Small Angle Scattering of Light by a Coulomb Field

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The angular distribution for small angles and the total cross section for Delbrück scattering (the elastic scattering of γ-rays by the Coulomb field of heavy nuclei) are calculated approximately. The method is a combination of an impact parameter and an analytic continuation method. It is valid for energies $\hbar\omega$ large compared to the electron rest energy mc^2. Curves are given for the shapes of the dispersive and absorptive parts of the differential cross section, valid for angles of order $mc^2/\hbar\omega$ or less.

I. METHOD OF IMPACT PARAMETERS

THE preceding paper[1] dealt with the forward scattering of light by the Coulomb field of heavy nuclei. Unfortunately, these results do not permit direct experimental verification. Exact calculations at other angles meet with considerable difficulty. However, recent preliminary experiments[2] seem to indicate that data for the scattering of high energy γ-rays into small angles can be obtained.

It is the purpose of this paper to calculate the shape and magnitude of the differential cross section for small angles in an approximation valid for high energies. Since at these energies nearly all of the incident wave is scattered into very small angles, this calculation will also give us an estimate of the total cross section. Achieser and Pomerantschuk pursued the same aim in a very complicated calculation.[3] Their results will be compared with ours in the last section of this paper.

We start our discussion from the well-known scattering formula of quantum mechanics which we apply to high energy γ-rays. The scattered amplitude in direction θ is

$$f(\theta)=-\tfrac{1}{2}i\lambda\!\!\!^{-} \sum_l (2l+1)(c_l e^{2i\delta_l}-1)P_l(\cos\theta), \tag{1}$$

where $\lambda\!\!\!^{-}$ is the wavelength of the γ-ray divided by 2π, δ_l is the phase shift for angular momentum l, and c_l is the absolute value of the amplitude of the outgoing radial wave l. If only scattering could occur, c_l would be equal to unity and (1) would reduce to the ordinary Rayleigh formula. Since the γ-rays can also be absorbed, forming electron pairs, c_l is less than 1, in fact,

$$1-c_l^2=\gamma_l \tag{2}$$

gives the probability of absorption for γ-rays of angular momentum l. If we succeed in calculating $f(\theta)$ from (1), its absolute square will give the differential cross section for potential scattering of γ-rays.

We shall first use the fact that the absorption probabilities γ_l and the phase shifts δ_l are small, of order $e^2/\hbar c$ or less [see Eq. (18), below]. Then we may write

$$-i(c_l e^{2i\delta_l}-1)\equiv\alpha_l+i\beta_l, \tag{3}$$

$$\alpha_l\approx 2\delta_l, \tag{3a}$$

$$\beta_l=1-c_l\approx\tfrac{1}{2}\gamma_l. \tag{3b}$$

Next we note that an extremely large number of terms l contribute to the sum in (1), the number being of the order of $a/\lambda\!\!\!^{-}$ where a is the radius of the atom. (For 100-Mev γ-rays, this is about 10,000 terms.) Therefore the sum in (1) may be replaced by an integral over the impact parameter,

$$b=\lambda\!\!\!^{-} l, \tag{4}$$

and the spherical harmonics may be replaced by their asymptotic expression for large l and small θ,

$$P_l(\theta)\approx J_0(l\theta)=J_0(bs), \tag{5}$$

where

$$s=\theta/\lambda\!\!\!^{-}=k\theta \tag{5a}$$

is the momentum change of the γ-ray, $k=1/\lambda\!\!\!^{-}$ is the wave number. Then the scattered amplitude will depend on the frequency ω and on s, and we may write

$$f(\theta)=a_1(s,\omega)+ia_2(s,\omega), \tag{6}$$

$$a_1(s,\omega)=k\int bdb\alpha(b,\omega)J_0(bs), \tag{7a}$$

$$a_2(s,\omega)=k\int bdb\beta(b,\omega)J_0(bs), \tag{7b}$$

where the quantities α and β are given in Eqs. (3a, b) and are now regarded as functions of the impact parameter b (and ω).

The quantity $\gamma(b,\omega)=2\beta(b,\omega)$ is the probability of pair production by a wave packet of γ-rays of frequency ω, traversing an atom at a distance b from the nucleus. This is a well-defined physical quantity which can in principle be obtained from the theory of pair production. We shall show in Sec. III how this may be done, while in Sec. II we shall give a simple approximation for γ. In any case, the total cross section for pair production may be written

$$\sigma_{\text{pair}}(\omega)=2\pi\int bdb\gamma(b,\omega), \tag{8}$$

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[1] F. Rohrlich and R. Gluckstern, Phys. Rev. **86**, 1 (1952).
[2] R. R. Wilson, private communication.
[3] A. Achieser and I. Pomerantschuk, Physik. Z. Sowjetunion **11**, 478 (1937).

so that γ is related to the partial pair cross section for impact parameter b.

There is no such simple physical meaning for the phase shift δ, and hence for $\alpha(b, \omega)$. However, we shall obtain α by the principle of analytic continuation which was already used successfully in the preceding paper. In fact, we shall assume that $\alpha(b, \omega)+i\beta(b, \omega)$ for fixed b, is an analytic function of ω. Then, from the theory of analytic continuation,

$$\alpha(b, \omega)=\frac{\omega}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\beta(b, \omega')d\omega'}{(\omega'-\omega)\omega'}, \tag{9}$$

where P denotes the principal part.

Physically, our assumption means that we regard a *part* of the atom, *viz.*, the cylindrical ring between b and $b+db$, by itself as a dispersive medium for γ-rays. Rays of any frequency may be absorbed or scattered by this ring, and its refractive index may be obtained from the absorption coefficient by the usual formula of dispersion theory which is identical with (9). That it is possible to consider this ring independently of the rest of the atom depends on the possibility of constructing wave packets of γ-rays which only hit the ring but not (appreciably) the rest of the atom. For the validity of (9) it is necessary that such wave packets can be constructed for all frequencies ω' for which the absorption probability $2\beta(b, \omega')$ is appreciable. Here the particular properties of the pair production cross section are very helpful: $\sigma_{\text{pair}}(\omega)$ and hence $\gamma(b\ \omega)$ for all b are zero for $\omega<2mc^2$ and still small for frequencies slightly greater than $2mc^2$. Hence our wave packets need to be constructed only for wavelengths small compared to the Compton wavelength. But it is well known that wave packets can be constructed which define the position to about one wavelength. Therefore, if we permit the width of the ring db to be a Compton wavelength or larger, the wave packets can be constructed for all relevant values of ω'. But the pair production arises equally from all impact parameters b between the Compton wavelength and the atomic radius, and therefore a definition of the impact parameter to a Compton wavelength is sufficient.[4]

In order to use the formula (9), we must decide how β behaves for negative frequencies. But it was shown in the preceding paper that $a_2(s=0, \omega)$ is an odd function of ω. Since $k=\omega/c$, it then follows from (7b) that $\beta(b, \omega)$ is an even function of ω. This gives

$$\alpha(b, \omega)=\frac{\omega}{\pi}\mathrm{P}\int_0^{\infty}\beta(b, \omega')\left(\frac{1}{\omega'-\omega}+\frac{1}{\omega'+\omega}\right)\frac{d\omega'}{\omega'}. \tag{10}$$

In this way, we have expressed the amplitude of the potential (Delbrück) scattering of γ-rays, $f(\theta)$, in terms of the probability (partial cross section) for pair production as a function of frequency and impact parameter. The differential cross section for Delbrück scattering is of course

$$d\sigma_{\text{scatt}}/d\Omega=|f(\theta)|^2=a_1^2(\omega, s)+a_2^2(\omega, s). \tag{11}$$

II. CALCULATION USING A SIMPLE ASSUMPTION

We must now determine the absorption probability $\gamma(b, \omega)$. It was shown by Bethe[5] that the total pair production cross section can be written as an integral over the momentum transfer to the nucleus q, namely,

$$\sigma_{\text{pair}}=\int_0^{\infty}(dq/q)\varphi(q), \tag{12}$$

where $\varphi(q)$ is constant between

$$q_{\text{min}}=1/a \text{ or } 1/kb \text{ (whichever is larger)}$$

and

$$q_{\text{max}}=1/b_0. \tag{13}$$

Here

$$b_0=\hbar/mc \tag{13a}$$

is the Compton wavelength and a is the radius of the Fermi atom. For $q<q_{\text{min}}$ and $q>q_{\text{max}}$, φ falls off fairly rapidly.

The momentum given to the nucleus is approximately related to the impact parameter by $q=1/b$. Thus we find approximately

$$\sigma_{\text{pair}}=A\int_{b_0}^{1/q_{\text{min}}}db/b, \tag{14}$$

where A is a constant. In the case of no screening, the upper limit in (14) is kb_0^2 and

$$\sigma_{\text{pair}}=A\ \ln kb_0=A\ \ln(\hbar\omega/mc^2). \tag{15}$$

Comparison with the known pair cross section[6] shows that

$$A=(28/9)\bar{\phi} \quad\text{where}\quad \bar{\phi}=Z^2r_0^2/137 \tag{15a}$$

is Heitler's cross-section unit. On the other hand, σ_{pair} may be written in terms of β, using (2) and (3):

$$\sigma_{\text{pair}}(\omega)=2\pi\int bdb2\beta(b, \omega). \tag{16}$$

Comparison with (14) gives in the case of no screening

$$\begin{aligned}&\beta=A/4\pi b^2\equiv C/b^2 && \text{if}\quad b_0<b<b_{\text{max}}=kb_0^2,\\ &\beta=0 && \text{if}\quad b>kb_0^2=(\hbar\omega/mc^2)b_0.\end{aligned} \tag{17}$$

In the case of screening, kb_0^2 is replaced by a, the atomic radius. For $b<b_0$, (14) would give $\beta=0$ but it is more accurate to take

$$\beta=C/b_0^2. \tag{17a}$$

[4] It is necessary for this argument to use the impact parameter b, not the angular momentum l. It would be impossible to construct a physical region of the atom (dispersive ring) which would correspond to the same l for all $\lambda\!\!\!^-$.

[5] H. A. Bethe, Proc. Cambridge Phil. Soc. **30**, 524 (1934).

[6] See W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, London, 1945). Note that the additive constants are here included in the logarithm.

The coefficient

$$C = A/4\pi = (7/9\pi)\bar{\phi}. \tag{17b}$$

Thus the maximum pair production probability, for $b < b_0$, is

$$\gamma = 2\beta = (14/9\pi)(Z^2 r_0^2/137 b_0^2) = (14/9\pi)(Z^2/137^3), \tag{18}$$

which is 1/600 for uranium. This justifies our assumption in Eq. (2) that γ is small.

For a given b we may consider β as a function of ω and obtain for $b_0 < b < a$:

$$\begin{aligned} \beta &= C/b^2 \quad \text{if} \quad \omega > \omega_{\min}(b) \\ \beta &= 0 \quad \text{if} \quad \omega < \omega_{\min}(b), \end{aligned} \tag{19}$$

where

$$\omega_{\min}(b) = ck_{\min}(b); \quad k_{\min}(b) = b/b_0^2. \tag{19a}$$

Further,

$$\beta = 0 \quad \text{for} \quad b > a, \quad \text{any } \omega \tag{19b}$$

$$\beta = C/b_0^2 \quad \text{for} \quad b < b_0, \quad \omega > c/b_0. \tag{19c}$$

Equation (19) may now be used to obtain α by the analytic continuation formula (10); this gives

$$\alpha(b,\omega) = \frac{C}{\pi b^2} \ln \frac{\omega + \omega_{\min}(b)}{|\omega - \omega_{\min}(b)|} \tag{20}$$

$$= \frac{C}{\pi b^2} \ln \frac{k + b/b_0^2}{|k - b/b_0^2|} \quad \text{for} \quad b_0 < b < a, \tag{20a}$$

$$\alpha = 0 \quad \text{for} \quad b > a,$$

and b is to be replaced by b_0 in (20) for $b < b_0$.

We can now insert α and β into (7) to calculate the amplitudes for a given scattering angle. Putting $b/b_0 = t$ and $sb_0 = x$, we find from (7b) and (19)

$$a_2(s,\omega) = Ck\left[\int_0^1 tdtJ_0(xt) + \int_1^\infty (dt/t)J_0(xt)\right]. \tag{21}$$

The upper limit of the integral has been put equal to ∞, whereas the correct limit would be, according to (17),

$$t_{\max} = kb_0 \quad \text{(no screening)},$$

$$t_{\max} = a/b_0 \quad \text{(screening)}.$$

This would make a difference only for very small $x(<1/t_{\max})$ and thus for very small scattering angles:

$$\theta = 1/(kb_0)^2 \quad \text{(no screening)},$$

$$\theta = 1/ka \quad \text{(screening)}.$$

However, the correct $t_{\max}$ would have to be used to obtain the scattering in the forward direction for comparison with the preceding paper.

The first integral is $J_1(x)/x$, whereas the second integral cannot be written in closed form. Since x is small we can expand in a power series and find

$$\int_x^\infty \frac{du}{u} J_0(u) = \text{const} - \ln x + \frac{x^2}{8} - \frac{x^4}{2^8} + \frac{x^6}{2^9 \cdot 3^3} - \cdots. \tag{21a}$$

The constant occurring here is found by considering the case of very small x,

$$\begin{aligned} \int_x^\infty \frac{du}{u} J_0(u) &= \frac{2}{\pi}\int_0^{\pi/2} d\phi \int_x^\infty \frac{du}{u}\cos(u\cos\phi) \\ &= (2/\pi)\int_0^{\pi/2} d\phi\, \mathrm{Ci}(x\cos\phi) \\ &= -(2/\pi)\int_0^{\pi/2} d\phi \ln(\gamma x\cos\phi) = \ln(2/\gamma x). \end{aligned} \tag{22}$$

The constant is therefore $\ln(2/\gamma) = 0.116$ and we have the result

$$a_2(s,\omega) = Ck\left(\ln\frac{2}{\gamma} - \ln x + \tfrac{1}{2} + \frac{x^2}{16} - \frac{x^4}{2^8 \cdot 3} + \frac{x^6}{2^{11} \cdot 3^3} - \cdots\right) \equiv CkF_2(x). \tag{23}$$

We now turn to the evaluation of $a_1(s,\omega)$. Using (20) and inserting $\omega_{\min}$ from (19) we get

$$a_1(s,\omega) = \frac{Ck}{\pi}\left[\int_0^1 tdtJ_0(xt) \ln\frac{kb_0 + t}{|kb_0 - t|} + \int_1^\infty \frac{dt}{t} J_0(xt) \ln\frac{kb_0 + t}{|kb_0 - t|}\right]. \tag{24}$$

For $\hbar\omega \gg mc^2$ the logarithm can be approximated by $2t/kb_0$ and

$$\begin{aligned} a_1(s,\omega) &= (C/\pi b_0)\left[(2/x)\int_0^\infty J_0(u)du + 2\int_0^x J_0(u)du(u^2x^{-3} - x^{-1})\right] \\ &= \frac{C}{\pi b_0}\left(\frac{2}{x} - \frac{4}{3} + \frac{x^2}{3 \cdot 5} - \frac{x^4}{2^4 \cdot 5 \cdot 7} + \cdots\right) \\ &\equiv (C/b_0)F_1(x). \end{aligned} \tag{25}$$

The total cross section is

$$\sigma = 2\pi \int_0^\pi |a_1(\theta) + ia_2(\theta)|^2 \sin\theta d\theta \equiv \sigma_1 + \sigma_2. \tag{26}$$

From the preceding results we find with

$$x = kb_0\theta = (\hbar\omega/mc^2)\theta, \tag{26a}$$

$$\sigma_1 = 2\pi(C/b_0)^2(mc^2/\hbar\omega)^2\int_0^\infty F_1^2(x)xdx,$$

$$\sigma_2 = 2\pi(C/b_0)^2\int_0^\infty F_2^2(x)xdx. \tag{27}$$

Obviously, σ_2 is by far larger than σ_1. The integral occuring in σ_2 can be evaluated, using the definition (21) and the orthogonality relation of the Bessel functions

$$\int xdxJ_0(xt)J_0(xt') = t^{-1}\delta(t'-t). \tag{27a}$$

This yields

$$\int F_2^2xdx = \int_0^1 tdt + \int_1^\infty dt/t^3 = 1. \tag{28}$$

In the case of σ_1, an approximate evaluation is elementary, *viz.*,

$$\int F_1^2xdx = (2/\pi)^2\int_{x_1}^\infty xdx/x^2 = 4\pi^{-2}\ln(1/x_1), \tag{28a}$$

where x_1 is the smallest value of x for which (21) is valid, *viz.*, $x_1 = b_0/a$ for complete screening and $x_1 = 1/kb_0$ without screening. Therefore we get, using (17b),

$$\begin{aligned}\sigma_2 &= 2\pi C^2/b_0^2 = (98/81\pi)(\bar{\phi}^2/b_0^2)\\ &= 0.385(\alpha Z)^4 r_0^2,\end{aligned} \tag{29a}$$

$$\sigma_1 = 3\pi^{-2}(mc^2/\hbar\omega)^2\sigma_2\ln(\hbar\omega/mc^2) \quad \text{without screening,} \tag{29b}$$

$$= 3\pi^{-2}(mc^2/\hbar\omega)^2\sigma_2\ln(a/b_0) \quad \text{complete screening.} \tag{29c}$$

The numerical factor in (29a) should not be taken too seriously because the integral in (28) depends appreciably on the region $t<1$ for which our theory is only approximate. Likewise, in (29b, c) an unknown constant should be added to the logarithms.

Within the range of validity of these formulas, i.e., for $\hbar\omega \gg mc^2$, the total cross section is almost entirely the result of the "absorptive" part of the scattering amplitude a_2. The ratio of the potential scattering to the pair production cross section is, from (15) and (17b):

$$\frac{\sigma_2}{\sigma_{\text{pair}}} = \frac{C}{2b_0^2\log} = \frac{7\alpha(Z\alpha)^2}{18\pi\log} = \frac{0.124\alpha(Z\alpha)^2}{\log}, \tag{30}$$

where log is the logarithm in the pair production cross section, Eq. (15).

III. THEORY OF THE IMPACT PARAMETER METHOD

We shall now give a more accurate mathematical theory of the impact parameter method which we have used, and we shall show how the scattered amplitudes $a_2(s, \omega)$ could, at least in principle, be calculated exactly.

For this purpose, we construct a γ-ray wave packet traversing the atom at a distance b from the nucleus by setting

$$\psi = (2\pi)^{-1}v^{-\frac{1}{2}}\int \exp[i(\mathbf{k}_0+\mathbf{u})\cdot\mathbf{r}]\varphi(\mathbf{u})du_xdu_y, \tag{31}$$

where φ may for instance be chosen as

$$\varphi \equiv \exp(-i\mathbf{u}\cdot\mathbf{b})\chi(u) = \alpha\pi^{-\frac{1}{2}}\exp[-i\mathbf{u}\cdot\mathbf{b} - \tfrac{1}{2}\alpha^2u^2]. \tag{32}$$

The wave vector $\mathbf{k}_0$ is fixed, the added wave vector $\mathbf{u}$ is assumed perpendicular to $\mathbf{k}_0$ which is taken in the z-direction. Evaluation of (31) gives

$$\psi = (\pi v)^{-\frac{1}{2}}\alpha^{-1}\exp[i\mathbf{k}_0\cdot\mathbf{r} - (\boldsymbol{\rho}-\mathbf{b})^2/2\alpha^2] \tag{33}$$

where $\boldsymbol{\rho}$ is the component of $\mathbf{r}$ in the xy-plane. Thus ψ is indeed confined to impact parameters near $\mathbf{b}$, within an accuracy α. Further, the magnitude of the wave vector $\mathbf{k}_0+\mathbf{u}$ is essentially the same as that of k_0 provided α is not too small; e.g., the choice $\alpha = b_0$ will make $\mathbf{k}_0+\mathbf{u}$ differ from k_0 only by an amount of relative order $(mc^2/\hbar\omega)^2$.

Generally, φ is assumed normalized,

$$\int |\varphi(u)|^2du_xdu_y = \int |\chi(u)|^2du_xdu_y = 1, \tag{34}$$

so that ψ is normalized to one particle incident per unit time. If then the pair production probability is calculated per unit time, as usual, this will give directly the probability $\gamma(b, \omega)$ used in Sec. I.

Consider now the matrix element for the production of a pair of electrons of momenta $\mathbf{p}_+$, $\mathbf{p}_-$ such that

$$E_+ + E_- = k_0. \tag{35}$$

Let us denote this matrix element by

$$M(\mathbf{k}, \mathbf{p}_+, \mathbf{p}_-, \mu, \sigma_+, \sigma_-). \tag{36}$$

It is a function of the quantum $\mathbf{k}$, and its polarization μ, of the spins of the two electrons σ_+, σ_- and, of course, of their momenta. Then the matrix element for producing the same electrons by the wave packet (31) is

$$N = (2\pi)^{-1} \times \int du_xdu_y\varphi(u)M(\mathbf{k}_0+\mathbf{u}, \mathbf{p}_+, \mathbf{p}_-, \mu, \sigma_+, \sigma_-). \tag{37}$$

To get the total pair production probability 2β for the wave packet (31), $|N|^2$ must be summed over σ_+ and σ_-, averaged over μ, and integrated over the directions of $\mathbf{p}_+$ and $\mathbf{p}_-$ and finally over the magnitude of one of them. This defines the $\beta(b, \omega)$ used in Sec. I in terms of mathematical operations which can all be carried out.

To get any further, let us introduce the vectors

$$\mathbf{q}_0 = \mathbf{p}_+ + \mathbf{p}_- - \mathbf{k}_0 \tag{38}$$

and

$$\mathbf{w} = (\mathbf{p}_+ - \mathbf{p}_-)_\perp, \tag{38a}$$

where $\perp$ denotes the component in the plane perpendicular to $\mathbf{k}_0$. It is easily seen that also $\mathbf{q}_0$ is nearly in this plane.[5] Then, if the magnitude of $\mathbf{p}_+$ [and hence also $\mathbf{p}_-$ from (35)] is given, $\mathbf{q}_0$ and $\mathbf{w}$ determine the vectors $\mathbf{p}_+$ and $\mathbf{p}_-$. The matrix element in (37) may then be written

$$M(\mathbf{q}_0 - \mathbf{u}, \mathbf{w}, \text{spins}), \tag{39}$$

and $\mathbf{w}$ is independent of $\mathbf{u}$. It is worth noting[5] that the actual differential cross section and matrix element depend most strongly on $(\mathbf{q}_0 - \mathbf{u})$ and less on $\mathbf{w}$.

The pair production probability for our wave packet is now

$$2\beta = (2\pi)^{-2} \int dE_+ \sum\nolimits_{\text{spin}} \times \int d^2\mathbf{u} d^2\mathbf{u}' d^2\mathbf{w} d^2\mathbf{q}_0 \varphi(\mathbf{u}) \varphi^*(\mathbf{u}') \times M(\mathbf{q}_0 - \mathbf{u}, \mathbf{w}, \text{spin}) M^*(\mathbf{q}_0 - \mathbf{u}', \mathbf{w}, \text{spin}), \tag{40}$$

where $d^2\mathbf{u}$ denotes integration over $du_x du_y$. Equation (40) could be evaluated, but it is easier to go directly to the scattered amplitude given in (7b), which becomes

$$a_2(s) = (2\pi)^{-1} k \int d^2\mathbf{b} \exp(i\mathbf{b}\cdot\mathbf{s}) \beta(\mathbf{b})$$

$$= \tfrac{1}{2} k (2\pi)^{-3} \int dE_+ \sum\nolimits_{\text{spin}} \int d^2\mathbf{b} d^2\mathbf{u} d^2\mathbf{u}' d^2\mathbf{w} d^2\mathbf{q}_0 \times \exp(i\mathbf{b}\cdot\mathbf{s}) \varphi(\mathbf{u}) \varphi^*(\mathbf{u}') \times M(\mathbf{q}_0 - \mathbf{u}, \mathbf{w}, \text{spin}) M^*(\mathbf{q}_0 - \mathbf{u}', \mathbf{w}, \text{spin}). \tag{41}$$

Remembering now (32), the factors containing b can be integrated:

$$\int d^2b \exp[i\mathbf{b}\cdot(\mathbf{s} + \mathbf{u}' - \mathbf{u})] = (2\pi)^2 \delta(\mathbf{s} + \mathbf{u}' - \mathbf{u}). \tag{42}$$

The integration over $\mathbf{u}'$ can then be carried out. At the same time, we introduce

$$\mathbf{q} = \mathbf{q}_0 - \tfrac{1}{2}(\mathbf{u} + \mathbf{u}'); \tag{43}$$

then we have

$$a_2(s) = (k/4\pi) \int dE_+ \sum\nolimits_{\text{spin}} \times \int d^2\mathbf{u} d^2\mathbf{w} d^2\mathbf{q} \chi(\mathbf{u}) \chi(\mathbf{u} - \mathbf{s}) \times M(\mathbf{q} - \tfrac{1}{2}\mathbf{s}, \mathbf{w}, \text{spin}) M^*(\mathbf{q} + \tfrac{1}{2}\mathbf{s}, \mathbf{w}, \text{spin}). \tag{44}$$

All reference to the auxiliary variable u has now been eliminated from the matrix elements M. The integral over $\mathbf{u}$ is elementary but can be further simplified by assuming that α in (32) is very small. This means very sharp definition of the γ-ray beam in space, and is indeed a necessary assumption if the first line of Eq. (41) is to hold exactly: if α were larger, the various wave packets (31) for different b would not be sufficiently orthogonal. Assuming α small, we can put $\chi(\mathbf{u} - \mathbf{s}) = \chi(\mathbf{u})$ and the u-integral reduces to the normalization integral (34). Thus we get the final result

$$a_2(s) = (k/4\pi) \int dE_+ \sum\nolimits_{\text{spin}} \times \int d^2\mathbf{q} d^2\mathbf{w} M(\mathbf{q} - \tfrac{1}{2}\mathbf{s}, \mathbf{w}, \text{spin}) \times M^*(\mathbf{q} + \tfrac{1}{2}\mathbf{s}, \mathbf{w}, \text{spin}), \tag{45}$$

which is entirely independent of the original wave packet assumption. For $s = 0$ we get simply

$$a_2(0) = (k/4\pi) \sigma_{\text{pair}}, \tag{46}$$

which was already used in the preceding paper.

Equation (45) could again be evaluated exactly, and $a_1(s)$ could then be obtained by analytic continuation. We shall here be content with the following rough calculation. The matrix element $M(q_1, w, \text{spin})$ is, for any important w and spin, found to be about inversely proportional to q_1 provided $q_{\min} < q_1 < q_{\max}$ as defined in (13), and decreases more rapidly (as q_1^{-2}) for $q > q_{\max} = 1/b_0$. Indeed, if this is assumed, the contribution to the pair cross section from a given q turns out to be proportional to

$$d^2\mathbf{q} \, |M(q, w, \text{spin})|^2 \sim q dq / q^2 \tag{46a}$$

as it should be according to (12). Then we can integrate (45) over all variables except q and obtain approximately

$$a_2(s) = Bk \int_{q_{\min}}^{q_{\max}} \frac{d^2\mathbf{q}}{|\mathbf{q} - \frac{1}{2}\mathbf{s}| \, |\mathbf{q} + \frac{1}{2}\mathbf{s}|} = Bk \int \frac{q dq d\phi}{[(q^2 + \frac{1}{4}s^2)^2 - q^2 s^2 \cos^2\phi]^{\frac{1}{2}}}, \tag{47}$$

where B is a constant and k the wave number. Now if $q_{\min} \ll s \ll q_{\max}$, the integral over ϕ gives a result of order $1/s^2$ for $q < s$, and of order $1/q^2$ for $q > s$. Hence

$$a_2(s) \sim k \left[\int_0^s \frac{q dq}{s^2} + \int_s^{1/b_0} \frac{q dq}{q^2} \right] = k \left[\ln \frac{1}{b_0 s} + \tfrac{1}{2} \right], \tag{48}$$

in agreement with Eq. (23). This justifies the assumptions made in the simple theory of Sec. II.

IV. DISCUSSION OF THE RESULTS

The angular distribution of the dispersive and absorptive amplitude is given by $F_1(x)$ and $F_2(x)$ of Eqs. (25) and (23), respectively. The shape of the differential cross section is plotted in Fig. 1 as a function of $x=sb_0=(\hbar\omega/mc^2)\theta$.

For very small angles, $\theta<\lambda/b_{\max}=(mc^2/\hbar\omega)b_0/b_{\max}$ with $b_{\max}$ given by (17), these functions do not give the correct angular distribution, since large impact parameters were incorrectly treated by extending the integral in (21) to infinity. This could easily be remedied if desired, and even screening could be included. In the limit of zero momentum transfer $x=0$, and if $t_{\max}=kb_0$ is inserted as the upper limit in (21), the results of the preceding paper[1] are approximately recovered.

At large angles, $\theta\gg mc^2/\hbar\omega$, Eqs. (25) and (23) will not be accurate because then the main contribution comes from small impact parameters $b<b_0$ and Eq. (17a) is not a good approximation. For the "absorptive" part $a_2(s)$, the method of Sec. III would still give correct results as long as θ is small compared with one radian. However, for the dispersive part this will not work because for the small b which are now important, the pair production starts at low energy and thus at wavelengths λ comparable to b; therefore the uncertainty principle will prevent definition of the impact parameter for frequencies ω' which contribute materially to the analytic-continuation integral (10), and a_1 cannot be reliably calculated. On the other hand, this does not make much difference for the scattering because a_1 is very small compared to a_2 which is calculable.

The differential cross section is determined essentially by F_2^2 in Fig. 1, because it was shown in (29), (25) that the dispersive part F_1^2 is relatively unimportant. As is seen from Fig. 1, F_2^2 increases substantially with decreasing x, namely by about a factor 20 between $x=1$ and $x=0.1$; according to Eq. (23), F_2^2 is proportional to $(\log 1/x)^2$. This increase is in contrast to the angular distribution of the pair electrons produced by the γ-ray which is essentially constant for $x<1$, i.e., for angles less than $mc^2/\hbar\omega$.

In Fig. 1, we have also plotted for comparison the diffraction scattering from a black sphere. The radius of this sphere was arbitrarily chosen as $7b_0$ because the part of the atom responsible for pair production and potential scattering extends from b_0 to the atomic radius, and $7b_0$ may be a reasonable average. The black-sphere scattering is then $(J_1(7x)/7x)^2$; the coefficient 40 in the figure was also chosen arbitrarily. It is seen from Fig. 1 that F_2^2 agrees with the black-sphere curve reasonably well in the mean, but that F_2^2 is higher both at very small and at large x. This is understandable because the scattering is really not from a black sphere of definite radius but from a "gray" sphere with opacity varying with radius; at small x, the outer parts of the atom become effective and increase the scattering because of the large radii involved; at large x, where a sphere of radius $7b_0$ would give very little scattering because of interference, the particularly black region near b_0 still gives appreciable scattering.

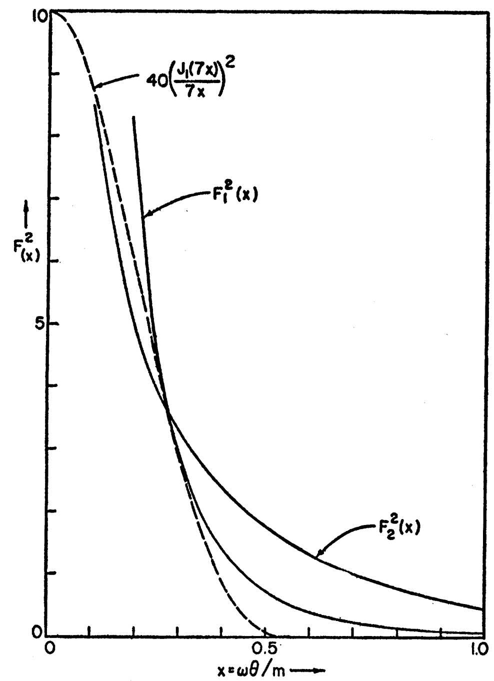

FIG. 1. Shape factor $F^2(x)$ of the angular distribution. The subscripts 1 and 2 refer to the dispersive and absorptive parts, respectively. For comparison the diffraction from a black sphere of radius $7b_0$ is plotted on arbitrary scale; it is $40[J_1(7x)/7x]^2$.

The dispersive scattering F_1^2 falls off still more rapidly with increasing θ than F_2^2. This is because, for any impact parameter b, the dispersive amplitude a_1 is largely determined by the ratio $\omega_{\min}(b)/\omega$ where $\omega_{\min}(b)$ is the frequency at which γ-rays passing at impact parameter b begin to produce pairs appreciably. The larger this ratio $\omega_{\min}(b)/\omega$ (as long as it is less than unity), the greater will be the dispersive effect; since $\omega_{\min}(b)$ is proportional to b according to Eq. (19a), the larger b give a greater dispersive effect as can be seen from (20); but the larger b will contribute to the scattering only at the smallest angles.

The energy dependence of the scattered amplitudes at a given finite s is

$$a_1(s)\sim\text{const},\quad a_2(s)\sim\omega\quad\text{if}\quad \hbar\omega\gg mc^2, \tag{49}$$

whereas at zero momentum transfer the preceding paper[1] gives the result

$$a_1(0)\sim\omega,\quad a_2(0)\sim\omega\ln\omega. \tag{50}$$

This is without screening; with screening this will be replaced by

$$a_1(0)\sim\omega_0,\quad a_2(0)\sim\omega\ln\omega_0, \tag{51}$$

where $\omega_0=\omega_{\min}(a)$ [see Eq. (19a)], with a the atomic radius. The energy dependence of (51) is the same as that of (49), but that of (50) is not. This is because in the absence of screening, the step increase of a_1, a_2 with decreasing x will continue down to $x_1 = b_0/b_{\max}(\omega)$, see Eq. (17), and will then flatten out; this, in combination with the angular dependence just discussed, will give the energy dependence (50).

Achieser and Pomerantschuk,[3] in a rather complicated paper, calculated only the dispersive part of the cross section. In the high energy limit, they find an angular dependence $1/\theta^2$ in agreement with our result $F_1^2 \sim 1/x^2$. Also the total dispersive cross section obtained by them,

$$\sigma_1 = \beta(\alpha Z)^4(mc^2/\hbar\omega)^2 r_0^2 \ln(\hbar\omega/mc^2) \qquad (52)$$

with β an unknown numerical factor, agrees with our result for this quantity, (29b).

The absolute value of the total cross section is given by (29a) which gives about 6 millibarns for uranium. According to (30), its ratio to the pair production cross section is about 1/8000. This factor is made up of a factor α, a factor $(\alpha Z)^2$, a small numerical factor of about $\frac{1}{8}$, and 1/log. Except for energies above about 10^{10} ev, the total cross section for potential scattering is much less than that for Compton scattering.

This is different for the differential cross section, because of the large forward maximum. According to (23), (17b), the differential cross section per unit solid angle is

$$\sigma(\theta) = (7/9\pi)^2(Z\alpha)^4 r_0^2(\hbar\omega/mc^2)^2 F_2^2(x) \qquad (53)$$

which, for uranium, is

$$\sigma(\theta) = 1.0(\hbar\omega/mc^2)^2 F_2^2(x) \text{ millibarns/steradian.}$$

For 300 Mev and $x = 0.1$, corresponding to $\theta = 0.01°$, this gives 3000 barns per steradian. For the Compton effect, the differential cross section is Zr_0^2 in the forward direction, independent of energy; this is 7 barns/steradian for uranium. Rayleigh scattering is $Z^2 r_0^2$ in the exact forward direction, and much less for $x = 0.1$ which represents a large angle for Rayleigh scattering. Therefore scattering of γ-rays at such high energies and small angles represents mainly potential scattering.

For the experimental observation, the main requirements are excellent collimation and elimination of all electrons from pair production and of the secondary γ-rays emitted by these electrons. The latter can best be done by using the different angular distribution. Unfortunately, since the scattering is mainly absorptive, its observation at high energy would not reveal anything about vacuum polarization, but would merely test the theory of the spatial distribution of pair production inside the atom.