10^{16}) in the energy required to effect the required angular contraction. The question of the number of mesons with energy $>10^{13}$ ev cannot be answered until we consider the results of our detailed analysis in the next section.

B. Comparison of Shower Observations with the Detailed Analysis of the Fermi Model

The observed air-shower effects that are considered here are attributed to the more energetic π^0 mesons ($E>10^{13}$ ev if we assume the showers originate near the top of the atmosphere), whereas the Fermi calculations were for mesons of all energies. Furthermore, the observed showers of a given minimum size are not necessarily caused predominantly by primary events whose average behavior corresponds to the minimum shower size; a more probable origin is one of the more abundant, lower-energy primaries that happens to make a collision with an impact parameter such that $\rho>0.959$ with a consequent hardening of the average spectrum of emitted mesons, or a collision (with any impact parameter) in which the spectrum is harder than average (with a consequent reduction in total number of emitted particles), or a collision in which π^0 mesons carry off more than their average share of the energy.

However, let us first consider an average collision, one with the median value of the impact parameter ($\rho=0.959$). Since the energy is very high, the case of nucleon emission will be considered and we have (from 30) $cp_0'=0.60\times10^8\gamma p_0(W/Mc^2)^{\frac{3}{4}}$ and $N_{\pi^0}=0.19(W/Mc^2)^{\frac{1}{4}}$. With the assumptions about the air showers that have previously been stated, we have $cp_0'\gtrsim10^{13}$ ev. We find a minimum value of W' by requiring that we have about two π^0 mesons with energy $\geqslant10^{13}$e v. With $W'=10^{15}$ ev, we have $N(0)=4$ in the forward cone and (Fig. 3) $N(10^{13})=2.8$, which is a reasonable number. The value of γp_0 is then 3 and η for $\frac{1}{2}$ intensity is 0.97. The resulting value of 0.24 for θ together with the relativistic contraction corresponding to $W'=10^{15}$ ev gives $\theta'=1.6\times10^{-4}$, which is only four times larger than the "observed" value, 4×10^{-5}, for air showers. On the other hand, let us first satisfy the angle requirement and then find the number of mesons. The value of η at $\frac{1}{2}$ intensity is about 0.97 for γp_0 in the range from 2 to 10 (Fig. 2). The primary energy required to contract the angle to the observed angle of 4×10^{-5} is $W'=1.8\times10^{16}$ ev. The total number of π^0 mesons is 8 in the forward cone and the number with energy $>10^{13}$ ev is also essentially 8. This number of π^0 mesons of energy $>10^{13}$ ev would produce a shower several times larger than the minimum size under consideration. In summary, an average collision treated according to the Fermi model will, in the limiting cases, give either a lateral spread that is about four times too great when the number of particles is correct or about four times too much energy to the shower component when the lateral spread is correct. An intermediate choice would, of course, give both too many particles and also too large an angle.

Returning to the idea of probable origins as expressed in the opening paragraph of this section, let us consider the results of distant collisions. For a semiquantitative discussion we shall choose $\rho=0.99$ as the impact parameter typifying collisions that contribute strongly to a given-size shower when the emitted particles have an equilibrium energy distribution. Table I shows that the angle at $\frac{1}{2}$ intensity is about 0.14 ($\eta\approx0.99$) for any γp_0 when $\rho=0.99$. The primary energy required to give $\theta'=4\times10^{-5}$ is, therefore, 6×10^{15} ev. The total number of π^0 mesons emitted in the forward direction in the c.m. system is three (Table I) and the average energy is so high (Fig. 3) that the number of secondary particles of energy $>10^{13}$ ev is 9/10 of the total,which is three. Therefore, collisions of this type produce a satisfactory number of particles and they are emitted at satisfactory angles.

The effect of fluctuations in the energy distribution of secondaries could be evaluated quantitatively only by returning to the analysis of the statistical model. Here we shall merely obtain a qualitative estimate by noting that, when we chose a primary energy of $1.8\times$

10^{16} ev for the median impact parameter case, the number of secondaries was too high (eight) but when we chose 10^{15} ev, the angular divergence was too large. Fluctuations of the meson production about the equilibrium might enable the relatively abundant lower energy primaries (10^{15} ev or less) to give showers which would correspond to the equilibrium production properties of less frequent, higher energy primaries. This might occur in the following way: There might be a reduction in the angular divergence in the c.m. system if unusually high energy π^0 mesons are emitted or if an unusually large fraction of the energy goes to π^0 mesons as compared with the other types of particles. On the other hand, since we are forced to choose smaller W' in order to keep the shower energy low enough, the relativistic contraction of angle will be less effective. Thus, there might be two opposing effects on the angular divergence and it is difficult to judge which is larger.

In summary, the detailed analysis of the Fermi model gives results that are almost compatible with the interpretation of existing observations. The main discrepancy is qualitatively similar to the case of an isotropic emission model but quantitatively much less serious, i.e., the primary energy required to give the desired relativistic contraction in angle between π^0 mesons results in too much energy to π^0 mesons.

C. Intensity of Energetic Primaries

In principle, a suitable model enables one to correlate observed shower events with originating primary particles. We have seen that the present model suggests primary energies of the order of 10^{16} ev for the creation of shower energies of the order of 10^{13} ev. The measured intensity[1] is about 2×10^{-8} cm^{-2} sec^{-1} sterad^{-1} for shower energies $\gtrsim10^{13}$ ev if we assume the initiating rays originate near the top of the atmosphere. If we assume a power law for the integral primary spectrum between 1.5×10^{10} ev (where rocket measurements give 0.028 cm^{-2} sec^{-1} sterad^{-1})[7] and 10^{16} ev, there results for the primary spectrum $F(E)=0.028\times(1.5\times10^{10}/E)^{1.06}$ cm^{-2} sec^{-1} sterad^{-1}. The exponent is large enough to escape an infinity in the total energy content even if the same exponent were assumed for greater energies. Actually, if we assume that primary energies are linearly related to average shower energies, the exponent has increased to 1.5–1.9 for primary energies greater than 10^{16} ev.[1]

[7] J. A. Van Allen and S. F. Singer, Phys. Rev. **78**, 819 (1950).