strahlung cross sections for 60-Mev electrons incident on a thin Pb foil give results which are (8±3) percent low compared with the Bethe-Heitler theory for bremsstrahlung.

The results obtained by Walker[12] on the relative pair production cross sections for incident quanta of 17.6 Mev as presented give discrepancies which are somewhat larger than those given by recent absorption measurements and the results of this experiment. However, the results of Walker can be made compatible by using the Wheeler and Lamb theory for triplet production in making the correction to the number of pairs observed to obtain the number of pairs created in the field of the nucleus. The triplet correction term obtained experimentally by Walker is $0.8/Z$, although his data are not inconsistent with a correction term $1.1/Z$ which is given by the Wheeler and Lamb theory. By applying this latter correction to his data, the relative pair production cross section discrepancy for Pb to Al, Sn to Al, and Cu to Al becomes respectively (13±3), (8±4), and (3±3) percent. Further correction to Walker's results should be made to account for the fact that he measured only the pairs produced in the central half of the differential cross section ($0.25<E_1/k_0<0.75$), where the effects of screening are greater in proportion to the effects of screening where $E_1k_0<0.25$ and $E_1/k_0>0.75$, especially in the case of the lower Z elements. The effect of a correction of this type would be to reduce his values of relative pair production cross sections by another percent or two. If one makes these corrections, the results of Walker should be compatible with the results of absorption measurements and those of this experiment.

[12] R. L. Walker, Phys. Rev. 76, 1440 (1949).

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An Investigation into the Nuclear Scattering of High Energy Protons

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The angular distributions of high energy protons, elastically scattered from various nuclei, have been considered theoretically. For this purpose, the optical model of the nucleus, hitherto developed on the basis of the nuclear scattering of high energy neutrons, has been suitably modified to account for the proton scattering. Appropriate proton wave equations have been established and solved exactly. However, these solutions have been found unsuitable for numerical computations. Next, the WKB and the Born approximations have been used, within their respective regions of validity, to yield appropriate expressions for the differential scattering cross section. Finally, the theoretical and the experimental diffraction patterns for the nuclear scattering of 340-Mev protons have been compared. It has been found that the above nuclear model gives only an approximate account of such a scattering. It is further indicated that the effective nuclear radii, appropriate for this energy, are about 10 percent smaller than those calculated on the basis of the nuclear scattering of 90-Mev neutrons.

I. INTRODUCTION

THE experimental observations and the corresponding theoretical considerations on the scattering of high energy nucleons by nuclei have provided valuable information about nuclear structure. In this connection, the theoretical investigations have been carried out by making use of the optical model of the nucleus, employing the concept of a partially transparent nucleus as introduced by Serber.[1] Fernbach, Serber, and Taylor[2] have accounted for the nuclear scattering of 90-Mev neutrons on this basis and Fernbach[3] has extended the same considerations to

[1] R. Serber, Phys. Rev. 72, 1114 (1947).
[2] Fernbach, Serber, and Taylor, Phys. Rev. 75, 1352 (1949).
[3] S. Fernbach, *The Scattering of High Energy Neutrons by Nuclei* (unpublished Ph.D. dissertation, Physics Department, University of California, 1951), pp. 16–18.

give an approximate account of the nuclear scattering of 280-Mev neutrons.

Richardson, Ball, Leith, and Moyer[4] have recently observed the angular distributions for 340-Mev protons elastically scattered from various nuclei. It is the purpose of the present investigation to examine whether the above nuclear model, with reasonable modifications, can be employed to account for the nuclear scattering of such high energy protons.

II. THE OPTICAL MODEL OF THE NUCLEUS

A. The Optical Model for Neutron Scattering

In the optical model for the transparent nucleus, one regards the nucleus as a uniform sphere of nuclear matter, characterized by a complex refractive index, in which the real part is due to the effect of the nuclear potential well and the imaginary part is considered to be due to inelastic collisions. Using optical considerations as well as the equivalent WKB approximation, Fernbach, Serber, and Taylor[2] have obtained expressions for the total cross section σ_t and its two components, namely, the absorption cross section σ_a and the cross section for elastic scattering σ_e. These expressions involve the nuclear parameters $r_0=R/(A^{\frac{1}{3}}\times 10^{-13})$, (k_1/k), and $(K/2k)$, where R is the nuclear radius

$$k=(E^2-\mu^2c^4)^{\frac{1}{2}}/\hbar c, \tag{1}$$

$$\frac{k_1}{k}=\left\{\left[\frac{(E-V_n)^2-\mu^2c^4}{E^2-\mu^2c^4}\right]^{\frac{1}{2}}-1\right\}, \tag{2}$$

and

$$\frac{K}{2k}=\frac{\epsilon}{2k}\left[\frac{D}{2}(\sigma_{nn}+\sigma_{np})\right]. \tag{3}$$

In these equations, E is the energy of the incoming neutron and μ is its rest mass; V_n is the nuclear potential and D is the nucleon density, inside the nucleus; σ_{nn} and σ_{np} are the (n,n) and (n,p) scattering cross sections, respectively, and ϵ is a factor, smaller than unity, introduced to account for the effects due to the exclusion principle. In terms of these parameters, the complex refractive index is

$$n=1+(k_1/k)+i(K/2k). \tag{4}$$

The above expressions have been collected here, since they are useful in the present investigation. It may also be noted that the nuclear scattering of 90-Mev neutrons is reasonably accounted for by using the nuclear parameters $r_0=1.39$ cm, $k_1=3.3\times10^{12}$ cm^{-1}, and $K=3.0\times10^{12}$ cm^{-1}. At 280 Mev an approximate account of the scattering is given by using $r_0=1.39$ cm, $k_1=0$, and $K=2.5\times10^{12}$ cm^{-1}, although the theory is less successful at this energy than at 90 Mev.

[4] Richardson, Ball, Leith, and Moyer, Phys. Rev. **83**, 859 (1951).

B. The Optical Model for the Proton Scattering

It will be assumed that the only modification in the nuclear model required to account for the scattering of protons will be the inclusion of the electrostatic potential due to the nucleus. Since the optical model of the nucleus is based upon the tacit assumption that the nucleon density within the nucleus is uniform, one can replace V_n by

$$V_p=Ze^2/r \text{ for } r>R,$$

and

$$V_p=V_n+(3R^2-r^2)Ze^2/2R^3 \text{ for } r<R, \tag{5}$$

where Z is the atomic number of the nucleus.

One can now define a quantity k_2, analogous to k_1 on the basis of Eq. (2), by replacing the potential V_n therein by the new potential V_p. One can also obtain an expression for the complex refractive index for the scattering of protons by substituting k_2 in place of k_1 in Eq. (4). Then the magnitude of the momentum, which is now complex, is

$$k_c=k+k_2+\tfrac{1}{2}iK. \tag{6}$$

This expression will be used to obtain the appropriate proton wave equations.

III. THE THEORETICAL NUCLEAR SCATTERING OF HIGH ENERGY PROTONS

A. The Proton Wave Equations

As a reasonable wave equation for high energy protons one may take the Dirac equation with an additional term, due to Pauli,[5] to account for the anomalous magnetic moment. However, if one neglects this term for the present, one obtains the usual Dirac equation. Next, following Williams,[6] one can square the Dirac operator and obtain a second-order equation in which the term containing all the spin effects naturally separates out. This term, being of the same order of magnitude as the Pauli term, may be similarly neglected for the present. Thus one obtains the Klein-Gordon equation

$$(E-V)^2\psi=(c^2p^2+\mu^2c^4)\psi. \tag{7}$$

One can now substitute for $[(E-V)^2-\mu^2c^4]$ in the above equation the quantity $(\hbar ck_c)^2$ from Eq. (6). Neglecting small second-order terms and separating into spherical coordinates, one obtains the proton wave equations for the regions outside and inside the nucleus, respectively:

$$\left[\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d}{d\rho}\right)+1-\frac{2\chi}{\rho}-\frac{l(l+1)}{\rho^2}\right]W_l^0=0, \tag{8}$$

$$\left[\frac{1}{\rho'^2}\frac{d}{d\rho'}\left(\rho'^2\frac{d}{d\rho'}\right)+1+\frac{\bar{\chi}}{\eta'^3}\rho'^2-\frac{l(l+1)}{\rho'^2}\right]W_l^i=0. \tag{9}$$

In these equations, $\rho=kr$, $\rho'=k'r$, $\eta=kR$, and $\eta'=k'R$.

[5] W. Pauli, Revs. Modern Phys. **13**, 223 (1941).

[6] E. J. Williams, Proc. Roy. Soc. (London) **A169**, 531 (1938).

We also have $\chi=\alpha\gamma k_0/k$ and $\bar{\chi}=\alpha\bar{\gamma}k_0/k$, where $\alpha=Z/137$, $\gamma=E/\mu c^2$, $\bar{\gamma}=(E-\bar{V})/\mu c^2$, $k_0=\mu c/\hbar$, and $k'=\bar{k}+\frac{1}{2}iK$, in which $\bar{k}=[(E-\bar{V})^2-\mu^2c^4]^{\frac{1}{2}}/\hbar c$ and $\bar{V}=V_n+(3Ze^2/2R)$.

The errors introduced in the phase shifts by neglecting the various terms so far mentioned have been estimated and found to be negligible. The above equations can also be written as a single equation

$$\left[\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d}{d\rho}\right)+1-U-\frac{l(l+1)}{\rho^2}\right]W_l=0, \tag{10}$$

where $U=2\chi/\rho$ for $\rho>\eta$, and $U=-[(k'^2-k^2)/k^2]-[(\alpha\bar{\gamma}k_0k')/(k\bar{k}\eta^3)]$ for $\rho<\eta$.

Equations (8) and (9) will be used to obtain the exact solutions, while Eq. (10) will be useful in the WKB and the Born approximations for the scattering.

B. The Exact Phase Shifts

Only a sketch of the results of the exact treatment will be presented here. It can be shown that Eqs. (8) and (9), with suitable substitutions, can be brought into the form of the equation for the confluent hypergeometric functions, in Kummer's notation, as defined by Whittaker and Watson.[7] Imposing the requirements of continuity at the boundary on the resulting solutions, one obtains an exact expression for the phase shifts

$$\delta_l^e=\eta_l+\lambda_l, \tag{11}$$

where $\eta_l=\arg\Gamma(l+1+i\chi)$ and $\lambda_l=-\tan^{-1}(A_l/B_l)$. A_l and B_l, are given by

$$A_l=F_\zeta^0F^i-aF^0F_\xi^i+bF^0F^i,\quad B_l=G_\zeta^0F^i-aG^0F_\xi^i+bG^0F^i,$$

where the subscripts denote differentiation with respect to the corresponding variables. One also has

$$F^0=F(p,q,\zeta),\quad G^0=G(p,q,\zeta),$$
$$F^i=F(p',q',\xi),\quad G^i=G(p',q',\zeta),$$

where F and G, respectively, are the regular and irregular confluent hypergeometric functions. The parameters of these functions are given by $p=l+1+i\chi$, $q=2l+2$, $p'=[(2l+3)-i(\eta'^3/\bar{\chi})^{\frac{1}{2}}]/4$, $q'=(2l+3)/2$, $\zeta=-2i\eta$, and $\xi=-i(\eta'\bar{\chi})^{\frac{1}{2}}$; while $a=(\eta'\bar{\chi})^{\frac{1}{2}}/\eta$ and $b=\frac{1}{2}(a-1)$.

We have not found it possible in this investigation to compute the exact phase shifts, because of a lack of tabulated values for the functions involved. Also, neither the ascending nor the descending series of these confluent hypergeometric functions converge rapidly enough for numerical computations. Moreover, the exact solutions considered above are not necessarily exact representations of the true solutions for a physical nucleus, where the boundary is surely not as abrupt as in the nuclear model employed here. Thus it is both necessary and reasonable to use approximation methods for the treatment of this problem.

C. The WKB Approximation

Since one has $\eta\gg1$ for high energy protons, one may expect that the WKB approximation may be valid. The Phase shifts, using Langer's modification,[8] are given by

$$\delta_l^L=\lim_{\rho\to\infty}\left[\int_{\rho_1}^{\rho}(\rho^2-\rho^2U-\nu^2)^{\frac{1}{2}}\frac{d\rho}{\rho}-\int_{\nu}^{\rho}(\rho^2-\nu^2)^{\frac{1}{2}}\frac{d\rho}{\rho}\right], \tag{12}$$

where $\rho_1^2-\rho_1^2U(\rho_1)-\nu^2=0$ and $\nu=l+\frac{1}{2}$.

There is no reliable criterion for the validity of the above expression for phase shifts. However, the potential for the proton scattering consists of the Coulomb potential and an approximate square well. Exact expressions for the phase shifts are known both for the pure Coulomb potential and for the square-well potential separately. Thus, one may use these potentials as test cases to obtain some insight into the errors introduced by using the above approximation. In the former case, an analytical comparison showed that the errors diminish for large values of l, while a numerical comparison showed that, for $l=0$, the errors at 340 Mev are quite small for all nuclei. On the other hand, for the square well, analytical as well as numerical comparisons showed that the errors are small for small values of l, while for larger values of l, significant errors are encountered. The contributions to these errors due to the steepness of the boundary would be smaller for the physical nucleus where one may expect the potential to change gradually. However, even when the boundary is not steep, experience has shown[9] that this approximation overestimates positive phase shifts, particularly when they are smaller than about 0.5 radian. These errors may have some effect on the shapes of the theoretical diffraction patterns. In the present approximate investigation, all these errors will be neglected.

Now, as is done by Fernbach,[3] one may simplify the expression for the phase shifts given by Eq. (12), to obtain

$$\delta_l=-\tfrac{1}{2}\lim_{\rho\to\infty}\left[\int_{\nu}^{\rho}(\rho^2-\nu^2)^{-\frac{1}{2}}U\rho d\rho\right]. \tag{13}$$

Appropriate numerical comparisons for suitable test cases have shown that this simplification introduces additional errors which become significant only when l is close to η. These errors may have some effect on the value of σ_l, but they can only affect the diffraction patterns at very small angles. Such errors also will be neglected in the present investigation.

Substituting the expression under Eq. (10) for U in Eq. (13), one obtains the phase shifts for the proton

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[7] E. T. Whittaker and G. M. Watson, *A Course of Modern Analysis* (The McMillan Company, New York, 1946), pp. 337–354.

[8] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1949), p. 127.

[9] See reference 8, p. 213.

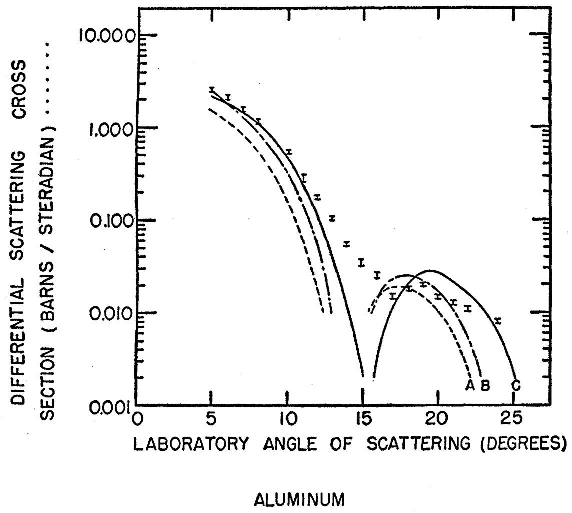

FIG. 1. The proton diffraction patterns for aluminum. Curves A and B, respectively, represent the WKB and the Born patterns with $r_0=1.39$ cm. Curve C represents the WKB pattern with $r_0=1.25$ cm.

scattering as

$$\delta_l=\delta_l'+i\delta_l'', \tag{14}$$

where, for $\nu>\eta$, $\delta_l'=\chi\ \ln\nu$, $\delta_l''=0$, and for $\nu<\eta$,

$$\delta_l'=\chi\ \ln[\eta+(\eta^2-\nu^2)^{\frac{1}{2}}]-(\eta^2-\nu^2)^{\frac{1}{2}}[u-v(\eta^2+2\nu^2)],$$
$$\delta_l''=w(\eta^2-\nu^2)^{\frac{1}{2}}.$$

In this, one has approximately

$$u=(\gamma k_0\bar{k}_0)/k^2, \quad v=(\alpha\gamma k_0)/(6k\eta^3),$$

and $w=K/(2k)$ with $\bar{k}_0=\bar{V}/\hbar c$.

Following the usual procedure[10] for such a modified Coulomb potential, one obtains the differential scattering cross section

$$\sigma(\theta)=[f(\theta)]^2+[g(\theta)]^2, \tag{15}$$

where

$$f(\theta)=-\frac{1}{2k}\Big\{\chi\csc^2\tfrac{1}{2}\theta\cos[\chi\ln(\csc^2\tfrac{1}{2}\theta)+2\eta_0] - \sum_{l=0}^{\bar{l}}(2l+1)[\exp(-2\delta_l'')\sin 2\delta_l' - \sin 2\eta_l]P_l(\cos\theta)\Big\},$$

$$g(\theta)=-\frac{1}{2k}\Big\{\chi\csc^2\tfrac{1}{2}\theta\sin[\chi\ln(\csc^2\tfrac{1}{2}\theta)+2\eta_0] + \sum_{l=0}^{\bar{l}}(2l+1)[\exp(-2\delta_l'')\cos 2\delta_l' - \cos 2\eta_l]P_l(\cos\theta)\Big\},$$

[10] Leonard I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), pp. 119–120.

and $\eta_l=\chi\ \ln\nu$ in this approximation, while $\bar{l}$ is the largest l less than η.

D. The Born Approximation

The usual criterion of validity for the Born approximation shows that this approximation can be used for the Coulomb scattering at 340 Mev in the case of light elements only. Moreover, experience has shown[9] that this approximation gives a good account of the scattering by a potential well when the phase shifts are smaller than about 0.5 radian. Positive phase shifts larger than this value are expected to be significantly underestimated. With reasonable values of the nuclear parameters, as obtained from the neutron scattering, one finds that a considerable number of phase shifts for light elements would be closely approximated by this approximation. Thus one may use the Born approximation for the proton scattering for such elements. In this case, one obtains an expression for the differential scattering cross section

$$\sigma(\theta)=[f(\theta)]^2+[g(\theta)]^2, \tag{16}$$

where

$$f(\theta)=\{[(a+3c\eta^2)\omega^2-6c]\sin(\omega\eta) - [(2\chi+a\eta+c\eta^3)\omega^3-6c\eta\omega]\cos(\omega\eta)\}/k\omega^5,$$

$$g(\theta)=\{b\omega^2\sin(\omega\eta)-b\eta\omega^3\cos(\omega\eta)\}/k\omega^5.$$

In these equations, $\omega=2\sin\frac{1}{2}\theta$, and, approximately, $a=-(2\gamma k_0\bar{k}_0)/k^2$, $b=K/k$ and $c=(\alpha\gamma k_0)/(k\eta^3)$.

One may also note here, that in this approximation σ_e for the neutron scattering can be expressed as

$$\sigma_e=\pi R^4(4k_1^2+K^2)/2, \tag{17}$$

where it is assumed that $\eta\gg 1$.

IV. COMPARISON WITH THE EXPERIMENTAL OBSERVATIONS

Aluminum and lead have been chosen as representative elements on which to compare the theoretical and the experimental diffraction patterns for 340-Mev protons. As an illustrative example, Fig. 1 shows such patterns for aluminum. The experimental values, together with probable errors, for $\frac{1}{2}°$ angular resolution, obtained by Richardson, Ball, Leith, and Moyer,[4] are indicated thereon. The WKB patterns for aluminum were computed with $r_0=1.39$ cm and various values of k_1 and K. It may be noted here that the potential V_n determines k_1. Curve A in Fig. 1 represents the WKB pattern for $k_1=0$ and $K=1.7\times 10^{12}$ cm^{-1} as a typical example. Curve B similarly represents the Born pattern with the same parameters.

One can immediately make the significant observation that the theoretical patterns exhibit rather deep minima, unlike the experimental patterns for both elements. If the absence of such deep minima in the observed patterns is not due to some undetermined experimental errors, one must ascribe this disagreement to some defect in the theoretical patterns. It seems unlikely that the use of approximation methods is responsible for this defect, Thus, one is led to the conclusion that the rather idealized nuclear model of a uniform sphere with a sharp boundary as used here, should be regarded only as an approximate representation of the physical nucleus.

In spite of this difficulty, it would be of some interest to obtain appropriate values of the nuclear parameters for this model. It has been found that both for aluminum and lead the theoretical patterns with the above parameters have their maxima at smaller angles than the corresponding maxima of the experimental patterns. In view of the errors discussed before, one would expect the WKB patterns to show some such deviation. However, it is difficult to understand why the Born pattern for aluminum deviates in the same direction. Moreover, this deviation for the WKB pattern of lead is not smaller than that for aluminum, and the deviation for lead at the third maximum is not smaller than that at the second maximum. This behavior is contrary to what one would expect if these deviations were due to errors in the WKB approximation. These arguments indicate that the above values of the nuclear parameters should be corrected. Modifications in the values of k_1 and K, within reasonable limits, have been found to have practically no influence on these deviations. However, such deviations can be removed by changing r_0. It has been estimated that within the limits of both theoretical and experimental uncertainties involved, it would be sufficient to take $r_0=1.25$ cm. This value somewhat overcompensates for this deviation in the case of aluminum, but it similarly undercompensates for it in case of lead.

With the above value of r_0, one can calculate K from Eq. (3), assuming $\sigma_{nn}=\sigma_{np}$ and using the experimental value of about 26 mb both for σ_{pp} and σ_{np}. Allowing less than 10 percent for the exclusion principle according to Fernbach's[3] prescription, one obtains $K=3.0\times10^{12}$ cm^{-1}, approximately. Since the Born approximation reasonably accounts for a number of the phase shifts for large values of l, which contribute to σ_e, one may use Eq. (17) with the above parameters together with $k_1=0$ to obtain $\sigma_e=95$ mb for the neutron scattering due to carbon, even though this nucleus is perhaps slightly small for the optical model to be strictly valid. The electrostatic potential inside the nucleus is small for this element, and thus would not affect this value significantly. Richardson[11] has estimated the corresponding σ_e by a graphical integration of the experimental pattern for carbon, after eliminating the small angle effects due to the electrostatic potential outside the nucleus. He obtains $\sigma_e=98$ mb in this case. This agreement shows that one may still take $k_1=0$, approximately.

[11] R. E. Richardson (to be published).

Extrapolating the curves of σ_t *versus* E prepared by Hildebrand,[12] approximate values of σ_t for the neutron scattering at 340 Mev were obtained for various elements. The corresponding WKB values of σ_t were computed with the above parameters from the formula given by Fernbach, Serber, and Taylor.[2] The calculated values are smaller by 18 percent for aluminum and by 10 percent for lead than the extrapolated experimental values. This underestimation may be largely due to the use of the rather simplified WKB approximation. Similar underestimation is shown by the comparison of the WKB and the exact calculations for 90-Mev neutrons scattered from aluminum as calculated by Pasternack and Snyder.[13] Thus, one may regard the above values of the nuclear parameters as appropriate for 340 Mev. Curve C in Fig. 1 shows the WKB pattern with the above parameters. In view of the various approximations involved, this curve gives a reasonable correlation with the experimental observations.

V. CONCLUSIONS

On the basis of the present investigation, one can conclude that the approximations inherent in the optical model of the nucleus, developed to account for the scattering of 90-Mev neutrons, become significant at higher energies of the order of 340 Mev. However, this model can be reasonably modified and appropriate nuclear parameters can be determined to give an approximate account of the scattering of 340-Mev protons. In this connection, it is indicated that the effective nuclear radii, appropriate for this energy, are about 10 percent smaller than those obtained from the neutron scattering at 90 Mev.

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[12] R. H. Hildebrand (private communication).

[13] S. Pasternack and H. S. Snyder, Phys. Rev. **80**, 921 (1950).