polar coordinates) are given in all three cases by an equation of the form

$$\pm(nA/r_0)J_n(\chi_1 r_0)J_n(\chi_2 r_0)=\alpha_2 J_n(\chi_1 r_0)J_n'(\chi_2 r_0) - \alpha_1 J_n(\chi_2 r_0)J_n'(\chi_1 r_0), \quad (1)$$

where in case (1)

$$A=\frac{\beta^2}{\beta_0}\frac{p_2^2-p_1^2}{\zeta_1\zeta_2}, \quad \alpha_s=\frac{\chi_s}{\zeta_s}[\beta^2-\beta_0^2+2\zeta_s R_p H_0-4R_p^2H_0^2] \quad (s=1,2),$$
$$\delta=(1-\mu_z/\mu), \quad R_p=(\alpha_H\beta_0/2\mu)(M_0/H_0),$$

and in cases (2) and (3)

$$A=(p_1^2-p_2^2)\beta_0, \quad \alpha_s=\chi_s[(\beta_0^2-\beta^2)\zeta_s+2\beta^2R_pH_0] \quad (s=1,2),$$
$$\delta=1-\epsilon_z/\epsilon, \quad R_p=\alpha_E\beta_0/2\epsilon,$$

and where for all three cases

$$\zeta_s=[(\beta^2-\beta_0^2)\delta-p_s^2]/2R_pH_0, \quad \chi_s^2=\beta_0^2-\beta^2+p_s^2 \quad (s=1,2),$$

$$p_{1,2}^2=-2R_p^2H_0^2-\frac{\delta}{2}(\beta_0^2-\beta^2)\pm 2R_pH_0 \times\left\{\beta^2-\beta_0^2\delta+\frac{1}{4}\left[2R_pH_0+\frac{\delta}{2R_pH_0}(\beta_0^2-\beta^2)\right]^2\right\}^{\frac{1}{2}}.$$

Equation (1) gives the β's for both TE limit and TM limit modes. For small H_0, these β's are easily found by expansion. It turns out that the deviations $\Delta\beta_{\pm n;m}$ from the β_{nm} of the TE_{nm} waves due to a small magnetic field H_0 [or to a small magnetization in case (1)] are proportional to H_0. The corresponding rotation $|\Delta\beta_{nm}/n|$ can, therefore, be specified by a new Verdet's constant. In case (1)

$$R_{\text{guide}}^{(nm)}=2\lambda_0 R_p/[(u_{nm}^2-n^2)\lambda_g], \quad (2)$$

and in cases (2) and (3)

$$R_{\text{guide}}^{(nm)}=2\lambda_g R_p/[(u_{nm}^2-n^2)\lambda_0], \quad (3)$$

where λ_g is the guide wavelength of the TE_{nm} mode at zero H_0, λ_0 the wavelength in the infinite medium, and u_{nm} is the mth zero of $J_n'(x)$.

Formulas (2) and (3) show that the guide causes dispersion in R additional to that of R_p. They have meaning only in the loss free case. When there is loss, the $\Delta\beta_{\pm nm}$ have opposite imaginary parts leading to progressive conversion from linear to circular polarization (accompanied by rotation). Equations (2) and (3) obviously fail very near resonances. The full equation (1) has then to be solved. Formula (3) also fails near cutoff ($\lambda_g=\infty$). These, and other matters are to be discussed in a later paper. The authors are indebted to Dr. C. L. Hogan and Dr. A. D. Perry of these Laboratories for discussion of their work on ferrites, and for acquainting the authors with their theoretical work on transverse H_0.

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Radiation from an Electron in a Magnetic Field

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THIS problem was recently considered by Parzen[1] who concluded that quantum corrections to the classical results of Schwinger[2] and Schiff[3] should be appreciable at an electron energy of 200 mc^2 in a field of 10^4 gauss. The form of his correction is such that for this magnetic field the energy loss per turn by radiation would only increase as $R^{\frac{1}{2}}$ with increasing energy, instead of increasing as E^4/R, thus removing the stringent radiation limitation on synchrotron design.

An examination of Parzen's calculation reveals an invalid approximation.[4] More significantly, the assertion that on $l=0$ to $l=0$ transitions are appreciable can be shown to be incorrect by directly summing the series [Eq. (36)]. This yields

$$I(n'l|n0)=\frac{(-\alpha^2/2)^{l/2}}{\sqrt{l!}}I(n'0|n0) \quad (1)$$

in which the important values of $(\alpha^2/2)$ are of order unity. This result may be verified more easily by use of the energy eigenfunctions in Cartesian coordinates,[5] by the use of which the summation over l is implicitly performed.

If one now examines the formula for the power radiated in the orbital plane and makes use of these results, the exponential correction factor in Eqs. (26) cancels out and the classical result is obtained as sketched in the following:

$$I(n'0|n0)=(n'!/n!)^{\frac{1}{2}}(-\alpha^2/2)^{\lambda/2}\exp(-\alpha^2/2)L_{n'}^{\lambda}(\alpha^2/2) \quad (2a)$$
$$=\frac{(-1)^{\lambda/2}}{(n!n'!)^{\frac{1}{2}}}\int_0^\infty dt e^{-t}t^{n'+\lambda/2}J_\lambda(\alpha(2t)^{\frac{1}{2}}). \quad (2b)$$

Combining Eq. (2b) with the relation[6]

$$J_\lambda(\alpha(2t)^{\frac{1}{2}})=(1+\tau/n)^{\lambda/2}\sum_{m=0}^{\infty}\frac{(-\alpha\tau)^m}{(2n)^{m/2}m!}J_{\lambda+m}(\alpha(2n)^{\frac{1}{2}}), \quad (3)$$

where $t=n+\tau$, one obtains

$$I(n'0|n0)\backsimeq\frac{(-1)^{\lambda/2}}{(2\pi n)^{\frac{1}{2}}}\exp(-\lambda^2/4n')\int_{-n}^{\infty}d\tau\exp(-\tau^2/2n) \times\sum_{m=0}^{\infty}\frac{(-\alpha\tau)^m}{(2n)^{m/2}m!}J_{\lambda+m}(\alpha(2n)^{\frac{1}{2}}). \quad (4)$$

The major contribution to this integral comes in the region $|\tau|<(2n)^{\frac{1}{2}}$. Also, $\alpha(2n)^{\frac{1}{2}}=\lambda\beta$ in the orbital plane. In the summation, only about α-terms contribute so that the relevant values of m are of order $\alpha=\lambda\beta/(2n)^{\frac{1}{2}}\ll\lambda$. For these m, $J_{\lambda+m}(\lambda\beta)=J_\lambda(\lambda\beta)$ for $1-\beta^2\ll 1$, so that (4) becomes

$$\frac{(-1)^{\lambda/2}J_\lambda(\lambda\beta)\exp(-\lambda^2/4n')}{(2\pi n)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\exp[-(\tau^2/2n)-\alpha\tau/(2n)^{\frac{1}{2}}]d\tau \backsimeq\exp(-\alpha^2/4)J_\lambda(\lambda\beta).$$

The power radiated in the orbital plane is proportional to the sum

$$\sum_{l=0}^{\infty}|I(n'l|n+1,0)+I(n'l|n-1,0)|^2,$$

which may now be evaluated using Eq. (1) to give $[2J_\lambda'(\lambda\beta)]^2$, the classical result.

It is a pleasure to acknowledge helpful discussions with Professor E. M. McMillan, who brought this matter to our attention. We also wish to thank Professor L. I. Schiff for sending us his wave packet arguments[7] which confirms Schwinger's criterion for validity of the classical approximation.

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Radiation Loss of Electrons in the Synchrotron

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IN a recent paper of Parzen,[1] the radiation of electrons in uniform circular motion was calculated by means of the exact wave functions in a homogeneous magnetic field.

The result would seem to indicate a total radiation loss considerably smaller than to be expected on classical theory.[2,3]

This is very surprising since the main loss is due to the soft quanta, i.e., emissions which cause a relatively small change of the enormous quantum numbers involved in an orbit of macroscopic size at these energies. By the correspondence principle it is

then hard to understand how the classically computed intensity could be much off.

Let us therefore inspect the classical and quantum-mechanical results: On classical theory the energy radiated per unit solid angle and second, with wave number vector $\mathbf{k}$, is

$$P_{\mathrm{cl}}=\frac{c}{2\pi}\left|\frac{\omega}{2\pi}\int_0^{2\pi/\omega}dt[e\boldsymbol{\beta}\times\mathbf{k}]\exp\{i[kct-\mathbf{k}\cdot\mathbf{R}(t)]\}\right|^2$$
$$=\frac{ce^2k^2}{2\pi}\{\cot^2\Theta J_N^2(N\beta\sin\Theta)+\beta^2J_N'^2(N\beta\sin\Theta)\},\quad(1)$$

where $\beta=$velocity$/c$, $\omega=$mechanical frequency, and $N=ck/\omega$.

This is then to be compared with the quantum-mechanical expectation value for the number of photons of frequency ck emitted per second, multiplied by their energy, $\hbar ck$. But that is just the quantity obtained in the first approximation of the conventional perturbation theory, *viz.*,

$$P_{\mathrm{q.m.}}=\hbar ck(2\pi/\hbar)\Sigma|\mathcal{H}_{FA'}|^2\rho(E_F)\quad(2)$$

where $\mathcal{H}_{FA'}$ is the matrix element of single photon emission. To get the total intensity the sum must be taken, not only over the polarizations of the quantum, but also over all the final electron states of the same energy.

In the notation of Parzen's paper[1] the matrix element is

$$\mathcal{H}_{FA'}=e\omega b\left(\frac{\pi\hbar}{ckL^3}\right)^{\frac{1}{2}}\{\epsilon_{k\lambda}^{-}(n+1)^{\frac{1}{2}}I(n',l'|n+1,l)$$
$$+\epsilon_{k\lambda}^{+}n^{\frac{1}{2}}I(n',l'|n-1,l)\}$$

where

$$b^2=\hbar/m\omega_0=\hbar c/eH,$$
$$I(n'l'|nl)=\int u_{n'l'}^{*}\exp(-ik\rho\sin\Theta\cos\varphi)u_{nl}\rho d\rho d\varphi,$$
$$u_{nl}=\left(\frac{l!}{2\pi b^2n!}\right)^{\frac{1}{2}}\cdot\exp(-\rho^2/4b^2)(\rho^2/2b^2)^{\frac{1}{2}(n-l)}$$
$$\cdot L_l^{n-l}(\rho^2/2b^2)\cdot\exp[i(n-l)\varphi].$$

u_{nl} are the wave functions of the electron in the magnetic field, corresponding to circular orbits of radius $R_n=(2n+1)^{\frac{1}{2}}b$ whose centers lie on a circle of radius $(2l+1)^{\frac{1}{2}}b$. The energy eigenvalues

$$E_n=c[p_z^2+m^2c^2+m\hbar\omega_0(2n\pm1+1)]^{\frac{1}{2}}$$

are independent of the quantum number l.

Putting aside for the moment the question of whether the narrowest possible location of the orbit center is a proper representation of the experimental situation, we shall only consider the initial state, $l=0$, treated by Parzen, although the general case is also tractable.[3]

The integral $I(n'l'|n0)$ can be evaluated in closed form:

$$I(n'l'|n0)=\left(\frac{n'!}{n!l'!}\right)^{\frac{1}{2}}\cdot i^{n-n'-l'}\cdot\left(\frac{\alpha^2}{2}\right)^{\frac{1}{2}(n-n'-l')}$$
$$\cdot\exp(-\alpha^2/2)L_{n'}^{n-n'}(\alpha^2/2),$$
$$\alpha=kb\sin\Theta=(n-n')(\hbar\omega/c)b\sin\Theta.$$

Now we need the products and squares of such integrals summed over the quantum numbers l' of the final state. True enough, the term $l'=0$ which was the only one taken into account by Parzen, is the largest, but the others cannot be neglected:

$$\sum_{l'=0}^{n'}|I(n'l'|n0)|^2\approx\sum_0^{\infty}$$
$$=\frac{n'!}{n!}\left[\left(\frac{\alpha^2}{2}\right)^{\frac{1}{2}(n-n')}\cdot\exp(-\alpha^2/4)L_{n'}^{n-n'}(\alpha^2/2)\right]^2.$$

Introducing the harmonic number $N=n-n'$ and a mean velocity $\bar{\beta}=(\omega/c)[(R_n^2+R_{n'}^2)/2]^{\frac{1}{2}}$, it is easy to show[4] that for large n, n' and $N\ll n$, n' the sum is

$$\Sigma|I|^2=J_N(N\bar{\beta}\sin\Theta)+(\alpha^2/2)^{5/4}\cdot O(n'^{-\frac{3}{4}}).$$

Insertion of such expressions in (2) gives the classical result (1), with an error which is certainly negligible in the most important region of angle and frequency. This conclusion can also be shown to hold for an initial state compounded of various l's, but the analysis is then more laborious.[3]

We would like to thank Professor Dyson for calling our attention to the importance of this question for synchrotron design and also for suggesting the probable cause of the discrepancy between Parzen's result and the previous ones, which is indeed chiefly due to the terms $l'>0$.

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Electrical Properties of the Sodium Tungsten Bronzes

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RECENTLY Brown and Banks[1] have determined the resistivity of the sodium tungsten bronzes (Na_xWO_3, $x<1$) as a function of x and of the temperature. They confirmed the metallic nature of the conduction proposed earlier on the basis of resistance-temperature[2] and magnetic susceptibility[3,4] measurements, and ascribed conduction primarily to the free electrons contributed by Na. However, their results disagree quantitatively with those to be expected for pure sodium.

It is the purpose of this note to indicate that their results are not inconsistent with those to be expected for pure Na, distributed uniformly in a cubic mesh within the WO_3, under high (positive or negative) pressure. The same model accounts for the values of the Hall coefficient (Huibregtse, Barker, and Danielson[2]) and of the magnetic susceptibility (Stubbin and Mellor[3]) obtained for these bronzes.

It is often noted that many properties of the alkali metals depend primarily on the atomic volume, and much less on crystal structure. The recently established continuity of the compressibility of liquid and solid sodium[5] is an example of this behavior. Let us assume that pure Na occupies a fraction α of the total volume of the bronze crystal. The volume available to Na in the cubic WO_3 is roughly the difference between the total volume and the ionic volume of the constituents. In the perovskite structure it is very likely that the tungsten completely fills the space between its six surrounding oxygens. Using the linear increase in lattice dimensions with x[6] and an average ionic radius $r(O^{-2})=1.37$A one finds the volume per Na atom $v(\mathrm{Na})=19.6/x+2.9$A^3, assuming the ionic volume to be relatively independent of x. The fraction of the total volume occupied by Na is nearly independent of x: $\alpha=0.36$. The value $x=0.55$ corresponds to the normal density of metallic Na. If sodium has its normal properties in the volume it occupies in the lattice, the volume susceptibility of the bronze for $x=0.55$ should differ from that for pure Na by a factor α. Kupka and Sienko[4] found the volume susceptibility of the bronze ($x=0.55$) to be 0.25×10^{-6} cgs. Using $\chi=0.63\times10^{-6}$ cgs for pure sodium, the above considerations would lead to a value $\chi=0.23\times10^{-6}$ cgs for this bronze.

The resistivity minimum of the bronze for $x=0.70$ found by Brown and Banks corresponds to a pressure of 24,000 kg/cm^2, assuming the normal properties of metallic sodium ($\Delta v/v_0=-0.19$). At its minimum the resistivity is 0.57 of its value at $x=0.55$ (0°C). Since the pressure value depends on the ionic volume assumed, only its general magnitude is significant. Bridgman[7] has predicted the resistivity minimum for pure Na at a pressure of 28,000 kg/cm^2. He expects the minimum resistivity to be 0.44 of its normal value. Furthermore, the specific temperature coefficient of resistivity $(1/\rho)(d\rho/dT)$ of pure sodium is independent of pressure. This is in good agreement with the values of this coefficient as a function of x.[8]

The measured values of the electrical properties of the sodium bronzes must now be reinterpreted on the basis that sodium has its normal properties in the volume it occupies in the lattice.