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Forward Scattering of Light by a Coulomb Field

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The complex scattering amplitude for the scattering of light by a Coulomb field (Delbrück scattering) in the forward direction is calculated exactly by two methods. First, the Feynman method is used and necessitates very tedious and complicated calculations. Then, the method of analytical continuation is applied to the pair production cross section and yields the same result much more easily. An exact analytical expression is obtained for the dispersive and absorptive parts of the amplitude. The result is plotted as a function of energy. The low energy limit agrees with previous calculations for this limit by Kemmer and Ludwig.

INTRODUCTION

ONE of the most interesting predictions made by quantum electrodynamics is the scattering of light by light.[1] This effect is in contradiction to the classical Maxwell equations which, because of their linearity, cannot account for it. Weisskopf[2] has shown that a nonlinear correction to the Maxwell equations will do justice to the phenomenon.

Experimental verification of this effect proved impossible because of its extremely small size in a crossed beam experiment. Considerably more fruitful is the idea of observing this process in a modified form in which light is scattered by a static electromagnetic field. The strong Coulomb fields of heavy nuclei seem by far the best targets. The occurrence of the scattering of light by a Coulomb field was first suggested by Delbrück,[3] and the effect is often referred to as Delbrück scattering. But until recently the effect remained of academic value only.

A few years ago renewed interest in the problem came from the new formulations of quantum electrodynamics. The only two observable processes involving closed electron loops in the Feynman diagrams are the polarization of the vacuum and the scattering of light by light. The former was verified experimentally beyond any doubt by the well-known Lamb-Retherford experiment on the level shift of the hydrogen atom and is of second order.

The Delbrück effect can be regarded as a photon self-energy effect in a Coulomb field. A method based on this view point could actually be used to calculate the process. But there seem to be no essential advantages in the method.

Very recent experiments by Wilson[4] in this laboratory seem to confirm the existence of the Delbrück effect. However, more complete calculations will be necessary for a quantitative analysis of these observations.

The foregoing remarks and the recent great improvements in calculating technique may justify the following very long and tedious calculations. Earlier investigations by Achieser and Pomerantschuk[5] and by Kemmer and Ludwig[6] yielded results which will be confirmed and augmented in this work and in a paper by Bethe and Rohrlich[7] immediately following this paper.

Finally, the present work shows and confirms a successful application of the extremely elegant method of

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[1] O. Halpern, Phys. Rev. 44, 855 (1933); H. Euler and B. Kockel, Naturwiss. 23, 246 (1935); H. Euler, Ann. Phys. 26, 398 (1936); R. Karplus and M. Neuman, Phys. Rev. 80, 380 (1950) and 83, 776 (1951).

[2] V. Weisskopf, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. XIV, No. 6 (1936).

[3] M. Delbrück, Z. Physik 84, 144 (1933).

[4] R. R. Wilson, private communication.

[5] A. Achieser and I. Pomerantschuk, Physik. Z. Sowjetunion 11, 478 (1937).

[6] N. Kemmer, Helv. Phys. Acta 10, 112 (1937); N. Kemmer and G. Ludwig, Helv. Phys. Acta 10, 182 (1937).

[7] H. A. Bethe and F. Rohrlich, Phys. Rev. 86, 10 (1951).

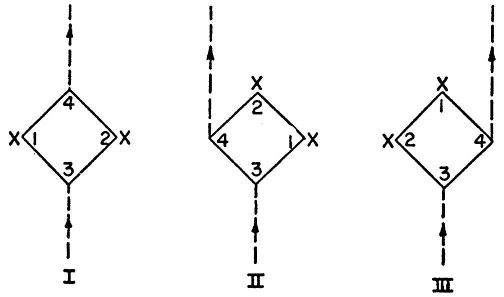

FIG. 1. Feynman diagrams for Delbrück scattering.

analytic continuation which was first suggested in a paper by Jost, Luttinger, and Slotnick,[8] on the basis of unpublished work by Wheeler and Toll.

Before turning to the actual calculations we note first that Delbrück scattering to first order in the external field vanishes identically because of Furry's theorem.[9] In carrying the calculation to second order in the Coulomb field, one finds soon that the manifold integrals involved cannot be expressed in terms of known functions and that a numerical solution is in general extremely impracticable. Valid approximations for intermediate angles also seem extremely obscure. We decided, therefore, to restrict ourselves first to forward scattering. This will enable us to carry out the calculation exactly and to express the final result in closed form. Also, we will be able to use the method of analytic continuation. Finally, we expect to provide a basis for calculations at other angles.

In the following we shall first give the Feynman method (method A). This is a very straightforward calculation, but extremely complicated and tedious in its actual performance. It seems to us that a brief sketch of this method and the essential technical points used should be sufficient. The analytic continuation method (method B) is given in much greater detail, since it is not so well known and is very much simpler than the other method. In B we assume the total pair production cross section as known and use a well-known theorem of optics to write down directly the absorptive (imaginary) part of the Delbrück scattering amplitude in the forward direction. The analyticity of the scattering matrix is then used to find the dispersive (real) part of this amplitude.

The final section contains a discussion of the results[10] and a comparison with earlier work.

Method A

The three different Feynman diagrams to be considered are shown in Fig. 1. The crosses stand for the action of the static potential

$$V(r)=Ze/r=(Ze/2\pi^2)\int e^{i\mathbf{q}\cdot\mathbf{r}}d_3q/\mathbf{q}^2. \qquad (1)$$

In the usual notation[11] the first diagram yields the following integral in a well-known manner,

$$A_4^e(\mathbf{q}_1)A_4^e(\mathbf{q}_2)\int\frac{\mathrm{Tr}[\gamma_\mu(i\mathfrak{p}-m)\gamma_4(i\mathfrak{p}-i\mathfrak{q}_1-m)\gamma_\nu(i\mathfrak{p}-i\mathfrak{q}_1-i\mathfrak{k}'-m)\gamma_4(i\mathfrak{p}+i\mathfrak{k}-m)]d_4p}{(p^2+m^2)[(p-q_1)^2+m^2][(p-q_1-k')^2+m^2][(p+k)^2+m^2]}, \qquad (2)$$

where μ and ν are the directions of polarization of the incoming and outgoing waves, $k=(\mathbf{k}, i\omega)$ and $k'=(-\mathbf{k}, -i\omega)$ are the corresponding wave number four-vectors, and $q_1=(\mathbf{q}_1, 0)$ and $q_2=(\mathbf{q}_2, 0)$ are the momentum transfers of the two actions of the potential. $A_4(\mathbf{q})=iV(\mathbf{q})$ is the Fourier transform of the Coulomb potential, Eq. (1). The expression (2) must still be integrated over $\mathbf{q}_1$ and $\mathbf{q}_2$, but reduces to the threefold integral over $\mathbf{q}_1$ only because of momentum conservation. The result is—apart from constants—the required scattering amplitude which depends on the polarizations, the energy $|\mathbf{k}|=|\mathbf{k}'|=\omega$ and the scattering angle $\vartheta=\cos^{-1}(\mathbf{k}\cdot\mathbf{k}'/\omega^2)$.

It will be convenient to use natural units ($\hbar=c=m=1$) and the following variables:

$$2\mathbf{Q}=\mathbf{q}_2-\mathbf{q}_1, \quad Q=(\mathbf{Q}, 0)$$
$$2\mathbf{P}=\mathbf{k}'-\mathbf{k}, \quad P=(\mathbf{P}, 0) \qquad (3)$$
$$2\mathbf{K}=\mathbf{k}'+\mathbf{k}, \quad K=(\mathbf{K}, i\omega),$$

which satisfy

$$\mathbf{K}\cdot\mathbf{P}=0, \quad \mathbf{K}^2+\mathbf{P}^2=\omega^2.$$

If we now restrict ourselves to forward scattering, we have $\mathbf{P}=0$, $\mathbf{K}=\mathbf{k}=\mathbf{k}'$. The scattering of an unpolarized beam is, in this case, equal to the scattering of a linearly polarized beam in the plane perpendicular to its polarization. No change of the direction of polarization can occur for forward scattering. Therefore, the traces can be easily carried out.

The method of integration over p is well known[12] and need not be given here. It introduces the auxiliary variables x, y, z. There result three kinds of terms which will be called $M_I^{(0)}$, $M_I^{(2)}$, and $M_I^{(4)}$. With the proper

[8] Jost, Luttinger, and Slotnick, Phys. Rev. **80**, 189 (1950).

[9] See F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

[10] Most of the results obtained in this paper were reported at the Washington meeting of the American Physical Society; see Phys. Rev. **83**, 218 (1951). Unfortunately, the numerical values quoted there for a_1 are in error. The corrected values are shown in Table I of the present paper.

[11] $x=(x_1, x_2, x_3, x_4)=(\mathbf{r}, x_4)=(x, y, z, it)$; $\mathfrak{p}=\Sigma\gamma_\mu p_\mu$; $d_4x=dxdydzdt$; $c=1$.

[12] See R. P. Feynman, Phys. Rev. **76**, 769 (1949). In order to assure gauge invariance, regulators can be used. The condition $\Sigma c_i=0$ is sufficient in this case.

constants, the contribution of the first diagram to the forward scattering amplitude (in units of $(\alpha Z)^2 r_0$) is

$$a_I(\omega,0)=\int (M_I^{(0)}+M_I^{(2)}+M_I^{(4)})d_3Q/\mathbf{Q}^4, \tag{4}$$

$$M_I^{(0)}=\int_0^1 dx\int_0^x dy\int_0^y dz P_I^{(0)}\Big/ b_I^2,$$

$$M_I^{(2)}=\int_0^1 dx\int_0^x dy\int_0^y dz P_I^{(2)}\Big/ b_I, \tag{5}$$

$$M_I^{(4)}=\int_0^1 dx\int_0^x dy\int_0^y dz P_I^{(4)} \ln b_I,$$

$$b_I=1+2\mathbf{K}\cdot\mathbf{Q}(z+y(y-x-z))+\mathbf{Q}^2 y(1-y). \tag{6}$$

$P_I^{(0)}$, $P_I^{(2)}$, and $P_I^{(4)}$ are essentially the traces which multiply the 0th, 2nd, and 4th power of p in the expression (2). They are linear combinations of ω^2, $\mathbf{Q}^2$, $\mathbf{K}\cdot\mathbf{Q}$, $(\mathbf{e}\cdot\mathbf{Q})^2$, $\mathbf{Q}^4$, $(\mathbf{K}\cdot\mathbf{Q})^2$, ω^2Q^2, $\omega^2(\mathbf{K}\cdot\mathbf{Q})$, $\mathbf{Q}^2(\mathbf{K}\cdot\mathbf{Q})$, and $\omega^2(\mathbf{e}\cdot\mathbf{Q})^2$; the coefficients are polynomials of up to fourth order in x, y, and z. $\mathbf{e}$ is a unit vector in the direction of polarization.

This whole procedure has to be repeated for the diagrams II and III of Fig. 1. The results are completely analogous to (4), (5), and (6).

The integrals in (5) deserve special attention. As was pointed out by several authors,[9,12] the method of integration over p actually implies that the singularities on the real p_0 axis, i.e., $p_0=\pm(\mathbf{p}^2+m^2)^{\frac{1}{2}}$ are moved slightly into the upper (or lower) half-planes. In other words, the path of integration around the singularities has to be properly taken into account. It contributes a purely imaginary term to be added to the principle part of the integral. If we restrict the integration first to the principle parts, the result will be the diffractive or real part, a_1, of the scattering amplitude. The absorptive or imaginary part, a_2, is found best by assuming the electron mass to have a small negative imaginary part. The integrals in (4) are now well defined, since the expression (5) for b_I should be augmented by a term $-i\epsilon$, where ϵ is eventually taken in the limit zero.

From here on the calculation is quite straightforward, but extremely complex and tedious, even though the specialization to $\vartheta=0$ simplifies the task considerably.

The integration over the three auxiliary variables x, y, and z can be reduced to a double integral by a judicious change of variables, having b_I, b_{II}, and b_{III} independent of one of the three new variables, and maintaining a simple region of integration. For example the sequence of variable changes required for b_I is

Variable			Region of integration	
x	y	z	$\int_0^1 dx\int_0^x dy\int_0^y dz$	(7a)
$u=1-x$	$v=y$	$w=y-z$	$\int_0^1 dv\int_0^{1-v} du\int_0^v dw$	(7b)
$u=(1-v)\zeta$	$v=v$	$w=v\xi$	$\int_0^1 v(1-v)dv\int_0^1 d\zeta\int_0^1 d\xi$	(7c)
$\zeta=\zeta$	$v=v$	$\eta=\xi-\zeta$	$\int_0^1 v(1-v)dv\int_0^1 d\eta\int_0^{1-\eta} d\zeta + \int_0^1 v(1-v)dv\int_{-1}^0 d\eta\int_{-\eta}^1 d\zeta.$	(7d)

At this stage we have

$$b_I=1+v(1-v)[\mathbf{Q}^2-2\mathbf{K}\cdot\mathbf{Q}\eta]. \tag{8a}$$

In the second term of the last step, (7d), let $\eta\rightarrow-\eta$ and $\mathbf{Q}\rightarrow-\mathbf{Q}$ simultaneously, leaving b_I unchanged and the region of integration becomes

$$\int_0^1 v(1-v)dv\int_0^1 d\eta\left(\int_0^{1-\eta} d\zeta_1+\int_\eta^1 d\zeta_2\right). \tag{9a}$$

If we finally let

$$Y=1-\eta,\quad \zeta_2=1-\zeta_3, \tag{7e}$$

we obtain

$$b_I=1+v(1-v)(\mathbf{Q}^2-2\mathbf{K}\cdot\mathbf{Q}+2\mathbf{K}\cdot\mathbf{Q}Y), \tag{8b}$$

with the region of integration

$$\int_0^1 v(1-v)dv\int_0^1 dY\left(\int_0^Y d\zeta_1+\int_0^Y d\zeta_3\right). \tag{9b}$$

The regions of integration for the two terms are now the same, but the integrand in the second term of Eq. (9b) must include the changes $\mathbf{Q}\rightarrow-\mathbf{Q}$, $\eta\rightarrow-\eta$, and $\zeta\rightarrow1-\zeta$ before combination with the integrand in the first term.

Similar changes bring the integrals in II and III into the same form,[13] allowing complete combination of all integrand polynomials. In all cases the ζ integration is trivial, and the required integrals reduce to the following

[13] Note that the integrals for diagrams II and III of Fig. 1 differ only in the sign of Q, since each integral is equal to its Hermitian adjoints.

general forms

$$\int_0^1 v^m dv \int_0^1 Y^n dY \ln b_I \quad m=1,2 \qquad n=1 \quad \text{(10a)}$$

$$\int_0^1 v^m dv \int_0^1 Y^n dY \Big/ b_I \quad m=1,2,3,4 \qquad n=1,2,3 \quad \text{(10b)}$$

$$\int_0^1 v^m dv \int_0^1 Y^n dY \Big/ b_I^2 \quad m=1,2,3,4,5,6 \quad n=1,2,3,4 \quad \text{(10c)}$$

with b_I given by Eq. (8b). The Y integration is straightforward, though complicated, and leads to algebraic forms in v as well as a logarithmic term. The algebraic terms can once again be integrated without difficulty, and in most cases the logarithmic term can be integrated by parts. The one integral which is not straightforward is

$$\tfrac{1}{2}L_a = \int_0^1 \frac{dv}{v} \ln[1+av(1-v)] \quad (11)$$

which is most easily performed by differentiating with respect to a.

$$\frac{\frac{1}{2}dL_a}{da} = \int_0^1 \frac{(1-v)dv}{1+av(1-v)} = \frac{1}{\sqrt{a}(a+4)^{\frac{1}{2}}} \ln\left[\frac{(4+a)^{\frac{1}{2}}+\sqrt{a}}{(4+a)^{\frac{1}{2}}-\sqrt{a}}\right]. \quad (12)$$

Very fortunately, the factor $1/[a(a+4)]^{\frac{1}{2}}$ is the derivative of the logarithmic term so that we have

$$L_a = \left[\ln\left(\frac{[4+a]^{\frac{1}{2}}+\sqrt{a}}{[4+a]^{\frac{1}{2}}-\sqrt{a}}\right)\right]^2. \quad (13)$$

The constant in the integrations over a is evaluated in the limit $a=0$ and vanishes.

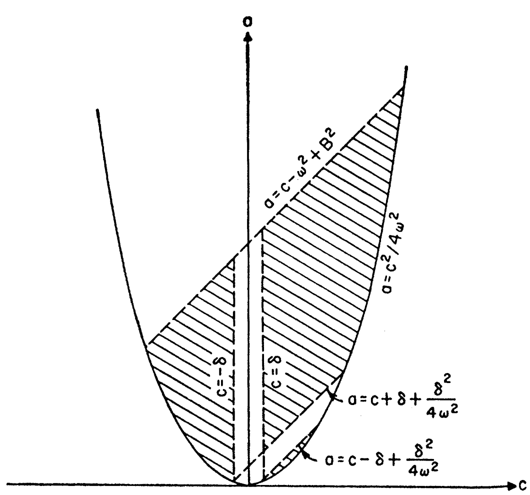

FIG. 2. Region for the a–c-integration.

At this point the quantity $(\mathbf{e}\cdot\mathbf{Q})^2$ is averaged over polarizations giving

$$\langle(\mathbf{e}\cdot\mathbf{Q})^2\rangle_{\mathrm{Av}} = \tfrac{1}{2}\mathbf{Q}^2 \sin^2\theta_{KQ} = \tfrac{1}{2}(\mathbf{Q}^2 + (\mathbf{K}\cdot\mathbf{Q})^2/\omega^2).$$

The result of the Y and v integrations can be written as

$$a(\omega,0) = \int [L_a f_1(a_1c) + L_b f_2(b_1c) + T_a f_3(a_1c) + T_b f_4(b_1c) + f_5(a_1c)] d_3Q/Q^4, \quad (14)$$

where L_a is given by Eq. (13),

$$T_a = \frac{(4+a)^{\frac{1}{2}}}{\sqrt{a}} \ln\left[\frac{(4+a)^{\frac{1}{2}}+\sqrt{a}}{(4+a)^{\frac{1}{2}}-\sqrt{a}}\right], \quad (15)$$

$$a = b+c = \mathbf{Q}^2, \quad (16a)$$

$$b = \mathbf{Q}^2 - 2\mathbf{K}\cdot\mathbf{Q}, \quad (16b)$$

$$c = 2\mathbf{K}\cdot\mathbf{Q}, \quad (16c)$$

and the f_i are algebraic functions of a, b, c (separable into simple polynomials and quotients, i.e., $1/c$, a^2/c^3, b^3/c^4, $1/c(a+4)$, etc.).

The d_3Q integration reduces to the double integral $\int \mathbf{Q}^2 dQ \int d(\mathbf{Q}\cdot\mathbf{K}/QK)$ because of the azimuthal symmetry of zero angle scattering. $\mathbf{Q}\cdot\mathbf{K}/QK$ is the cosine of the angle of $\mathbf{Q}$ referred to the polar direction $\mathbf{K}$. The terms in (14) separate readily into some terms expressed as functions of a and c, and others expressed as functions of b and c. If the integrand is expressed in terms of a and c the region of integration is

$$\int_0^\infty \frac{da}{4\omega} \int_{-2\omega\sqrt{a}}^{2\omega\sqrt{a}} dc, \quad (17)$$

whereas if the integrand is in terms of b and c the region may be obtained by substituting $a=b+c$ in (17):

$$\int_0^\infty \frac{da}{4\omega} \int_{-2\omega\sqrt{a}}^{2\omega\sqrt{a}} dc = \int_{-\infty}^\infty \frac{dc}{4\omega} \int_{c^2/4\omega^2}^\infty da = \int_{-\infty}^\infty \frac{dc}{4\omega} \int_{c^2/4\omega^2 - c}^\infty db = \int_{-\omega^2}^\infty \frac{db}{4\omega} \int_{2\omega^2 - 2\omega(\omega^2+b^2)^{\frac{1}{2}}}^{2\omega^2+2\omega(\omega^2+b)^{\frac{1}{2}}} dc. \quad (18)$$

The required integrations are therefore

$$\int_0^\infty \frac{da}{4\omega a^2} L_a \int_{-2\omega\sqrt{a}}^{2\omega\sqrt{a}} dc f_1(a_1c) + \int_0^\infty \frac{da}{4\omega a^2} T_a \int_{-2\omega\sqrt{a}}^{2\omega\sqrt{a}} dc f_3(a_1c) + \int_0^\infty \frac{da}{4\omega a^2} \int_{-2\omega\sqrt{a}}^{2\omega\sqrt{a}} dc f_5(a_1c), \quad (19)$$

and

$$\int_0^\infty \frac{zdz}{2\omega} L_{z^2-\omega^2}\int_{2\omega^2-2\omega z}^{2\omega^2-2\omega z}\frac{dc}{(c+z^2-\omega^2)^2} f_2(z^2-\omega^2,c)$$
$$+\int_0^\infty \frac{zdz}{2\omega}T_{z^2-\omega^2}\int_{2\omega^2-2\omega z}^{2\omega^2+2\omega z}\frac{dc}{c+z^2-\omega^2} f_4(z^2-\omega^2,c), \quad (20)$$

where $z^2=\omega^2+b$. Since we have arranged to do integrations over a and b (or z) last in (19) and (20), respectively, the c-integration (over simple but lengthy algebraic forms) offers no problem.

However, we have created a new problem for ourselves by the separation of (14) into (19) and (20) for integration purposes. Although (14) in its entirety is convergent (and integrable) throughout the range of integration, neither (19) nor (20) is convergent. Poles have been introduced at

$$a=0 \text{ and } \infty, \quad b=0 \text{ and } \infty, \quad \text{and} \quad c=0.$$

They have been brought in by the separation and appear in the f's as well as in the result of the c integration. We have used a consistent pair of cutoffs, δ and B as shown in Figs. 2 and 3, to make (19) and (20) finite. The integrations in each case are carried out over the shaded region only, each integration having to be broken up into several "pieces." Of course, after the evaluation of (19) and (20) in the limit $\delta\rightarrow 0$, $B\rightarrow\infty$, the cut-off dependent terms must and do vanish.

After the c-integration in (19) has been performed, there remain the integrations over a of the form

$$\int daL_a/a^n, \quad \int daT_a/a^n, \quad (21)$$

and of other less complicated forms which can be readily evaluated. The two forms in (21) may be evaluated for $n\neq 1$ ($n=1$ does not occur) by integration by parts.

After the c-integration in (20), it becomes convenient to express the final integration once again in terms of b, and many of the resulting forms are the same as in (21). With the aid of several contour integrations in the upper (or lower) half b plane the forms may all be expressed in terms of the following four functions:

$$C_1(p)=\mathrm{Re}\int_0^{1/p}\frac{\sin^{-1}x}{x}\cosh^{-1}\frac{1}{px}dx, \quad (22a)$$

$$D_1(p)=\mathrm{Re}\int_0^{1/p}\frac{\cosh^{-1}(1/px)}{(1-x^2)^{\frac{1}{2}}}dx, \quad (22b)$$

$$E_1(p)=\mathrm{Re}\int_0^{1/p}\left(\frac{1-p^2x^2}{1-x^2}\right)^{\frac{1}{2}}dx, \quad (22c)$$

$$F_1(p)=\mathrm{Re}\int_0^{1/p}\frac{dx}{[(1-x^2)(1-p^2x^2)]^{\frac{1}{2}}}. \quad (22d)$$

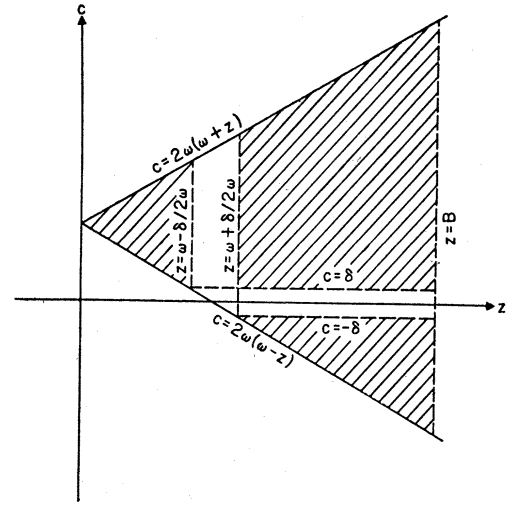

FIG. 3. Region for the b–c-integration.

The result of the combination of all terms gives for the real part of the matrix element.

$$a_1(\omega,0)=(p/\pi)(2C_1(p)-D_1(p)) +(1/27\pi p)[(109+64p^2)E_1(p) -(67+6p^2)(1-p^2)F_1(p)]-p^2/9-9/4, \quad (23)$$

where $p=2/\omega$, and $a(\omega,0)=a_1(\omega,0)+ia_2(\omega,0)$.

The imaginary part of the matrix element arises from the b-integration alone and only from the L_b and T_b terms. These two functions defined in (13) and (15) are real for $b>-4$ and complex for $b<-4$. Since the b-integration goes from $-\omega^2$ to $+\infty$, there will be no imaginary contribution for $\omega\leqslant 2$, but there will be one for $\omega>2$, the contribution coming directly from the range $-\omega^2\leqslant b\leqslant -4$. The complete expressions for L_a and T_a are

$$\begin{aligned} L_a&=2\lim_{\epsilon\rightarrow 0}\int_0^1 \ln(1-i\epsilon+az(1-z))dz/z \\ &=(2\sinh^{-1}\tfrac{1}{2}\sqrt{a})^2 \quad (a\geqslant 0) \quad (24)\\ &=-(2\sin^{-1}\tfrac{1}{2}\sqrt{-a})^2 \quad (-4\leqslant a\leqslant 0)\\ &=(2\cosh^{-1}\tfrac{1}{2}\sqrt{-a})^2-\pi^2-4\pi i\cosh^{-1}\tfrac{1}{2}\sqrt{-a} \quad (a\leqslant -4)\end{aligned}$$

$$\begin{aligned} T_a&=2\lim_{\epsilon\rightarrow 0}\int_0^1 dz\ln(1-i\epsilon+az(1-z)) \\ &=2(1+4/a)^{\frac{1}{2}}\sinh^{-1}\tfrac{1}{2}\sqrt{a} \quad (a\geqslant 0) \quad (25)\\ &=2(-1-4/a)^{\frac{1}{2}}\sin^{-1}\tfrac{1}{2}\sqrt{-a} \quad (-4\leqslant a\leqslant 0)\\ &=(1+4/a)^{\frac{1}{2}}(2\cosh^{-1}\tfrac{1}{2}\sqrt{-a}-i\pi) \quad (a\leqslant -4).\end{aligned}$$

To obtain the imaginary part we start with the b-integration in Eq. (18) between the limits $-\omega^2$ and -4. There are, therefore, no cut-off considerations to complicate the algebra. The integrations otherwise proceed as before and the result is

$$
\begin{aligned}
a_2(\omega,0)&=0 \quad (\omega\leqslant 2)\\
&=(p/\pi)[2C_2(p)-D_2(p)]-(1/27\pi p)\\
&\quad\times[(109+64p^2)E_2(p)-(67+6p^2)(1-p^2)F_2(p)], \quad (\omega>2) \qquad (26)
\end{aligned}
$$

where

$$C_2(p)=\int_1^{1/p}\frac{\cosh^{-1}x}{x}\cosh^{-1}\frac{1}{px}dx, \qquad (27a)$$

$$D_2(p)=\int_1^{1/p}\frac{\cosh^{-1}(1/px)}{(x^2-1)^{\frac{1}{2}}}dx, \qquad (27b)$$

$$E_2(p)=\int_1^{1/p}\left(\frac{1-p^2x^2}{x^2-1}\right)^{\frac{1}{2}}dx, \qquad (27c)$$

$$F_2(p)=\int_1^{1/p}\frac{dx}{[(x^2-1)(1-p^2x^2)]^{\frac{1}{2}}}, \qquad (27d)$$

Comparison of Eqs. (23) and (26) shows that the real and imaginary parts are identical in form except for the last two terms of a_1, which are missing in a_2. Similarly, the functions C_1 and C_2, etc., are closely related.

For actual calculations the functions C and D can easily be expanded in power series, whereas the functions E_1, E_2, and F_1, F_2, can be expressed in terms of the complete elliptic integrals of the first and second kind, $F(x)$ and $E(x)$. We have

$$
\begin{aligned}
E_1(p)&=E(p) \quad (p\leqslant 1),\\
&=pE(1/p)+(1/p-p)F(1/p) \quad (p\geqslant 1),\\
F_1(p)&=F(p) \quad (p\leqslant 1),\\
&=(1/p)F(1/p) \quad (p\geqslant 1),\\
E_2(p)&=F[(1-p^2)^{\frac{1}{2}}]-E[(1-p^2)^{\frac{1}{2}}] \quad (p\leqslant 1),\\
F_2(p)&=F[(1-p^2)^{\frac{1}{2}}] \quad (p\leqslant 1).
\end{aligned}
$$

The power series expansions break down near $p=1$ where one has to proceed numerically. In particular one finds

$$C_1(1)=1.62876, \qquad (28a)$$

$$D_1(1)=1.83193. \qquad (28b)$$

$D_1(1)$ is twice Catalan's constant.[14]

Method B

Consider Cauchy's theorem for an analytic function

$$w(z)=\frac{1}{2\pi i}\oint\frac{w(\zeta)d\zeta}{\zeta-z}.$$

Let z be on the real axis, $z=x$, and assume that w is regular in the upper half-plane. We can choose the path of integration to follow the real axis from $-R$ to $+R$ with a small half-circle in the positive direction around the point x, and to close in a large half-circle R over the upper half-plane of the complex variable $\zeta=\xi+i\eta$. We thus find

$$w(x)=\frac{1}{2\pi i}\mathrm{P}\int_{-R}^{R}\frac{w(\xi)d\xi}{\xi-x}+\tfrac{1}{2}w(x)+\frac{1}{2\pi i}\int_R\frac{w(\zeta)}{\zeta}d\zeta, \qquad (29)$$

where in the last integral x has been neglected. We further assume that $w(\zeta)/\zeta$ is regular at $\zeta=0$, so that

$$\int_R\frac{w(\zeta)}{\zeta}d\zeta=-\int_{-R}^{R}\frac{w(\xi)}{\xi}d\xi.$$

We separate $w(\xi)$ into its real and imaginary part and find from Eq. (29) and the last equation in the limit $R\to\infty$,

$$u(\xi)=\frac{x}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{v(\xi)d\xi}{\xi(\xi-x)}, \qquad (30a)$$

$$v(\xi)=\frac{-x}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{u(\xi)d\xi}{\xi(\xi-x)}. \qquad (30b)$$

The complex scattering amplitude $a(\omega,0)=a_1(\omega,0)+ia_2(\omega,0)$ satisfies the assumptions[15] made about $w(\xi)$ when ξ is identified with the energy ω.

A well-known theorem of optics[16] relates the total absorption cross section $\sigma_{\mathrm{abs}}(\omega)$ and the forward scattering amplitude $a(\omega,0)$

$$\sigma_{\mathrm{abs}}(\omega)=4\pi\lambda\!\!\!^{-}\,\mathrm{Im}a(\omega,0)=2\pi p a_2(\omega,0). \qquad (31)$$

The absorption process that corresponds to the elastic scattering of light by a Coulomb field is pair production. The total pair production cross section is well known. We will use the analytical form obtained by Jost, Luttinger, and Slotnick.[17]

In our notation we therefore find without calculation

$$
\begin{aligned}
a_2(\omega,0)&=0 \quad (\omega\leqslant 2)\\
&=(p/\pi)[2C_2(p)-D_2(p)]\\
&\quad+(1/27p\pi)[-(109+64p^2)E(\{1-p^2\}^{\frac{1}{2}})\\
&\quad+(42+125p^2+6p^4)F(\{1-p^2\}^{\frac{1}{2}}) \quad (\omega\geqslant 2). \qquad (32)
\end{aligned}
$$

Making use of the fact that $\sigma_{\mathrm{abs}}(\omega)$ is an even function of ω we find from Eq. (30a)

$$a_1(\omega,0)=(\omega^2/4\pi^2)\mathrm{P}\int_{-\infty}^{\infty}\frac{\sigma_{\mathrm{abs}}(\omega')d\omega'}{\omega'(\omega'-\omega)} \qquad (33a)$$

$$=(\omega^2/2\pi^2)\mathrm{P}\int_2^{\infty}\frac{\sigma_{\mathrm{abs}}(\omega')d\omega'}{\omega'^2-\omega^2} \qquad (33b)$$

$$=\frac{1}{\pi^2}\mathrm{P}\int_0^1\frac{\sigma_{\mathrm{abs}}(p')dp'}{p^2-p'^2}. \qquad (33c)$$

[14] See E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, Inc., New York, 1943), p. 80.

[15] See also the remarks at the end of this paper.

[16] Recently, a very general proof was given by M. Lax, Phys. Rev. **78**, 306 (1950). Further references may be found in this paper.

[17] See reference 8. Two misprints in their paper should be noted: in Eq. (56) the sign of $82\alpha^2/3$ and in Eq. (57) the sign of $2F_1(\alpha)$ are wrong.

Although the functions C_2, D_2, E, and F are quite complicated, the integration indicated in Eq. (33c) can be carried out without serious difficulties and the result can be found in terms of a closed expression.

In the remainder of this section we will briefly indicate how this integration can be carried out.

The integral containing the function $C_2(p)$ is

$$P\int_0^1 \frac{p'^2 C_2(p')dp'}{p^2-p'^2}$$
$$=P\int_0^1\left(-1+\frac{p^2}{p^2-p'^2}\right)dp'\int_1^{1/p}\frac{\cosh^{-1}x}{x}\cosh^{-1}\frac{1}{p'x}dx$$
$$=P\int_0^1\frac{dy}{y}\cosh^{-1}\frac{1}{y}\int_0^y\left(-1+\frac{p^2}{p^2-p'^2}\right)\cosh^{-1}\frac{y}{p'}dp',$$

where we used $y=p'x$ and interchanged the order of the integration. An elementary integration yields for the first term $-\pi^2/4$. The second term becomes by partial integration

$$p^2 P\int_0^1\frac{dy}{y}\cosh^{-1}\frac{1}{y}\int_0^y\frac{dp'}{p^2-p'^2}\cosh^{-1}\frac{y}{p'}$$
$$=\frac{p}{2}\int_0^1 dy\cosh^{-1}\frac{1}{y}\int_0^y \ln\left|\frac{p+p'}{p-p'}\right|\frac{dp'}{p'(y^2-p'^2)^{\frac{1}{2}}}$$
$$=\frac{p}{2}\pi\int_0^1\frac{dy}{y}\cosh^{-1}\frac{1}{y}\,\mathrm{Re}\sin^{-1}\frac{y}{p}.$$

Here, we made use of the equality[18]

$$\mathrm{Re}\int_0^1\ln\frac{1+ax}{1-ax}\frac{dx}{x(1-x^2)^{\frac{1}{2}}}=\pi\,\mathrm{Re}\sin^{-1}a. \qquad (34)$$

Combining these results we find

$$P\int_0^1\frac{p'^2C_2(p')dp'}{p^2-p'^2}=-\frac{\pi^2}{4}+\tfrac{1}{2}\pi pC_1(p), \qquad (35)$$

where $C_1(p)$ is identical with the function defined in (22a) as it resulted in method A.

The term in $D_2(p)$ of Eq. (33c) is treated similarly:

$$P\int_0^1\frac{p'^2D_2(p')dp'}{p^2-p'^2}$$
$$=P\int_0^1\left(-1+\frac{p^2}{p^2-p'^2}\right)dp'\int_1^{1/p}\frac{\cosh^{-1}(1/p'x)}{(x^2-1)^{\frac{1}{2}}}dx.$$

[18] D. Bierens de Haan, *Nouvelles Tables d'Integrales Definies* (Stechert, New York, 1939), Table 122 (2).

The first term gives $-\pi^2/4$ as before; the second term is

$$P\int_0^1\frac{p^2dp'}{p^2-p'^2}\int_{p'}^1\frac{\cosh^{-1}(1/y)}{(y^2-p'^2)^{\frac{1}{2}}}dy$$
$$=P\int_0^1\cosh^{-1}\frac{1}{y}dy\int_0^y\frac{p^2dp'}{(p^2-p'^2)(y^2-p'^2)^{\frac{1}{2}}}$$
$$=\frac{\pi p}{2}\mathrm{Re}\int_0^1\cosh^{-1}\frac{1}{y}\frac{dy}{(p^2-y^2)^{\frac{1}{2}}}$$
$$=\frac{p\pi}{2}\mathrm{Re}\int_0^{1/p}\frac{dx}{(1-x^2)^{\frac{1}{2}}}\cosh^{-1}\frac{1}{px}dx.$$

The result for the term in D_2 is therefore

$$P\int_0^1\frac{p'^2D_2(p')dp'}{p^2-p'^2}=-\frac{\pi^2}{4}+\tfrac{1}{2}\pi pD_1(p), \qquad (36)$$

where $D_1(p)$ is identical with the function defined in Eq. (22b).

The integration of the terms of Eq. (33c) which involve complex elliptic integrals of the first and second kind may be done as follows:[19]

$$\int_0^1 F[(1-p^2)^{\frac{1}{2}}]dp$$
$$=\int_0^1\frac{dx}{(1-x^2)^{\frac{1}{2}}}\int_0^1\frac{dp}{(1-x^2+p^2x^2)^{\frac{1}{2}}}$$
$$=\int_0^1\frac{dx}{x(1-x^2)^{\frac{1}{2}}}\tan^{-1}x=\frac{\pi^2}{4}, \qquad (37)$$

$$\int_0^1 p^2F[(1-p^2)^{\frac{1}{2}}]dp=\tfrac{1}{2}\int_0^1\frac{dx}{x}\frac{1}{(1-x^2)^{\frac{1}{2}}}$$
$$\times\left[\tanh^{-1}x+\frac{1}{x}\left(1-\frac{\tan^{-1}x}{x}\right)\right]$$
$$=\tfrac{1}{2}(\pi^2/4-\pi^2/8)=\pi^2/16, \qquad (38)$$

$$\int_0^1\frac{F[(1-p'^2)^{\frac{1}{2}}]}{p^2-p'^2}dp'$$
$$=\frac{1}{2p}\mathrm{Re}\int_0^1\frac{dx}{(1-x^2)^{\frac{1}{2}}}\frac{1}{(1-(1-p^2)x^2)^{\frac{1}{2}}}$$
$$\times\ln\frac{1+p^{-1}(1-(1-p^2)x^2)^{\frac{1}{2}}}{1-p^{-1}(1-(1-p^2)x^2)^{\frac{1}{2}}}$$
$$=\frac{\pi}{2p}\mathrm{Re}F\left(p,\sin^{-1}\frac{1}{p}\right)=\frac{\pi}{2p}F_1(p). \qquad (39)$$

[19] See reference 18, Table 122, (10).

The last integration can again be found in Bierens de Haan.[20]

In a similar way one finds

$$\int_0^1 E[(1-p^2)^{\frac{1}{2}}]dp=\pi^2/8, \tag{40}$$

$$\begin{aligned}\int_0^1 \frac{E[(1-p'^2)^{\frac{1}{2}}]}{p^2-p'^2}dp' &= -\int_0^1 \frac{x^2dx}{(1-x^2)^{\frac{1}{2}}}\int_0^1 \frac{dp'}{(1-x^2+p'^2x^2)^{\frac{1}{2}}} \\ &\quad +\int_0^1 \frac{1-x^2+p^2x^2}{(1-x^2)^{\frac{1}{2}}}dx\cdot \mathrm{P}\int_0^1 \frac{dx}{(p^2-p'^2)(1-x^2+p'^2x^2)^{\frac{1}{2}}} \\ &= -\frac{\pi}{2}+\frac{1}{2p}\,\mathrm{Re}\int_0^1 dx\left(\frac{1-x^2+p^2x^2}{1-x^2}\right)^{\frac{1}{2}} \\ &\qquad \times \ln\frac{1+p^{-1}(1-x^2+p^2x^2)^{\frac{1}{2}}}{1-p^{-1}(1-x^2+p^2x^2)^{\frac{1}{2}}}.\end{aligned}$$

We can use the identity

$$\begin{aligned}\mathrm{Re}\int_0^1 dx\left(\frac{1-r^2x^2}{1-x^2}\right)^{\frac{1}{2}} \ln\frac{1+q(1-r^2x^2)^{\frac{1}{2}}}{1-q(1-r^2x^2)^{\frac{1}{2}}} \\ \equiv \mathrm{Re}\frac{\pi}{q}\int_0^1 \frac{dx}{x^2}\left[\frac{1}{(1-q^2x^2)^{\frac{1}{2}}[1-(1-r^2)q^2x^2]^{\frac{1}{2}}}-1\right],\end{aligned}$$

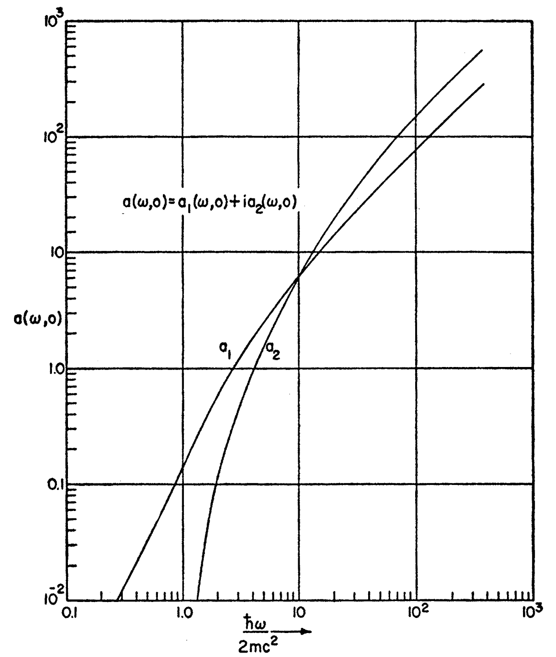

FIG. 4. Delbrück scattering amplitude for forward scattering in units of $(\alpha Z)^2r_0$ as a function of energy.

[20] See reference 18, Table 122 (15).

and find with $y=1/x$

$$\begin{aligned}-\frac{\pi}{2}+\frac{\pi}{2}\,\mathrm{Re}\int_0^1 \frac{dx}{x^2}\left(\frac{p}{[(1-x^2)(p^2-x^2)]^{\frac{1}{2}}}-1\right) \\ = -\frac{\pi}{2}+\frac{\pi}{2}\,\mathrm{Re}\left(\frac{1}{2}\int_{-\infty}^{\infty}-\frac{1}{2}\int_{-1}^{1}\right) \\ \times dy\left(\frac{py^2}{[(y^2-1)(p^2y^2-1)]^{\frac{1}{2}}}-1\right) \\ = -\frac{\pi}{2}-\frac{\pi}{2}\,\mathrm{Re}\int_0^1 dy\left(\frac{py^2}{-[(1-y^2)(1-p^2y^2)]^{\frac{1}{2}}}-1\right),\end{aligned}$$

since the first integral vanishes. Thus[21]

$$\mathrm{P}\int_0^1 \frac{E[(1-p'^2)^{\frac{1}{2}}]}{p^2-p'^2}dp'=\frac{\pi}{2p}[F_1(p)-E_1(p)], \tag{41}$$

where F_1 and E_1 are identical with the functions defined in (22c and d).

It is easily seen that the real part of the forward scattering amplitude, $a_1(\omega,0)$ as defined by (33c), (31),

TABLE I. The forward scattering amplitude $a(\omega,0)=a_1(\omega,0)+ia_2(\omega,0)$ in units of $(\alpha Z)^2r_0$ and the forward differential cross section in lead for some characteristic energy values.

Energy (Mev)	$a_1(\omega,0)$	$a_2(\omega,0)$	$d\sigma(\omega,0)/d\Omega$ for Pb
0.411	0.0205	0	0.00428 mb/sterad
1.33	0.241	0.0058	0.591 mb/sterad
2.62	0.912	0.265	9.18 mb/sterad
17.6	11.87	14.5	3.57 b/sterad
200	150	395	1.82 kb/sterad

and (32), can be written with the aid of our intermediate results [Eqs. (35) to (41)], in terms of C_1, D_1, E_1, and F_1, as defined by (22). The final result is identical with the result in Eq. (23) of method A. It is easily seen that Eqs. (26) and (32) are also identical.

DISCUSSION OF RESULTS

The dispersive and absorptive parts of the forward scattering amplitude are plotted *versus* energy in Fig. 4. Up to about 10 Mev the dispersive part dominates, whereas for higher energies the absorptive part is the larger of the two and dominates by orders of magnitude for very high energies. The divergence of the scattering amplitudes for very high energies is not surprising; the vacuum is an infinite source of pairs, and their production is limited only by the available energy. Compared with known phenomena involving an index of refraction, the vacuum has its "resonance" at $\omega=\infty$, such that there is no change of sign of the refractive index, and a_1 is positive throughout.

The differential Delbrück scattering cross section per unit solid angle, $d\sigma(\omega,0)/d\Omega$ for $\vartheta=0$ is

$$d\sigma(\omega,0)/d\Omega=|a_1(\omega,0)+ia_2(\omega,0)|^2(\alpha Z)^4r_0^2, \tag{42}$$

[21] See reference 18, Table 12 (9).

since our scattering amplitude is given in units of $(\alpha Z)^2 r_0$ where r_0 is the classical electron radius.

In the high energy limit one finds easily

$$a_1(\omega,0)=7\omega/18 \qquad (\omega\gg 1), \tag{43a}$$

$$a_2(\omega,0)=(7\omega/9\pi)\ln(2\omega) \qquad (\omega\gg 1). \tag{43b}$$

The latter is obviously in agreement with $\omega/4\pi$ times the high energy limit of the total pair production cross section.

Near the pair production threshold, $\omega=2$, the absorption part is

$$a_2(\omega,0)=\tfrac{1}{3}(1-2/\omega)^3 \quad (\omega\gtrsim 2), \tag{44}$$

which vanishes, of course, at the threshold.

In the low energy limit the Delbrück scattering amplitude becomes

$$a(\omega,0)=a_1(\omega,0)=(73/72)(\omega^2/32). \tag{45}$$

This function is plotted in Fig. 5 together with the exact expression. The close agreement shows that nearly up to the pair production cross section Delbrück scattering is qualitatively almost entirely classical dipole scattering, the cross section being proportional to ω^4. The classical aspect is seen in the fact that the amplitude is independent of Planck's constant. Abandoning natural units for a moment, we find for the low energy limit

$$\begin{aligned} d\sigma(\omega,0)/d\Omega &= (73/72)^2(1/32)^2(\hbar\omega/mc^2)^4(\alpha Z)^4 r_0^2 \\ &= (73/72)^2(1/32)^2(\omega/c)^4 Z^4(e^2/mc^2)^6. \end{aligned} \tag{45'}$$

The only calculation in the literature on forward Delbrück scattering is a low energy limit by Kemmer and Ludwig.[6] Their work corresponds exactly to the calculations of Euler[1] on the scattering of light by light. They assume an arbitrary potential, which must fall off fast enough, however; therefore, they do not carry out the Q integration indicated in our Eq. (14). Taking the limit in this integral we find

$$a(\omega,0)=(11/135)32\pi\omega^2\int dQ$$

in agreement with Kemmer and Ludwig.

A few characteristic numerical values for forward scattering are listed in Table I.

Experiments on the elastic scattering of light by heavy elements necessarily yield cross sections which are the result of interference of the following three phenomena: Thomson scattering by the nucleus, Rayleigh scattering by the electrons (primarily K-electrons, unless the energies are very low), and Delbrück scattering by the nuclear Coulomb field. All three

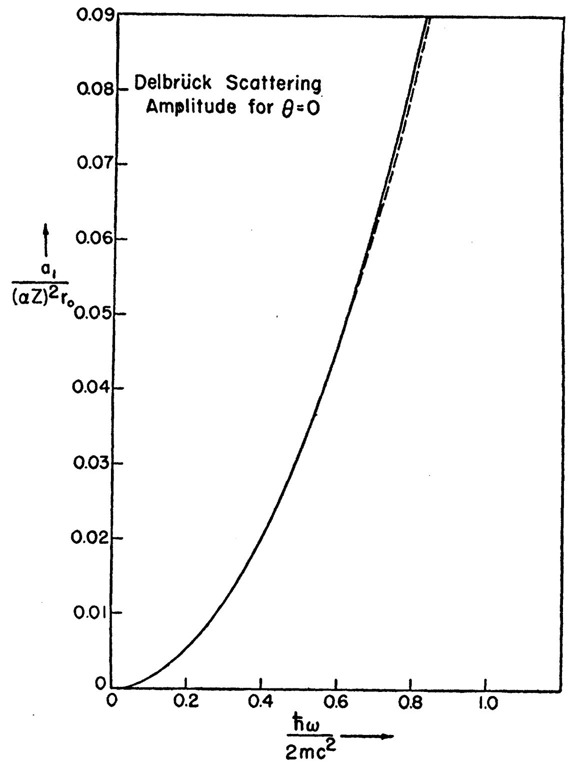

FIG. 5. Low energy limit of the Delbrück scattering amplitude (dashed line) compared with the exact curve (solid line).

effects are coherent. Since both Rayleigh and Delbrück scattering have complex scattering amplitudes—the photoeffect being the absorptive process associated with Rayleigh scattering—an analysis of these experiments cannot be made until the differential cross sections of these processes are known quantitatively.

In conclusion, we draw attention to the very elegant method of analytic continuation. Its applicability, however, needs further study. We assumed here that $a(\omega)$ fulfills all the conditions imposed on $w(\zeta)$. On the other hand, the validity of the assumption that $w(\zeta)/\zeta$ is regular at $\zeta=0$ is guaranteed by gauge invariance for all processes involving real photons.

An application other than the one given here is, for example, the calculation of the elastic Rayleigh scattering for zero angle from the photoelectric absorption cross section. Also, convergence arguments may be used to infer from dispersion integrals like Eq. (33) an upper limit on the high energy behavior of the absorption process. Finally, Eq. (30b) can be used to infer the absorption cross section from the dispersive forward scattering.