of only one function aaa which belongs to the irreducible representation [3]. The manifold (aab) consists of three functions, aab, aba, and baa, which can be combined into three linearly independent functions ($aab+baa+aba$), belonging to one-dimensional representation [3], ($aab-baa$) and ($aab+baa-2aba$) both belonging to the irreducible representation [2+1]. From the six functions in the manifold (abc), six linearly independent functions can be formed, two of which belong to each of the two two-dimensional representations [2+1], and one each to the one-dimensional representations [3] and [1+1+1].

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Electrodynamic Displacement of Atomic Energy Levels. I. Hyperfine Structure

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The vacuum fluctuations of the photon and pair fields modify the interaction of an electron with an electromagnetic field. The effects on the energy levels are conveniently described in terms of the mass operator and the vacuum polarization potential. A gauge-covariant expansion of the mass operator for the motion of an electron in a weak external electromagnetic field is derived; the expression contains terms quadratic in the field but includes only the lowest order electrodynamic correction. The modification in the Fermi formula is then computed by specializing the external field to consist of the Coulomb and magnetic dipole fields of the nucleus and by taking the matrix element of the operators in an S-state of a hydrogen-like atom. All changes can be described as a correction $\Delta g=-2Z\alpha^2(5/2-\ln 2)$ in the gyromagnetic ratio of the electron. The value of the fine structure constant deduced from measurements of the hyperfine structure becomes $\alpha^{-1}=137.0364$.

I. INTRODUCTION

THE success of the covariant reformulations of quantum electrodynamics is predicated in the first instance on their unambiguous prediction of the observable effects associated with the coupling of an electron to the vacuum fluctuations of the photon field. The experimental investigation of these phenomena depends primarily on experiments performed on hydrogen-like atoms. By this means, it has been possible to test the predictions of the theory concerning the effective static magnetic moment of the electron[1,2] and the electrodynamic shift of energy levels (Lamb shift).[3,4] In addition, the use of the magnetic moment result in the corrected Fermi formula,[5] in conjunction with a precise determination of the hyperfine structure splitting of the ground state of hydrogen,[6] has been the most accurate way of determining α, the fine structure constant.[5]

With the exception of the calculation of the static magnetic moment, which has been carried out to the inclusion of fourth-order (α^2) electrodynamic corrections,[2] the present theoretical predictions must be termed incomplete in several well-defined respects. These are the following: the dependence on the nuclear field has been obtained only through linear terms and for a slowly varying field; the α^2 electrodynamic correction has not been fully ascertained; and the nucleus has been treated as a structureless particle possessing a charge and a magnetic dipole moment.

It is the purpose of the present paper and of one that is to follow to describe methods of treating the prescribed nuclear field (or any external electromagnetic field) to higher approximation. The results will find twofold application. First we shall obtain corrections to the Fermi formula which arise from interference between the Coulomb and dipole fields of the nucleus. Second, we shall compute the contribution to the Lamb shift formula of terms which are quadratic in the Coulomb field.

The investigation proceeds from a Dirac equation modified to include the electron self-energy and the polarization potential induced in the vacuum.[7] The former is described by the mass operator, an integral operator whose general structure has been analyzed elsewhere.[7,8] We restrict our discussion to the electrodynamic correction of order α. The explicit field dependence of the mass operator is contained only in the Green's function for the electron in the prescribed field. Several procedures have been developed for the representation of this dependence to the desired order of approximation.[8,9] In this paper, we describe one of

[1] Koenig, Prodell, and Kusch, Phys. Rev. **83**, 687 (1951).
[2] R. Karplus and N. Kroll, Phys. Rev. **77**, 536 (1950).
[3] W. E. Lamb and R. C. Retherford, Phys. Rev. **81**, 222 (1951).
[4] Bethe, Brown, and Stehn, Phys. Rev. **77**, 370 (1950). Further references are given here.
[5] J. W. M. Dumond and E. R. Cohen, "A Least-Squares Adjustment of the Atomic Constants as of December 1950" (A report to the Natl. Research Council). See also Phys. Rev. **82**, 555 (1951).
[6] A. G. Prodell and P. Kusch, Phys. Rev. **79**, 1009 (1950).
[7] J. Schwinger, Proc. Natl. Acad. **7**, 432, 455 (1951).
[8] J. Schwinger, Phys. Rev. **82**, 664 (1951). We follow the notation of this paper, hereafter referred to as I.
[9] Some of these are discussed by R. P. Feynman, Phys. Rev. **84**, 108 (1951).

these methods which is suited to the treatment of a weak field, in which case a power series expansion is permissible. The construction of the Green's function is related to the construction of a transformation function and its associated momentum operators.[8] By direct expansion and extensive rearrangement, these are exhibited as gauge-covariant quantities correct to second order in the external field. In the resulting mass operator, a separation is effected between the infinite field independent part which is the mass renormalization and the finite field dependent part, which forms the starting point for further discussion and application.

The assumption of a weak field is adequate for the treatment of the coupling of the electron to the dipole field of the proton.[10] The largest contribution, which is the well-known $(\alpha/2\pi)$ correction to the static magnetic moment, is obtained by neglecting the high Fourier components of the magnetic field in the linear field term. For an S-state, the associated magnetic energy depends only on the density of the electron wave function at the origin, as in the Fermi formula. The essentially new results involve the corrections of relative order $Z\alpha$ compared to the above. From the structure of the mass operator, it will be seen that these arise from the behavior of the electron within a neighborhood of its Compton wavelength from the nucleus and are thus present only for an S-state. In a P-state, for example, there will be another factor of $Z\alpha$ arising from the reduced probability density near the origin.

The present considerations do not suffice for the treatment of the Lamb shift. There one encounters the well-known infrared difficulties arising from the fact that the bulk of the effect is not confined to the neighborhood of the nucleus. The emission of soft virtual quanta is thus given full play, with the resultant breakdown of an expansion in powers of the field. The presentation of a modified approach, which subverts this difficulty, the evaluation of results, as well as other methodological advances in the treatment of the mass operator are reserved for a subsequent paper.

II. PRELIMINARY CONSIDERATIONS

A. The Mass Operator; Green's Functions

The description of the motion of an electron in a prescribed electromagnetic field, including vacuum polarization and self-energy effects, will be based on a modified Dirac equation of the form[7, 8]

$$\gamma_\mu(-i\partial_\mu - e\bar{A}_\mu(x))\psi(x)+\int M(x,x')\psi(x')d^4x'=0. \quad (2.1)$$

[10] As will be seen below, this interaction is effective in an S-state only when the electron is within a Compton wavelength $(1/m)$ of the nucleus. At this distance, the kinetic energy of the electron, $p^2/2m\sim m$, whereas the potential energy (in the Coulomb field), α/r, $\sim\alpha m$. The magnetic coupling is smaller still by a factor of the order of the ratio of the electron to the proton mass. Thus, both interactions can be treated as small perturbations in this problem. It is also clear why we shall never consider terms which are quadratic in the magnetic coupling.

Here $\bar{A}_\mu(x)$ is the four-potential of the external field augmented by the potential induced in the vacuum. $M(x, x')$ is the mass operator which formally contains self-energy effects to all orders. To the lowest order in e (second-order electrodynamic correction), it is given by

$$M(x,x')=m_0\delta(x-x')+ie^2\gamma_\mu G_+(x,x')\gamma_\mu D_+(x-x'), \quad (2.2)$$

where $G_+(x, x')$ is a Green's function for the Dirac equation in the external field, which is specified more precisely below, and $D_+(x-x')$ is a photon Green's function represented by (note that $ab=a_\mu b_\mu=\mathbf{a}\cdot\mathbf{b}-a_0b_0$)

$$\begin{aligned}D_+(x-x')&=\int(2\pi)^{-4}d^4k e^{ik(x-x')}(k^2)^{-1}\\&=\int(2\pi)^{-4}d^4k e^{ik(x-x')}i\int_0^\infty dt\exp[-itk^2]\\&=(4\pi)^{-2}\int_0^\infty t^{-2}dt\exp[i(x-x')^2/4t]. \quad (2.3)\end{aligned}$$

In Eq. (2.3) it is implicitly understood that for k^2, one should read $k^2-i\epsilon$, with ϵ small and positive so that the Green's function $D_+(x-x')$ contains only outgoing waves in the remote past and future.

The procedure for extracting the physical consequences of Eq. (2.1) rests, in the approximation considered, on a perturbation calculus which takes the wave equation in the given field as starting point and treats the polarization potential and the second term of Eq. (2.2) as the perturbing elements. Of these, the former has already been computed to the accuracy to which we shall require it and will be stated and used when needed. On the other hand, previous treatments of self-energy effects have, with the exception of Bethe's nonrelativistic calculation, been confined to a consideration of Eq. (2.2) in first Born approximation. Our first aim will be to explore means of representing the mass operator in explicit form to a higher order of approximation in the external field. We consider here one such means based on the direct expansion of $G_+(x, x')$, applicable when the field can be considered weak. The method in question was devised so as to exhibit the Green's function and any operator of which it is part in gauge-covariant form before any specialization of electromagnetic potential is made. It has already been applied in several forms in I to the problem of vacuum polarization by a slowly varying field of arbitrary strength. We briefly recall those aspects which are relevant to the present case.

The particle Green's function satisfies the operator equation

$$(\gamma\Pi+m)G_+=1. \quad (2.4)$$

The solution of Eq. (2.4) can be written in the symmetrical form

$$\begin{aligned}G_+&=\tfrac{1}{2}\{m-\gamma\Pi,[m^2-(\gamma\Pi)^2]^{-1}\}\\&=i\int_0^\infty ds\exp[im^2s]\tfrac{1}{2}\{m-\gamma\Pi,\exp[i(\gamma\Pi)^2s]\}. \quad (2.5)\end{aligned}$$

The introduction of the integral representation (2.5) implies that m^2 has a small negative imaginary part and that the Green's function G_+ is therefore the one which propagates with increasing phase in any time-like direction from the source. We might emphasize here once and for all that we shall be interested in the real part of any matrix element of the mass operator since we are concerned with energy level shifts. It will thus be unnecessary during our future work to take cognizance of the presence of the small imaginary addition to the mass that makes Eq. (2.5) well defined and of the corresponding addition to k^2 in Eq. (2.3). When we eventually perform an integration over the parameter s, it will be correct merely to treat oscillatory exponentials as if they were decaying exponentials. More formally we could make the substitution of variables $s'=is$ and integrate with respect to s' along its positive real axis.

The electron Green's function is the matrix element of the operator Eq. (2.5) with respect to space-time coordinates,

$$G_+(x',x'')=(x'|G_+|x''). \tag{2.6}$$

We employ the notation

$$(x'|\exp[i(\gamma\Pi)^2 s]|x'') \equiv (x'|U(s)|x'')\equiv(x'(s)|x''(0)) \tag{2.7}$$

which will be termed the "transformation function";

$$(x'|\Pi_\mu U(s)|x'')\equiv(x'(s)|\Pi_\mu(s)|x''(0)) =(-i\partial_\mu'-eA_\mu')(x'(s)|x''(0)) \tag{2.8}$$

and

$$(x'|U(s)\Pi_\mu|x'')\equiv(x'(s)|\Pi_\mu(0)|x''(0)) =(i\partial_\mu''-eA_\mu'')(x'(s)|x''(0)) \tag{2.9}$$

are the corresponding matrix elements of the momentum operators. The Green's function can thus be written[11]

$$G_+(x',x'')=i\int_0^\infty ds\,\exp[-im^2s][m(x'(s)|x''(0)) -\tfrac{1}{2}\gamma(x'(s)|\Pi(s)|x''(0)) -\tfrac{1}{2}(x'(s)|\Pi(0)|x''(0))\gamma]. \tag{2.10}$$

The procedure is now to expand $(x'(s)|x''(0))$ about its value for a free particle up to terms quadratic in the external field, the result to be exhibited in gauge-covariant form. The matrix elements of the momentum operators are then obtained by means of Eqs. (2.7) and (2.8) and the results amalgamated to form the mass operator, Eq. (2.2). The details of this calculation are contained in Sec. III.

[11] The combination of momentum operator matrix elements that appears in Eq. (2.10) will henceforth be written for convenience as

$$-\tfrac{1}{2}(x'(s)|\gamma\Pi(s)+\Pi(0)\gamma|x''(0))$$

with the understanding that the more precise rendering is that of (2.10).

B. The Hyperfine Structure

In order to lay the basis for the application of the results of Sec. III to the calculation of corrections to the Fermi hyperfine structure formula, we shall briefly discuss pertinent background material.

The full vector potential of Eq. (2.1) has the form

$$\bar{A}_\mu(x)=A_\mu{}^E(x)+A_\mu{}^M(x)+A_\mu{}^{EP}(x)+A_\mu{}^{MP}(x), \tag{2.11}$$

where the superscripts E and M indicate electric and magnetic, respectively, and the additional superscript P indicates the corresponding vacuum polarization potential. $A^E(x)$ is the Coulomb potential of the nucleus,

$$\mathbf{A}^E=0,\quad A_0{}^E=-Ze/4\pi r \tag{2.12}$$

(e is the electronic charge), and belongs to the unperturbed problem. $A^M(x)$ is the vector potential of the proton considered as a point dipole. It is given by

$$\mathbf{A}^M=\boldsymbol{\mu}\times\mathbf{r}/4\pi r^3=\nabla\times(\boldsymbol{\mu}/4\pi r),\quad A_0{}^M=0, \tag{2.13}$$

where $\boldsymbol{\mu}$ is the proton dipole moment operator. The Fermi formula is the diagonal matrix element of the interaction of A^M with the electron current in the ground state of hydrogen. Since only the polarization current of the electron is relevant here, the interaction energy ΔE_0 has the form

$$\Delta E_0=-(e/2m)\int d\mathbf{r}\bar{\psi}_c(\mathbf{r})\boldsymbol{\sigma}\psi_c(\mathbf{r})\cdot\mathbf{H}(\mathbf{r}). \tag{2.14}$$

For an S-state the spherical symmetry reduces the magnetic field to a δ-function,

$$\mathbf{H}=\nabla\times\mathbf{A}=\nabla\times(\nabla\times(\boldsymbol{\mu}/4\pi r))=\nabla\nabla\cdot(\boldsymbol{\mu}/4\pi r) -\nabla^2(\boldsymbol{\mu}/4\pi r)\rightarrow-\tfrac{2}{3}\nabla^2(\boldsymbol{\mu}/4\pi r)=\tfrac{2}{3}\boldsymbol{\mu}\delta(\mathbf{r}). \tag{2.15}$$

An adequate representation of the large and small components of the Coulomb wave function $\psi_c(\mathbf{r})$ by means of the corresponding Schrödinger wave function $\varphi_0(\mathbf{r})$ then gives the Fermi formula with the Breit correction[12,13]

$$\Delta E_0=-\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(1+\tfrac{3}{2}(Z\alpha)^2). \tag{2.16}$$

The corrections we shall obtain will be exhibited as multiples of the leading term in Eq. (2.16).

The largest addition of electrodynamic origin is, of course, the relative change $(\alpha/2\pi)$ in the spin density of the electron, which will emerge again from our calculation. Further corrections are at least of relative order $Z\alpha^2$ compared to the reference term and may all be said to arise from the spatially distributed nature of quantum electrodynamic corrections. This statement is perhaps most clearly illustrated by preliminary consideration of the effects due to the interaction of the electron current with the vacuum polarization potentials. In this connection it proves convenient to intro-

[12] E. Fermi, Z. Physik **60**, 320 (1930).
[13] G. Breit, Phys. Rev. **35**, 1447 (1949).

duce the four-dimensional Fourier integral representation

$$A_\mu(x)=\int(2\pi)^{-2}d^4kA_\mu(k)e^{ikx}, \qquad (2.17)$$

with analogous definitions for the current $J_\mu(x)$ and the field tensor $F_{\mu\nu}(x)$. The transforms that will be of particular interest in the following work are

$$\begin{aligned}
2\pi A_0{}^E(k)&=-\delta(k_0)Ze/k^2,\\
2\pi\mathbf{A}^M(k)&=\delta(k_0)i\mathbf{k}\times\boldsymbol{\mu}/k^2,\\
2\pi\mathbf{E}(k)&=\delta(k_0)Zei\mathbf{k}/k^2,\\
2\pi\mathbf{H}(k)&=\delta(k_0)i\mathbf{k}\times(i\mathbf{k}\times\boldsymbol{\mu})/k^2\rightarrow\tfrac{2}{3}\boldsymbol{\mu}\delta(k_0),\\
2\pi J_0{}^E(k)&=-\delta(k_0)Ze,\\
2\pi\mathbf{J}^M(k)&=\delta(k_0)i\mathbf{k}\times\boldsymbol{\mu}.
\end{aligned} \qquad (2.18)$$

The last expression for $\mathbf{H}(k)$ is the Fourier transform of the result Eq. (2.12). In terms of the definition Eq. (2.17)[14]

$$A_\mu{}^P(k)=\frac{\alpha}{4\pi}J_\mu(k)\int_0^1\frac{dvv^2(1-\frac{1}{3}v^2)}{m^2+\frac{1}{4}k^2(1-v^2)}. \qquad (2.19)$$

From Eq. (2.18) it follows that

$$\begin{aligned}
\mathbf{H}^P(k)&=i\mathbf{k}\times\mathbf{A}^{MP}(k)\\
&=\frac{\alpha}{4\pi}\frac{\delta(k_0)}{2\pi}k^2\mathbf{H}(\mathbf{k})\int_0^1\frac{dvv^2(1-\frac{1}{3}v^2)}{m^2+\frac{1}{4}k^2(1-v^2)}.
\end{aligned} \qquad (2.20)$$

The corresponding interaction energy is

$$-\mu_0\int d\mathbf{r}\bar{\psi}_c(\mathbf{r})\sigma\psi_c(\mathbf{r})\cdot\tfrac{2}{3}\boldsymbol{\mu}\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{r}}\frac{\alpha}{4\pi}k^2 \times\int_0^1\frac{dvv^2(1-\frac{1}{3}v^2)}{m^2+\frac{1}{4}k^2(1-v^2)}. \qquad (2.21)$$

The magnetic field is no longer confined to the origin, but, as is evident from its form, is spread out over a distance of the electron Compton wavelength, $1/m$. In fact, if one approximates $\psi_c(\mathbf{r})$ by $\varphi_0(0)$, expression (2.21) is easily seen to vanish. To obtain a nonvanishing result one requires the more accurate representation

$$\begin{aligned}
\psi_c(\mathbf{r})&\rightarrow\varphi_0(\mathbf{r})\cong\varphi_0(0)(1-Z\alpha rm),\\
\varphi^*(\mathbf{r})\varphi(\mathbf{r})&\cong|\varphi_0(0)|^2(1-2Z\alpha rm)
\end{aligned} \qquad (2.22)$$

of the wave function in the neighborhood of the origin.

The matrix element of Eq. (2.21) now becomes

$$\tfrac{2}{3}\mu_0\langle\sigma\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2\frac{Z\alpha^2}{2\pi}\int\frac{d\mathbf{r}d\mathbf{k}}{(2\pi)^3}rk^2m \times\int_0^1\frac{dvv^2(1-\frac{1}{3}v^2)}{m^2+\frac{1}{4}k^2(1-v^2)}. \qquad (2.23)$$

Further consideration of expression (2.23) is reserved for Sec. VI, where an analogous contribution from the Coulomb polarization potential taken in conjunction with a wave function modified by the magnetic field will also be considered. We merely note there that Eq. (2.23) is of prototype form, since it contains one factor of α of electrodynamic origin, one factor of $Z\alpha$ of Coulombic origin, and is linear in the magnetic coupling.

The $Z\alpha^2$ corrections that are contributed by the matrix elements of the mass operator will be considered in Secs. IV and V.

III. MASS OPERATOR

An expression for the mass operator in a weak, arbitrarily varying external electromagnetic field will now be derived up to quadratic terms in this field. Since the formulas are quite involved, it is useful to obtain as a preliminary result, the matrix element of the transformation function $U(s)$ to the necessary order of accuracy. The calculation of the momenta Π_μ from the transformation function then furnishes all the necessary ingredients for the Green's function which contains the entire dependence of the mass operator on the prescribed field.

In order to exhibit the gauge covariance of the transformation function, it will be expressed in the form

$$\begin{aligned}
(x'(s)|x''(0))&=(x'|U(s)|x'')\\
&=-i(4\pi s)^{-2}\Phi(x',x'')\\
&\quad\times\exp[i(x'-x'')^2/4s]U'(s;x',x''),
\end{aligned} \qquad (3.1)$$

where the function $U'(s;x',x'')$ must depend on the field in a gauge invariant way and must reduce to unity as the field vanishes. An expansion of the form discussed in Sec. VI of reference 8 yields the transformation operator

$$\begin{aligned}
U(s)=&\exp[-isp^2]+ise\int_{-1}^1\tfrac{1}{2}dv\exp[-isp^2\tfrac{1}{2}(1-v)]\\
&\times(pA+Ap+\tfrac{1}{2}\sigma F-eA^2)\exp[-isp^2\tfrac{1}{2}(1+v)]\\
&+(ise)^2\int_{-1}^1\tfrac{1}{2}dv_1\int_{-1}^{v_1}\tfrac{1}{2}dv_2\\
&\times\exp[-isp^2\tfrac{1}{2}(1-v_1)](pA+Ap+\tfrac{1}{2}\sigma F)\\
&\times\exp[-isp^2\tfrac{1}{2}(v_1-v_2)](pA+Ap+\tfrac{1}{2}\sigma F)\\
&\times\exp[-isp^2\tfrac{1}{2}(1+v_2)],
\end{aligned} \qquad (3.2)$$

[14] J. Schwinger, Phys. Rev. **76**, 790 (1949).

which has the matrix element

$$(x'|U(s)|x'')$$

$$=\int(2\pi)^{-4}d^4pe^{ip(x'-x'')}\Big\{\exp[-isp^2]$$

$$+ise\int_{-1}^{1}\tfrac{1}{2}dv\int(2\pi)^{-2}d^4ke^{\frac{1}{2}ik(x'+x'')}$$

$$\times\exp[-is(p+\tfrac{1}{2}k)^2\tfrac{1}{2}(1-v)]\Big[2pA+\tfrac{1}{2}\sigma F$$

$$-e\int(2\pi)^{-2}d^4k'A(k')A(k-k')\Big]$$

$$\times\exp[-is(p-\tfrac{1}{2}k)^2\tfrac{1}{2}(1+v)]+(ise)^2\int_{-1}^{1}\tfrac{1}{2}dv_1$$

$$\times\int_{-1}^{v_1}\tfrac{1}{2}dv_2(2\pi)^{-4}d^4k_1d^4k_2e^{\frac{1}{2}i(k_1+k_2)(x'+x'')}$$

$$\times\exp[-is(p+\tfrac{1}{2}(k_1+k_2))^2\tfrac{1}{2}(1-v_1)]$$

$$\times[(2p+k_2)A^1+\tfrac{1}{2}\sigma F^1]\exp[-is(p-\tfrac{1}{2}k_1+\tfrac{1}{2}k_2)^2$$

$$\times\tfrac{1}{2}(v_1-v_2)][(2p-k^1)A^2+\tfrac{1}{2}\sigma F]$$

$$\times\exp[-is(p-\tfrac{1}{2}k_1-\tfrac{1}{2}k_2)^2\tfrac{1}{2}(1+v_2)]\Big\},$$

$$\left(A_\mu(k_i)=A_\mu{}^i=\int(2\pi)^{-2}d^4xe^{-ik_ix}A_\mu(x)\right). \quad (3.3)$$

The translations

$$p_\mu\rightarrow(p+\tfrac{1}{2}kv)_\mu \quad\text{and}\quad p_\mu\rightarrow(p+\tfrac{1}{2}(k_1v_1+k_2v_2))_\mu \quad (3.4)$$

of the momentum coordinates in the linear and in the quadratic expansion terms serve to eliminate all scalar products (pk) from the exponents and yield the common factor $\exp[-isp^2]$. It is then possible to carry out the integration over the variable p_μ by noting that

$$\int(2\pi)^{-4}d^4pe^{ip(x'-x'')}p_\mu\exp[-isp^2]$$

$$=\int(2\pi)^{-4}d^4pe^{ip(x'-x'')}\tfrac{1}{2}is^{-1}(\partial/\partial p_\mu)\exp[-isp^2]$$

$$=\int(2\pi)^{-4}d^4pe^{ip(x'-x'')}(2s)^{-1}(x'-x'')_\mu\exp[-isp^2]$$

$$=-i(4\pi s)^{-2}(2s)^{-1}(x'-x'')_\mu\exp[i(x'-x'')^2/4s] \quad (3.5)$$

on an integration by parts. With the notation $(x'-x'')_\mu=\Delta x_\mu$, it then follows that

$$\int(2\pi)^{-4}d^4pe^{ip\Delta x}\exp[-isp^2]\{1;\,p_\mu;\,p_\mu p_\nu;\,p_\mu p_\nu p_\lambda\}$$

$$=-i(4\pi s)^{-2}\exp[i(\Delta x)^2/4s]\{1;\,(2s)^{-1}\Delta x_\mu;$$

$$(2s)^{-2}\Delta x_\mu\Delta x_\nu-i(2s)^{-1}\delta_{\mu\nu};\,(2s)^{-3}\Delta x_\mu\Delta x_\nu\Delta x_\lambda$$

$$-i(2s)^{-2}(\delta_{\mu\nu}\Delta x_\lambda+\delta_{\mu\lambda}\Delta x_\nu+\delta_{\nu\lambda}\Delta x_\mu)\}. \quad (3.6)$$

The function $U'(s;x',x'')$ defined in Eq. (3.1) can now be written

$$U'(s;x',x'')=[\Phi(x',x'')]^{-1}\Big\{1+ise\int_{-1}^{1}\tfrac{1}{2}dv$$

$$\times\int(2\pi)^{-2}d^4ke^{ik\xi}\exp[-is\tfrac{1}{4}k^2(1-v^2)]$$

$$\times\Big[((\Delta x/s)+kv)A+\tfrac{1}{2}\sigma F-e\int(2\pi)^{-2}d^4k'A(k')$$

$$\times A(k-k')\Big]+(ise)^2\int_{-1}^{1}\tfrac{1}{2}dv_1\int_{-1}^{v_1}\tfrac{1}{2}dv_2e^{ik_1\xi_1}$$

$$\times e^{ik_2\xi_2}(2\pi)^{-4}d^4k_1d^4k_2\exp[-is\tfrac{1}{4}K^2]$$

$$\times\{[((\Delta x/s)+k_2(1+v_2)+k_1v_1)A^1$$

$$+\tfrac{1}{2}\sigma F^1][((\Delta x/s)+k_2v_2-k_1(1-v_1))A^2$$

$$+\tfrac{1}{2}\sigma F^2]-2iA^1A^2/s\}\Big\}, \quad (3.7)$$

where

$$\xi_{i\mu}=\tfrac{1}{2}[x_\mu'(1+v_i)+x_\mu''(1-v_i)]$$

and $\qquad (3.7')$

$$K^2=k_1^2(1-v_1^2)+2k_1k_2(1-v_1)(1+v_2)+k_2^2(1-v_2^2).$$

By a fairly extensive rearrangement it can now be shown that up to second-order terms in the external field the function $U'(s;x',x'')$ really depends only on gauge invariant quantities. This fact will be verified explicitly for the first-order terms

$$ise\int_{-1}^{1}\tfrac{1}{2}dv(2\pi)^{-2}d^4ke^{ik\xi}\{[(\Delta x/s)+kv]A$$

$$\times\exp[-is\tfrac{1}{4}k^2(1-v^2)]-(\Delta x/s)A\}. \quad (3.8)$$

The last contribution in the bracket has come from an expansion of the gauge dependent exponential $[\Phi(x',x'')]^{-1}$. An integration by parts with respect to v of the middle term

$$\int_{-1}^{1}\tfrac{1}{2}dve^{ik\xi}v\exp[-is\tfrac{1}{4}k^2(1-v^2)]$$

$$=(isk^2)^{-1}e^{ik\xi}(\exp[-is\tfrac{1}{4}k^2(1-v^2)]-1)\Big|_{-1}^{1}$$

$$-(k\Delta x/sk^2)\int_{-1}^{1}\tfrac{1}{2}dve^{ik\xi}(\exp[-is\tfrac{1}{4}k^2(1-v^2)]-1) \quad (3.9)$$

shows the identity of Eq. (3.8) with

$$ise\int_{-1}^{1}\tfrac{1}{2}dv(2\pi)^{-2}d^4ke^{ik\xi}(\Delta xJ/sk^2)\times(\exp[-is\tfrac{1}{4}k^2(1-v^2)]-1), \quad (3.10)$$

since

$$k^2A_\mu-k_\mu(kA)=-ik_\nu F_{\nu\mu}=J_\mu. \quad (3.11)$$

By a similar treatment the second-order terms from the expansion of $[\Phi(x',x'')]^{-1}$, from the linear expansion term, and from the quadratic expansion term can be combined into an expression that depends only on field strengths and current densities and their derivatives. With the abbreviations

$$e_i(s)=\exp[-is\tfrac{1}{4}k_i^2(1-v_i^2)],\quad E(s)=\exp[-is\tfrac{1}{4}K^2] \quad (3.12)$$

for the ubiquitous exponential factors, as well as

$$\int d^4\bar{k}_j=\int_{-1}^{1}\tfrac{1}{2}dv_j\int(2\pi)^{-2}d^4k_je^{ik_j\xi_j}$$

and (3.13)

$$\int d^8\bar{K}=\int_{-1}^{1}\tfrac{1}{2}dv_1\int_{-1}^{v_1}\tfrac{1}{2}dv_2\int(2\pi)^{-4}d^4k_1d^4k_2e^{ik_1\xi_1}e^{ik_2\xi_2}$$

for the Fourier integral operators, the function $U'(s;x'x'')$ becomes

$$\begin{aligned}U'(s;x',x'')&=1+ise\int d^4\bar{k}\{(\Delta xJ/sk^2)(e(s)-1)+\tfrac{1}{2}\sigma Fe(s)\}\\&\quad+(ise)^2\int d^8\bar{K}\{(\Delta xJ^1/sk_1^2)(\Delta xJ^2/sk_2^2)\\&\quad\times[E(s)-e_1(s)-e_2(s)+1]\\&\quad+i[(J^2F^1\Delta x/sk_2^2)(1-v_1)\\&\quad-(J^1F^2\Delta x/sk_1^2)(1+v_2)]E(s)\\&\quad+\tfrac{1}{2}(J^1J^2/sk_1^2k_2^2)[k_2\Delta x(1+v_2)-k_1\Delta x(1-v_1)\\&\quad+sk_1^2v_1(1-v_1)-sk_2^2v_2(1+v_2)]E(s)\\&\quad+\tfrac{1}{2}\sigma F^1(\Delta xJ^2/sk_2^2)[E(s)-e_1(s)]\\&\quad+\tfrac{1}{2}\sigma F^2(\Delta xJ^1/sk_1^2)[E(s)-e_2(s)]\\&\quad+[\tfrac{1}{2}\sigma F^1(k_1J^2/k_2^2)(1+v_2)-\tfrac{1}{2}\sigma F^2(k_2J^1/k_1^2)\\&\quad\times(1-v_1)]E(s)-(1-v_1)(1+v_2)\tfrac{1}{2}F^1F^2E(s)\\&\quad+\tfrac{1}{2}\sigma F^1\tfrac{1}{2}\sigma F^2E(s)\}. \quad (3.14)\end{aligned}$$

We may now proceed to an evaluation of the matrix elements of the momentum operators:

$$\begin{aligned}(x'(s)|\Pi_\mu(s)|x''(0))&=(-i\partial_\mu'-eA_\mu(x'))(x'(s)|x''(0))\\&=-i(4\pi s)^{-2}\exp[i(\Delta x)^2/4s]\Phi(x',x'')\\&\quad\times\left\{\left[(\Delta x_\mu/2s)+e\int d^4\bar{k}\tfrac{1}{2}(1+v)(F\Delta x)_\mu\right]\right.\\&\quad\left.\times U'(s;x',x'')-i\partial_\mu'U'(s;x',x'')\right\} \quad (3.15)\end{aligned}$$

and

$$\begin{aligned}(x'(s)|\Pi_\mu(0)|x''(0))&=(i\partial_\mu''-eA_\mu(x''))(x'(s)|x''(0))\\&=-i(4\pi s)^{-2}\exp[i(\Delta x)^2/4s]\Phi(x',x'')\\&\quad\times\left\{\left[(\Delta x_\mu/2s)-e\int d^4\bar{k}\tfrac{1}{2}(1-v)(F\Delta x)_\mu\right]\right.\\&\quad\left.\times U'(s;x',x'')+i\partial_\mu''U'(s;x',x'')\right\}. \quad (3.16)\end{aligned}$$

Since the momenta enter the mass operator in the combination $\frac{1}{2}(\gamma\Pi(s)+\Pi(0)\gamma)$, only this quantity will be given in detail. The fact that the operator is a Dirac matrix suggests that the contributions to the momenta and to the transformation function be classified according to their spinor character. Thus we have the scalar part (a multiple of the unit matrix), the "spin" part (a multiple of $\sigma_{\mu\nu}$), and the pseudoscalar part (a multiple of γ_5) which is introduced by the decomposition

$$\tfrac{1}{2}\sigma F^1\tfrac{1}{2}\sigma F^2=\tfrac{1}{2}F^1F^2-i\,\mathrm{tr}[F^1\sigma F^2]+\gamma_5\tfrac{1}{2}F^1F^{2*} \quad (3.17)$$

and involves the dual of $F_{\mu\nu}$,

$$F_{\mu\nu}{}^*=\tfrac{1}{2}i\epsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}\quad\text{(cf. Eqs. (I, 3.25–3.27)).} \quad (3.18)$$

In terms of integral operators that will be defined below, we can now write the Green's function for the Dirac field:

$$\begin{aligned}G_+(x',x'')&=i\int_0^\infty ds\exp[-im^2s]\\&\quad\times(x'(s)|m-\tfrac{1}{2}(\gamma\Pi(s)+\Pi(0)\gamma)|x''(0))\\&=(4\pi)^{-2}\int_0^\infty s^{-2}ds\exp[-im^2s+i(\Delta x)^2/4s]\Phi(x',x'')\\&\quad\times([m-(\gamma\Delta x/2s)][1+A(s)+C(s)]\\&\quad-\gamma_\mu[A_{+\mu}(s)+C_{+\mu}(s)]-\tfrac{1}{2}\{\gamma B_+(s)\\&\quad+[(\gamma\Delta x/2s)-m]B(s),\tfrac{1}{2}\sigma F\}-\tfrac{1}{2}[\gamma B_-(s),\tfrac{1}{2}\sigma F]\\&\quad-\tfrac{1}{2}\{\gamma D_+{}^2(s)+[(\gamma\Delta x/2s)-m]D^2(s),\tfrac{1}{2}\sigma F^1\}\\&\quad-\tfrac{1}{2}[\gamma D_-{}^2(s),\tfrac{1}{2}\sigma F^1]-\tfrac{1}{2}\{\gamma D_+{}^1(s)\\&\quad+[(\gamma\Delta x/2s)-m]D^1(s),\tfrac{1}{2}\sigma F^2\}\\&\quad-\tfrac{1}{2}[\gamma D_-{}^1(s),\tfrac{1}{2}\sigma F^2]+\tfrac{1}{2}i\{\gamma G_+(s)\\&\quad+[(\gamma\Delta x/2s)-m]G(s),\mathrm{tr}[F^1\sigma F^2]\}\\&\quad+\tfrac{1}{2}i[\gamma G_-(s),\mathrm{tr}[F^1\sigma F^2]]\\&\quad+[\gamma G_-(s)+mG(s)]\tfrac{1}{2}\gamma_5F^1F^{2*}). \quad (3.19)\end{aligned}$$

The symbols have the following significance:

$$A(s)=ise\int d^4\bar{k}(\Delta xJ/sk^2)[e(s)-1]; \quad (3.20a)$$

$$A_{+\mu}(s)=ise\int d^4\bar{k}\tfrac{1}{2}v[-i(F\Delta x/s)_\mu - vJ_\mu]e(s); \quad (3.20b)$$

$$B(s)=ise\int d^4\bar{k}e(s)$$

$$\left(\text{whence } B(s)\tfrac{1}{2}\sigma F=ise\int d^4\bar{k}\tfrac{1}{2}\sigma Fe(s)\right); \quad (3.20c)$$

$$B_{+\mu}(s)=ise\int d^4\bar{k}\tfrac{1}{2}vk_\mu e(s); \quad (3.20d)$$

$$B_{-\mu}(s)=ise\int d^4\bar{k}\tfrac{1}{2}k_\mu e(s); \quad (3.20e)$$

$$\begin{aligned}C(s)=(ise)^2\int d^8\bar{K}\{&(\Delta xJ^1/sk_1^2)(\Delta xJ^2/sk_2^2)\\&\times[E(s)-e_1(s)-e_2(s)+1]+\tfrac{1}{2}(J^1J^2/sk_1^2k_2^2)\\&\times[k_2\Delta x(1+v_2)-k_1\Delta x(1-v_1)\\&+sk_1^2v_1(1-v_1)-sk_2^2v_2(1+v_2)]E(s)\\&+i[(J^2F^1\Delta x/sk_2^2)(1-v_1)\\&-(J^1F^2\Delta x/sk_1^2)(1+v_2)]E(s)\\&+(v_1-v_2+v_1v_2)\tfrac{1}{2}F^1F^2E(s)\}, \quad (3.20f)\end{aligned}$$

$$\begin{aligned}C_{+\mu}(s)=(ise)^2\int d^8\bar{K}\{&-\tfrac{1}{2}(v_1/s)J_\mu^1(\Delta xJ^2/sk_2^2)\\&\times[E(s)(v_1+(k_1k_2/k_1^2)(1+v_2))-v_1e_1(s)]\\&-\tfrac{1}{2}(v_2/s)J_\mu^2(\Delta xJ^1/sk_1^2)\\&\times[E(s)(v_2-(k_1k_2/k_2^2)(1-v_1))-v_2e_2(s)]\\&+(sk_1^2k_2^2)^{-1}[\tfrac{1}{2}J^1J^2(k_2(1+v_2)-k_1(1-v_1))_\nu\\&+ik_1^2(J^2F^1)_\nu(1-v_1)-ik_2^2(J^1F^2)_\nu(1+v_2)]\\&\times[\Delta x_\nu\tfrac{1}{2}(k_1v_1+k_2v_2)_\mu-i\delta_{\mu\nu}]E(s)\\&-\tfrac{1}{2}iv_1(F^1\Delta x)_\mu(\Delta xJ^2/sk_2^2)[E(s)-e_1(s)]\\&+\tfrac{1}{2}iv_2(F^2\Delta x)_\mu(\Delta xJ^1/sk_1^2)[E(s)-e_2(s)]\\&+[\tfrac{1}{2}(J^1J^2/k_1^2k_2^2)(k_1^2v_1(1-v_1)\\&-k_2^2v_2(1+v_2))+(v_1-v_2+v_1v_2)\tfrac{1}{2}F^1F^2]\\&\times\tfrac{1}{2}(k_1v_1+k_2v_2)_\mu E(s)\}; \quad (3.20g)\end{aligned}$$

$$\begin{aligned}D^1(s)=(ise)^2\int d^8\bar{K}\{&(\Delta xJ^1/sk_1^2)(E(s)-e_2(s))\\&+(1+v_2)(k_2J^1/k_1^2)E(s)\}; \quad (3.20h)\end{aligned}$$

$$\begin{aligned}D^2(s)=(ise)^2\int d^8\bar{K}\{&(\Delta xJ^2/sk_2^2)(E(s)-e_1(s))\\&-(1-v_1)(k_1J^2/k_2^2)E(s)\}; \quad (3.20i)\end{aligned}$$

$$\begin{aligned}D_{+\mu}^1(s)=(ise)^2\int d^8\bar{K}\{&\tfrac{1}{2}v_2k_{2\mu}(\Delta xJ^1/sk_1^2)(E(s)-e_2(s))\\&-\tfrac{1}{2}iv_1(F^1\Delta x/s)_\mu E(s)\\&-\tfrac{1}{2}v_1J_\mu^1[v_1+(k_1k_2/k_1^2)(1+v_2)]E(s)\\&+\tfrac{1}{2}(k_1v_1+k_2v_2)_\mu(k_2J^1/k_1^2)E(s)\}; \quad (3.20j)\end{aligned}$$

$$\begin{aligned}D_{+\mu}^2(s)=(ise)^2\int d^8\bar{K}\{&\tfrac{1}{2}v_1k_{1\mu}(\Delta xJ^2/sk_2^2)(E(s)-e_1(s))\\&-\tfrac{1}{2}iv_2(F^2\Delta x/s)_\mu E(s)\\&-\tfrac{1}{2}v_2J_\mu^2[v_2-(k_1k_2/k_2^2)(1-v_1)]E(s)\\&-\tfrac{1}{2}(k_1v_1+k_2v_2)_\mu(k_1J^2/k_2^2)E(s)\}; \quad (3.20k)\end{aligned}$$

$$\begin{aligned}D_{-\mu}^1(s)=(ise)^2\int d^8\bar{K}\{&\tfrac{1}{2}k_{2\mu}(\Delta xJ^1/sk_1^2)(E(s)-e_2(s))\\&-\tfrac{1}{2}i(F^1\Delta x/s)_\mu E(s)\\&-\tfrac{1}{2}J_\mu^1[v_1+(k_1k_2/k_1^2)(1+v_2)]E(s)\\&+\tfrac{1}{2}(k_1+k_2)_\mu(k_2J^1/k_1^2)E(s)\}; \quad (3.20l)\end{aligned}$$

$$\begin{aligned}D_{-\mu}^2(s)=(ise)^2\int d^8\bar{K}\{&\tfrac{1}{2}k_{1\mu}(\Delta xJ^2/sk_2^2)(E(s)-e_1(s))\\&-\tfrac{1}{2}i(F^2\Delta x/s)_\mu E(s)\\&-\tfrac{1}{2}J_\mu^2[v_2-(k_1k_2/k_2^2)(1-v_1)]E(s)\\&-\tfrac{1}{2}(k_1+k_2)_\mu(k_1J^2/k_2^2)E(s)\}; \quad (3.20m)\end{aligned}$$

$$G(s)=(ise)^2\int d^8\bar{K}E(s); \quad (3.20n)$$

$$G_{+\mu}(s)=(ise)^2\int d^8\bar{K}\tfrac{1}{2}(k_1v_1+k_2v_2)_\mu E(s); \quad (3.20o)$$

$$G_{-\mu}(s)=(ise)^2\int d^8\bar{K}\tfrac{1}{2}(k_1+k_2)_\mu E(s). \quad (3.20p)$$

We may note that the A and C are scalar parts, that the B, D, and G are spin parts, and that the G are pseudoscalar parts.

There still remains the task of constructing the entire mass operator and of separating that part of it which merely represents an addition to the rest mass of the electron. In terms of the proper-time representation of the Green's function of the photon field, the mass operator is written

$$M(x',x'')=m_0\delta(x'-x'')+ie^2(2\pi)^{-4}\times\Phi(x',x'')\int_0^\infty s^{-2}ds\int_0^s w^{-2}dw\times\exp[-im^2s+i(\Delta x)^2/4w]\gamma_\lambda(\quad)\gamma_\lambda,\quad(3.21)$$

where ( ) stands for the same bracket of Eq. (3.19) and the variable $w=(s^{-1}+t^{-1})^{-1}$ has replaced t of Eq. (2.3).

We may now observe that the leading terms in the correction to the rest mass,

$$-i(4\pi w)^{-2}\exp[i(\Delta x)^2/4w][-4m-2(\gamma\Delta x/2s)]\quad(3.22)$$

are also the leading terms in the expansion of the operator

$$(x'(w)|-4m-(w/s)(\gamma\Pi(w)+\Pi(0)\gamma)|x''(0))=\delta m(w;x',x'')\quad(3.23)$$

of which the wave function satisfying

$$(\gamma\Pi+m)\psi=0\quad(3.24)$$

is an eigenfunction,

$$\int\delta m(w;x',x'')\psi(x'')d^4x''=-2m(2-w/s)\times\exp[im^2w]\psi(x').\quad(3.25)$$

If we adopt a procedure that treats the electrodynamic correction as a perturbation to be evaluated to first order, wave functions satisfying Eq. (3.24) may be used and the content of Eq. (3.21) can be rewritten

$$\int M(x',x'')\psi(x'')d^4x''=\int(m\delta(x'-x'')+\bar{M}(x',x''))\psi(x'')d^4x''\quad(3.26)$$

in terms of the observed mass of a free electron[7]

$$m=m_0+(\alpha m/2\pi)\int_0^\infty s^{-2}ds\int_0^s dw(2-w/s)\times\exp[-im^2(s-w)]\quad(3.27)$$

and of the finite operator that describes the effect of the vacuum fluctuations of the field on the behavior of an electron in an external field,

$$\bar{M}(x',x'')=-(\alpha/4\pi)\int_0^\infty ds\exp[-im^2s]\int_0^s dw\times\{s^{-2}(x'(w)|4m+(w/s)(\gamma\Pi(w)+\Pi(0)(\gamma))|x''(0))+w^{-2}\gamma_\lambda(x'(s)|m-\tfrac{1}{2}(\gamma\Pi(s)+\Pi(0)\gamma)|x''(0))\gamma_\lambda\times\exp[\tfrac{1}{4}i(\Delta x)^2(w^{-1}-s^{-1})]\}.\quad(3.27')$$

By consulting Eq. (3.19) for the spinor character of the matrix element, one can combine the two parts of Eq. (3.27′). The following identities are useful in this connection:

$$\gamma_\lambda^2=-4;\quad\gamma_\lambda\gamma_5\gamma_\lambda=4\gamma_5\quad(3.28a)$$

$$\gamma_\lambda\gamma_\mu\gamma_\lambda=2\gamma_\mu;\quad\gamma_\lambda\gamma_\mu\gamma_5\gamma_\lambda=-2\gamma_\mu\gamma_5\quad(3.28b)$$

$$\gamma_\lambda\sigma_{\mu\nu}\gamma_\lambda=0;\quad(3.28c)$$

$$\gamma_\lambda[\gamma_\mu,\sigma_{\nu\rho}]\gamma_\lambda=-2[\gamma_\mu,\sigma_{\nu\rho}];\quad\gamma_\lambda\{\gamma_\mu,\sigma_{\nu\rho}\}\gamma_\lambda=2\{\gamma_\mu,\sigma_{\nu\rho}\}.\quad(3.28d)$$

The expansion of the mass operator Eq. (3.27′) analogous to the expression (3.19) for the Green's function may now be written by inspection with the help of the operators $A(w)$, $B(w)$, etc.:

$$\begin{aligned}\bar{M}(x',x'')=&-i\alpha(4\pi)^{-3}\int_0^\infty s^{-2}ds\int_0^s w^{-2}dw\\&\times\exp[-im^2s+i\Delta x^2/4w]\Phi(x',x'')\\&\times([4m+\gamma\Delta x/s][A(s)-A(w)+C(s)-C(w)]\\&+2\gamma_\mu[A_{+\mu}(s)-A_{+\mu}(w)+C_{+\mu}(s)-C_{+\mu}(w)]\\&+4m[B(w)\tfrac{1}{2}\sigma F+D^1(w)\tfrac{1}{2}\sigma F^2+D^2(w)\tfrac{1}{2}\sigma F^1\\&-iG(w)\operatorname{tr}[F^1\sigma F^2]+(G(s)+G(w))\gamma_5\tfrac{1}{2}F^1F^{2*}]\\&-\{\gamma B_+(s)+(w/s)\gamma B_+(w)+(\gamma\Delta x/2s)(B(s)\\&+B(w)),\tfrac{1}{2}\sigma F\}+[\gamma B_-(s)-(w/s)\gamma B_-(w),\tfrac{1}{2}\sigma F]\\&-\{\gamma D_+^1(s)+(w/s)\gamma D_+^1(w)+(\gamma\Delta x/2s)\\&\times(D^1(s)+D^1(w)),\tfrac{1}{2}\sigma F^2\}-\{\gamma D_+^2(s)\\&+(w/s)\gamma D_+^2(w)+(\gamma\Delta x/2s)(D^2(s)+D^2(w)),\tfrac{1}{2}\sigma F^1\}\\&+i\{\gamma G_+(s)+(w/s)\gamma G_+(w)+(\gamma\Delta x/2s)\\&\times(G(s)+G(w)),\operatorname{tr}[F^1\sigma F^2]\}\\&+[\gamma D_-^1(s)-(w/s)\gamma D_-^1(w),\tfrac{1}{2}\sigma F^2]\\&+[\gamma D_-^2(s)-(w/s)\gamma D_-^2(w),\tfrac{1}{2}\sigma F^1]\\&-i[\gamma G_-(s)-(w/s)\gamma G_-(w),\operatorname{tr}[F^1\sigma F^2]]\\&+2[\gamma G_-(s)-(w/s)\gamma G_-(w)]\gamma_5\tfrac{1}{2}F^1F^{2*}]).\quad(3.29)\end{aligned}$$

It may be noticed that this operator has appreciable values only for $(\Delta x)^2\lesssim m^{-2}$, for otherwise the exponential factors oscillate rapidly and average to zero on integration over the proper time parameters.

IV. FIRST-ORDER PART

Since we desire to obtain a result accurate to terms quadratic in the field, it is clear that the matrix elements of those terms in Eq. (3.29) which involve the field only linearly—all operators A and B—must be calculated more carefully than the quadratic terms. In particular, the field dependent terms that are contained in the relationships between $\gamma\Pi$ and $\gamma\Delta x/2s$, Eqs. (3.15) and (3.16), must be retained in the former while they may be ignored in the latter. Similarly, in the latter terms the factor $\Phi(x',x'')$ may be approximated by unity, while in the former it must be expanded to include linear terms. These rearrangements that must be carried out will now be considered without restricting the generality of the vector potential A_μ. We shall suppose, however, that a matrix element of the mass operator is taken between wave functions that satisfy Eq. (3.24); hence operators $\gamma(-i\partial'-eA(x'))$ and $\gamma(i\partial''-eA(x''))$ acting from the left or right, respectively, may be transferred to the wave function by an integration by parts and then replaced by $(-m)$.

The procedure is illustrated by its application to the first term in Eq. (3.29):[15]

$$
\begin{aligned}
&4m\exp[i(\Delta x)^2/4w]\Phi(x',x'')ise\\
&\quad\times\int d^4\bar{k}(-\tfrac{1}{2}\{\gamma\Delta x,\gamma J\}/sk^2)(e(s)-e(w))\\
&=4mise\int(-w/s)\bigg[\gamma\bigg(-i\partial'-eA(x')-\tfrac{1}{2}k(1+v)\\
&\qquad-e\int d^4k_1\tfrac{1}{2}(1+v_1)(F^1\Delta x)\bigg)\gamma J\\
&\qquad+\gamma J\gamma\bigg(i\partial''-eA(x'')+\tfrac{1}{2}k(1-v)\\
&\qquad+e\int d^4k_1\tfrac{1}{2}(1-v_1)(F^1\Delta x)\bigg)\bigg]\Phi(x',x'')\\
&\qquad\times\exp[i(\Delta x)^2/4w]d^4\bar{k}(e(s)-e(w))/k^2\\
&=4m\exp[i(\Delta x)^2/4w]\Phi(x',x'')\\
&\quad\times\bigg\{ie\int d^4\bar{k}[2m\gamma J-k^2\tfrac{1}{2}\sigma F][e(s)-e(w)]\Big/k^2\\
&\qquad+(ie)^2w\int d^4\bar{k}_1d^4\bar{k}_2\bigg[i\gamma J^2(\gamma F^1\Delta x)\tfrac{1}{2}(1-v_1)\\
&\qquad-i(\gamma F^1\Delta x)\gamma J^2\tfrac{1}{2}(1+v_2)\bigg](e_2(s)-e_2(w))\Big/k_2^2\bigg\}. \qquad (4.1)
\end{aligned}
$$

In carrying out this reduction as well as in the treatment of the other first-order terms the following identities help to simplify the results:

$$\{\gamma J,\gamma k\}=0; \qquad (4.2a)$$

$$\tfrac{1}{2}[\gamma J,\gamma k]=k^2\tfrac{1}{2}\sigma F; \qquad (4.2b)$$

$$\tfrac{1}{2}\{\gamma_\mu,\tfrac{1}{2}\sigma F\}=i\gamma_5(\gamma F^*)_\mu;\quad \tfrac{1}{2}\{\gamma k,\tfrac{1}{2}\sigma F\}=0; \qquad (4.2c)$$

$$\tfrac{1}{2}[\gamma_\mu,\tfrac{1}{2}\sigma F]=i(\gamma F)_\mu;\qquad \tfrac{1}{2}[\gamma k,\tfrac{1}{2}\sigma F]=\gamma J. \qquad (4.2d)$$

[15] Note that $d^4\bar{k}$ conceals a dependence on the coordinates x' and x''.

If the gauge dependent factor is now expanded up to first-order terms, one obtains the contributions to the mass operator that are linear in the external field,

$$
\begin{aligned}
&\bar{M}_1(x',x'')\\
&=(\alpha/2\pi)\int_0^\infty s^{-2}ds\int_0^s dw\exp[im^2(s-w)]\\
&\quad\times\int(2\pi)^{-4}d^4pe^{ip\Delta x}\exp[-iw(p^2+m^2)]ie\\
&\quad\times\int d^4\bar{k}\{[2m^2w(2-w/s)+iw/s]\\
&\quad\times[e(s)-e(w)]\gamma J/k^2+\tfrac{1}{2}[s+w-v^2(s-w)]\\
&\qquad\times e(s)\gamma J-mw(1-w/s)e(s)\tfrac{1}{2}\sigma F\}. \qquad (4.3)
\end{aligned}
$$

We have returned here to a Fourier representation of

$$
\begin{aligned}
&-i(4\pi w)^{-2}\exp[i(\Delta x)^2/4w]\\
&\qquad=\int(2\pi)^{-4}d^4pe^{ip\Delta x}\exp[-iwp^2]. \qquad (4.4)
\end{aligned}
$$

The remaining parts of the terms we have considered are explicitly second order in the field,

$$
\begin{aligned}
&\bar{M}_2{}^I(x',x'')\\
&=(\alpha/2\pi)\int_0^\infty s^{-2}ds\int_0^s dw\exp[-im^2(s-w)]\\
&\quad\times\int(2\pi)^{-4}d^4pe^{ip\Delta x}\exp[-iw(p^2+m^2)](ie)^2\\
&\quad\times\int d^4\bar{k}_1d^4\bar{k}_2\{\tfrac{1}{2}iw(v_1-v_2)(\gamma F^1\Delta x)(\Delta xJ^2)\\
&\quad\times[e_2(s)-e_2(w)]/k_2^2+imw(2-w/s)\\
&\quad\times(v_2J^2F^1\Delta x+\tfrac{1}{2}[\gamma J^2,\gamma F^1\Delta x])\\
&\quad\times[e_2(s)-e_2(w)]/k_2^2-\tfrac{1}{4}iw(v_2\{\gamma F^1\Delta x,\tfrac{1}{2}\sigma F^2\}\\
&\quad+v_1v_2[\gamma F^1\Delta x,\tfrac{1}{2}\sigma F^2])[e_2(s)-(w/s)e_2(w)]\\
&\quad-\tfrac{1}{4}iw(v_1\{\gamma F^1\Delta x,\tfrac{1}{2}\sigma F^2\}+[\gamma F^1\Delta x,\tfrac{1}{2}\sigma F^2])\\
&\quad\times[e_2(s)+(w/s)e_2(w)]+(\gamma J^2)(A^1\Delta x)\\
&\quad\times[2m^2w(2-w/s)+iw/s][e_2(s)-e_2(w)]/k_2^2\\
&\quad+\tfrac{1}{2}(\gamma J^2)(A^1\Delta x)[s+w-v_2{}^2(s-w)]e_2(s)\\
&\qquad-m(\tfrac{1}{2}\sigma F^2)(A^1\Delta x)w(1-w/s)e_2(s)\}. \qquad (4.5)
\end{aligned}
$$

They will be considered later together with the second-order terms C, D, and G in Eq. (3.29).

In the evaluation of the matrix element of Eq. (4.3) in an S-state of a hydrogenic atom,

$$\bar{M}_1=\int d\mathbf{r}'d\mathbf{r}''dt''\bar{\psi}_0(x')\bar{M}_1(x',x'')\psi_0(x''), \qquad (4.6)$$

it must be remembered that only corrections of orders α and $Z\alpha^2$ with respect to the matrix element of the hyperfine energy density $\frac{1}{2}\sigma F$ are required. Since a coefficient α appears explicitly, all other factors need be treated only to order $Z\alpha$. Thus the energy of the electron in this state can be approximated by the rest energy, so that the integration over t'' can be carried out when it is noticed that for a static potential

$$A_\mu(k)=A_\mu(\mathbf{k})\delta(k_0) \text{ and } \psi_0(x)=\psi_0(\mathbf{r})e^{-imt}. \quad (4.7)$$

Then one obtains

$$\bar{M}_1=\int d\mathbf{r}'d\mathbf{r}''\bar{\psi}_0(\mathbf{r}')(\alpha/2\pi)\int_0^\infty s^{-2}ds\int_0^s dw \times\exp[-im^2(s-w)]\int(2\pi)^{-3}d\mathbf{p}e^{i\mathbf{p}\cdot\Delta\mathbf{r}} \times\exp[-iwp^2]ie\int_{-1}^1\tfrac{1}{2}dv\int(2\pi)^{-2}d\mathbf{k} \times e^{\frac{1}{2}i\mathbf{k}\cdot[\mathbf{r}'(1+v)+\mathbf{r}''(1-v)]}\{\quad\}\psi_0(\mathbf{r}'') \quad (4.8)$$

where $\{\quad\}$ stands for the bracket of Eq. (4.3).

The presence of the exponentials $e(s)$ and $e(w)$ in every term together with factor $e^{\frac{1}{2}i\mathbf{k}\cdot(\mathbf{r}+\mathbf{r}')}$ implies that most of the contribution to $\bar{M}_1$ comes from the region of space within one electronic Compton wavelength $1/m$ of the origin. The crudest approximation to $\bar{M}_1$ is therefore obtained when the small components of ψ_0 are neglected and the function itself is replaced by the value $\varphi_0(0)$ of the Pauli wave function at the origin. Only the last term in Eq. (4.3) fails to vanish, whence, with $u=1-w/s$,

$$\bar{M}_1=-(\alpha/2\pi)\int_0^\infty im^2ds\int_0^1du\exp[-im^2su] \times 2u(1-u)\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2 =-\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(\alpha/2\pi). \quad (4.9)$$

This formula leads to an addition to the hyperfine structure separation due to the anomalous magnetic moment $\mu_0\alpha/2\pi$.

To obtain the corrections to this result one must use more accurate solutions of Eq. (3.24). The first-order effects of the magnetic field are included in the expression

$$\psi(x)=\psi_c(x)+e\int G_c(x,x')\boldsymbol{\gamma}\cdot\mathbf{A}^M(\mathbf{r}')\psi_c(x')d^4x', \quad (4.10)$$

which depends on the exact Coulomb wave function $\psi_c(x)$ and on the Coulomb Green's function G_c. The last term may be approximated quite crudely because the vector potential appearing in it, taken with the mass operator that is linear in the field, gives explicitly quadratic field-dependent terms. The Green's function is therefore replaced by that of the free particle,

$$G(x,x')=\int(2\pi)^{-4}d^4ke^{ik(x-x')}(m-\gamma k)/(m^2+k^2), \quad (4.11)$$

and the energy of the state is approximated by the rest energy

$$\psi(x)=\psi(\mathbf{r})e^{-imt}, \quad \psi_c(x)=\psi_c(\mathbf{r})e^{-imt}. \quad (4.12)$$

After the integration over t', the spatial dependence of the wave function is[16]

$$\psi(\mathbf{r})=\psi_c(\mathbf{r})+\int(2\pi)^{-3}(k^2)^{-1}d\mathbf{k}d\mathbf{r}'e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \times[m(1+\gamma_0)-\boldsymbol{\gamma}\cdot\mathbf{k}]\boldsymbol{\gamma}\cdot\mathbf{A}^M(\mathbf{r}')\psi_c(\mathbf{r}'). \quad (4.13)$$

In terms of the Pauli wave function, the large components of this wave function for an S-state are

$$\varphi_0(\mathbf{r})+e\int(2\pi)^{-3}(k^2)^{-1}d\mathbf{k}d\mathbf{r}'e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\boldsymbol{\sigma}\cdot k\boldsymbol{\sigma}\cdot\mathbf{A}^M(\mathbf{r}')\varphi_0(\mathbf{r}') \cong\varphi_0(\mathbf{r})+e\int(2\pi)^{-3}(k^2)^{-1}d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}i\boldsymbol{\sigma}\cdot(\mathbf{k}\times\mathbf{A}^M(k))\varphi_0(0) \rightarrow\varphi_0(\mathbf{r})+\tfrac{2}{3}e\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\varphi_0(0)/4\pi r, \quad (4.14)$$

where the second step results from the recognition that the constant $\varphi_0(0)$ is a sufficient approximation to the slowly varying wave function, and the third step anticipates the spherically symmetrical averaging in the matrix element. Finally, the small components of the wave function Eq. (4.13) are

$$-i(\boldsymbol{\sigma}\cdot\nabla/2m)\varphi_0(\mathbf{r}) \quad (4.15)$$

to the required accuracy. With these wave functions, the evaluation of the Dirac matrices in Eq. (4.8) leads to the four kinds of terms,

$$\bar{\psi}_0(\mathbf{r}')e\gamma J^M\psi_0(\mathbf{r}'') \rightarrow\varphi_0^*(\mathbf{r}')[e\boldsymbol{\sigma}\cdot J^M(\mathbf{k})\boldsymbol{\sigma}\cdot(\mathbf{p}-\tfrac{1}{2}\mathbf{k}(1-v)) +\boldsymbol{\sigma}\cdot(\mathbf{p}+\tfrac{1}{2}\mathbf{k}(1+v))e\boldsymbol{\sigma}\cdot\mathbf{J}^M(\mathbf{k})]\varphi_0(\mathbf{r}'')/2m \rightarrow\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle k^2\varphi_0^*(\mathbf{r}')\varphi_0(\mathbf{r}'')/2\pi, \quad (4.16)$$

$$\bar{\psi}_0(\mathbf{r}')e\gamma J^E\psi_0(\mathbf{r}'') \rightarrow\tfrac{2}{3}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle 2Z\alpha\tfrac{1}{2}[\varphi_0^*(\mathbf{r}')\varphi_0(0)/r'' +\varphi_0^*(0)\varphi_0(\mathbf{r}'')/r']/2\pi, \quad (4.17)$$

$$\bar{\psi}_0(\mathbf{r}')e\tfrac{1}{2}\sigma F^M\psi_0(\mathbf{r}'')\rightarrow\tfrac{2}{3}e\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle\varphi_0^*(\mathbf{r}')\varphi_0(\mathbf{r}'')/2\pi, \quad (4.18)$$

$$\bar{\psi}_0(\mathbf{r}')e\tfrac{1}{2}\sigma F^E\psi_0(\mathbf{r}'')\rightarrow 0. \quad (4.19)$$

The symmetry of $\bar{M}_1(x',x'')$ in r' and r'' and the fact that it is large only near the origin imply that the ex-

[16] We are indebted to Norman M. Kroll for calling to our attention the necessity for this magnetic correction.

pansion of the Coulomb wave function

$$\begin{aligned}\varphi_0(\mathbf{r})&=\varphi_0(0)(1-Z\alpha rm),\\ \varphi_0^*(\mathbf{r}')\varphi_0(\mathbf{r}'')&=|\varphi_0(0)|^2(1-2Z\alpha r'm),\end{aligned}\tag{4.20}$$

is sufficiently accurate to be used in Eq. (4.8). With all these approximations the matrix element simplifies to

$$\begin{aligned}\bar{M}_1=&\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(\alpha/2\pi)\int_0^\infty im^2ds\int_0^1 du\\ &\times\exp[-ism^2u]\int_{-1}^1\tfrac{1}{2}dv\int(2\pi)^{-6}d\mathbf{p}d\mathbf{k}d\mathbf{r}'d\mathbf{r}''\\ &\times e^{i\mathbf{p}\cdot(\mathbf{r}'-\mathbf{r}'')}e^{i\frac{1}{2}\mathbf{k}\cdot[\mathbf{r}'(1+v)+\mathbf{r}''(1-v)]}\\ &\times\{[1-2Z\alpha r'm+4Z\alpha m/r'k^2][(2(1-u^2)\\ &-(1-u)/im^2s)(e(s)-e(s(1-u)))\\ &+\tfrac{1}{2}(2-u-uv^2)(k^2/m^2)e(s)]\\ &-(1-2Z\alpha r'm)2u(1-u)e(s)\}.\end{aligned}\tag{4.21}$$

The integrations of $\mathbf{r}''$ and $\mathbf{p}$ are trivial because

$$\int(2\pi)^{-3}d\mathbf{r}''\exp[i\mathbf{r}''\cdot(\tfrac{1}{2}\mathbf{k}(1-v)-\mathbf{p})]=\delta(\mathbf{p}-\tfrac{1}{2}\mathbf{k}(1-v)).\tag{4.22}$$

The integrations over r' and k are of the form

$$\begin{aligned}\int&(2\pi)^{-3}d\mathbf{r}'d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}'}\exp[-\tfrac{1}{4}isk^2\lambda_n]\\ &\times\{1;k^2;r';r'k^2;(r'k^2)^{-1};(r')^{-1}\}\\ &=\{1;0;2(i\lambda_ns/\pi)^{\frac{1}{2}};-4(i\lambda_ns\pi)^{-\frac{1}{2}};\\ &\quad-(i\lambda_ns/\pi)^{\frac{1}{2}};2(i\lambda_ns\pi)^{-\frac{1}{2}}\},\quad(n=1,2)\end{aligned}\tag{4.23}$$

where

$$\begin{aligned}\lambda_1&=(1-v)(2-u(1-v))\text{ from terms involving }e(s),\\ \lambda_2&=2(1-v)(1-u)\text{ from terms involving }e(s(1-u)).\end{aligned}\tag{4.24}$$

The proper time integrations are Γ functions. These operations leave the following integral over the two parameters u and v:

$$\begin{aligned}\bar{M}_1=&\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(\alpha/2\pi)\int_0^1du\int_{-1}^1\tfrac{1}{2}dv\\ &\times\{-2(1-u)(1-2Z\alpha u^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}})\\ &+[-2Z\alpha+4Z\alpha(-\tfrac{1}{2})][2(u^{-\frac{3}{2}}(1-u^2)\\ &-u^{-\frac{1}{2}}(1-u))(\lambda_1^{\frac{1}{2}}-\lambda_2^{\frac{1}{2}})\\ &-2(2-u-uv^2)u^{-\frac{1}{2}}\lambda_1^{-\frac{1}{2}}]\}.\end{aligned}\tag{4.25}$$

With the help of a table of integrals Eq. (4.25) is evaluated to

$$\bar{M}_1=-\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2\{(\alpha/2\pi)-\tfrac{1}{4}Z\alpha^2(60\ln 2-23)\}.\tag{4.26}$$

V. SECOND-ORDER TERMS

The host of second-order terms in Eqs. (3.29) and (4.5) must now be treated in a manner similar to the treatment of the first-order terms, except that an order of accuracy can be sacrificed in the expressions for the wave functions which are now approximated by their value at the position of the nucleus,

$$\psi_0(x)\cong\varphi_0(0)e^{-imt}.\tag{5.1}$$

To carry out the coordinate integrations it is convenient to return to momentum space with the aid of Eqs. (4.4) and the inverse of Eq. (3.6),

$$\begin{aligned}&\Delta x_\mu\to2wp_\mu,\\ &\Delta x_\mu\Delta x_\nu\to4w^2p_\mu p_\nu+2iw\delta_{\mu\nu},\\ &\Delta x_\mu\Delta x_\nu\Delta x_\lambda\to8w^3p_\mu p_\nu p_\lambda+4iw^2(\delta_{\mu\nu}p_\lambda+\delta_{\mu\lambda}p_\nu+\delta_{\nu\lambda}p_\mu).\end{aligned}\tag{5.2}$$

The coordinate integrations now lead to Dirac δ-functions relating the momentum variables k_1, k_2, and p as follows:

$$\begin{aligned}\bar{M}_2=&(\alpha/4\pi)\int_0^\infty s^{-2}ds\int_0^s dw\exp[-im^2(s-w)]\\ &\times\int(2\pi)^{-1}d\mathbf{k}_1d\mathbf{k}_2d^4p\exp[-iw(p^2+m^2)]\\ &\times(ie)^2\int_{-1}^1\tfrac{1}{2}dv_1\int_{-1}^{v_1}\tfrac{1}{2}dv_2\delta(\mathbf{k}_1+\mathbf{k}_2)\\ &\times\delta(\mathbf{p}+\tfrac{1}{2}\mathbf{k}_1(v_1-v_2))\delta(p_0-m)\\ &\times\varphi_0^*(0)\{\quad\}\varphi_0(0),\end{aligned}\tag{5.3}$$

where $\{\quad\}$ stands for the details of Eqs. (3.29) and (4.5); the terms of the latter have been symmetrized in the arguments v_1 and v_2. This bracket will now be evaluated in the light of the fact that it is of interest only when the arguments of the δ-functions vanish, when

$$\mathbf{k}_1=-\mathbf{k}_2,\quad\mathbf{p}=\tfrac{1}{2}\mathbf{k}_1\Delta v=-\tfrac{1}{2}\mathbf{k}_2\Delta v,\quad\Delta v=v_1-v_2.\tag{5.4}$$

These relations, coupled with the facts that the two field vectors are the product of an electric with a magnetic field and that in the final integration over k_1 only spherically symmetrical quantities can survive, greatly simplify Eq. (5.3). It is easy to verify that in addition to the anticommutators in Eqs. (3.29) and (4.5) only two terms,

$$\varphi_0^*(0)\tfrac{1}{2}\sigma F^1A^1\Delta x\varphi_0(0)\to\tfrac{2}{3}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(2Z\alpha)(2mw/2\pi k_1^2)\tag{5.5a}$$

and

$$\varphi_0^*(0)[\gamma J^2, \gamma F^1\Delta x]\varphi_0(0) \rightarrow \tfrac{2}{3}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(2Z\alpha)(4imw/2\pi) \quad (5.5b)$$

in Eq. (4.5), fail to vanish.[17] The anticommutators are evaluated as follows (for the sake of brevity, the symbols of Eq. (3.20) are used to represent the integrands given there, and it is supposed that the substitution (5.2) is made):

$$\varphi_0^*(0)[\{(\gamma\Delta x/2s)(D^1(s)+D^1(w)), \tfrac{1}{2}\sigma F^2\} + \{(\gamma\Delta x/2s)(D^2(s)+D^2(w)), \tfrac{1}{2}\sigma F^1\}]\varphi(0) \rightarrow \tfrac{2}{3}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(2Z\alpha)4(m^2w^2-iw) \times[2E(s)-e_1(s)-e_2(s)+(w/s)(2E(w)-e_1(w)-e_2(w))]/2\pi k_1^2; \quad (5.5c)$$

$$\varphi_0^*(0)[\{\gamma(D_+^1(s)+(w/s)D_+^1(w)), \tfrac{1}{2}\sigma F^2\} + \{\gamma(D_+^2(s)+(w/s)D_+^2(w)), \tfrac{1}{2}\sigma F^1\}]\varphi_0(0) \rightarrow \tfrac{2}{3}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(2Z\alpha)(-2s) \times[(\Delta v)^2 w(E(s)+(w/s)^2E(w)) + \Delta v(1-\Delta v)s(E(s)+(w/s)^3E(w))]/2\pi; \quad (5.5d)$$

$$-i\varphi_0^*(0)\{(\gamma\Delta x/2s)(G(s)+G(w)), \operatorname{tr}(F^1\sigma F^2)\}\varphi_0(0) \rightarrow \tfrac{2}{3}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(2Z\alpha)2sw\Delta v \times(E(s)+(w/s)^2E(w))/2\pi; \quad (5.5e)$$

$$-i\varphi_0^*(0)\{\gamma(G_+(s)+(w/s)G_+(w)), \operatorname{tr}(F^1\sigma F^2)\}\varphi_0(0) \rightarrow \tfrac{2}{3}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(2Z\alpha)(-2s^2\Delta v) \times(E(s)+(w/s)^3E(w))/2\pi; \quad (5.5f)$$

$$\varphi_0^*(0)\{\gamma F^1\Delta x, \tfrac{1}{2}\sigma F^2\}\varphi_0(0) \rightarrow \tfrac{2}{3}\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(2Z\alpha)(4iw\Delta v/2\pi). \quad (5.5g)$$

One exponential factor simplifies,

$$E(s)=\exp[-\tfrac{1}{4}isk_1^2\Delta v(2-\Delta v)], \quad (5.6)$$

while the others can be combined (see Eq. (5.5c)),

$$\int_{-1}^{1}\tfrac{1}{2}dv_1\int_{-1}^{v_1}\tfrac{1}{2}dv_2(e_1(s)+e_2(s)) \rightarrow \int_{-1}^{1}\tfrac{1}{2}dv_1\int_{-1}^{1}\tfrac{1}{2}dv_2 e_2(s) \quad (5.7)$$

so that they will be taken together with the contributions (5.5a, b, g) from Eq. (4.5).

Then, after the parameter $u=1-w/s$ is introduced, the remaining momentum integration is carried out. The relevant formulas are

$$\int d\mathbf{k}\exp[-\tfrac{1}{4}isk^2\lambda_i]\{1; (k^2)^{-1}\} = \{8\pi^{\frac{3}{2}}(i\lambda_i)^{-\frac{3}{2}}; 4\pi^{\frac{3}{2}}(i\lambda_i)^{-\frac{1}{2}}\}, \quad (5.8)$$

where the λ_i have the following significance:

$$\begin{aligned} \lambda_1&=\Delta v(2-\Delta v) \text{ from } E(s),\\ \lambda_2&=2\Delta v(1-u) \text{ from } E(w),\\ \lambda_3&=1-v_2^2+(\Delta v)^2(1-u) \text{ from } e_2(s),\\ \lambda_4&=(1-v_2^2+(\Delta v)^2)(1-u) \text{ from } e_2(w). \end{aligned} \quad (5.9)$$

The proper time integration, as before, involves half-integral Γ-functions. It leaves the two contribution to the hyperfine energy

$$\bar{M}_2{}^A=\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(2Z\alpha^2/2\pi) \times\int_{-1}^{1}\tfrac{1}{2}dv_1\int_{-1}^{v_1}\tfrac{1}{2}dv_2\int_0^1 du\{4(1-u)^2u^{-\frac{3}{2}} \times(\lambda_1^{-\frac{1}{2}}-\lambda_2^{-\frac{1}{2}})+4(1-u)u^{-\frac{1}{2}} \times(2\lambda_1^{-\frac{1}{2}}-(1-u)\lambda_2^{-\frac{1}{2}})-4(u^{\frac{1}{2}}\Delta v(1-\Delta v) +u^{-\frac{1}{2}}\Delta v)\lambda_1^{-\frac{3}{2}}-4u^{-\frac{1}{2}}(1-u)^2\Delta v\lambda_2^{-\frac{3}{2}}\} =\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2 4Z\alpha^2(1-\ln 2) \quad (5.10)$$

and

$$\bar{M}_2{}^B=\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(2Z\alpha^2/2\pi) \times\int_{-1}^{1}\tfrac{1}{2}dv_1\int_{-1}^{1}\tfrac{1}{2}dv_2\int_0^1 du4u^{-\frac{1}{2}} \times\{-(1-u)\lambda_3^{-\frac{1}{2}}+(1-u)^2\Delta v(v_1+v_2)\lambda_3^{-\frac{3}{2}} -(1-u)^2\lambda_4^{-\frac{1}{2}}+(1-u)^3(\Delta v)^2\lambda_4^{-\frac{3}{2}}\} =\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2 Z\alpha^2(5-12\ln 2). \quad (5.11)$$

The sum of the three corrections is

$$\bar{M}=-\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2 \times\{(\alpha/2\pi)-\tfrac{1}{4}Z\alpha^2(13-4\ln 2)\}. \quad (5.12)$$

VI. VACUUM POLARIZATION

We shall now obtain the contribution of the vacuum polarization effects discussed in some detail in Sec. II. An equivalent to Eq. (2.14),

$$M^P=-e\int\bar{\psi}(\mathbf{r})\gamma\cdot\mathbf{A}^P\psi(\mathbf{r})d\mathbf{r} \quad (6.1)$$

in conjunction with Eq. (2.19), permits direct utilization of the matrix elements calculated in Eqs. (4.16) and (4.17). The resultant expression,

$$M^P=\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(Z\alpha^2/2\pi) \times\int(2\pi)^{-3}d\mathbf{r}d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}(rk^2-2/r)m \times\int_0^1 dv\frac{v^2(1-\frac{1}{3}v^2)}{m^2+\frac{1}{4}k^2(1-v^2)}, \quad (6.2)$$

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[17] Take, for example, the first term in C, Eq. (3.20f)

$\varphi_0^*(0)\Delta xJ^1\Delta xJ^2\varphi_0(0)\rightarrow|\varphi_0(0)|^2(4w^2(pJ^1)(pJ^2)+2iw(J^1J^2))$;

Now $J_\mu{}^EJ_\mu{}^M=0$ because $J^E=(0, J_0{}^E)$, $J^M=(\mathbf{J}^M, 0)$; Further, $(pJ^E(\mathbf{k}_1))(pJ^M(\mathbf{k}_2))\rightarrow p_0J_0{}^E(\mathbf{k}_1)\mathbf{p}\cdot\mathbf{J}^M(\mathbf{k}_2) \rightarrow -mJ_0{}^E(\tfrac{1}{2}\Delta v\mathbf{k}_2\cdot\mathbf{J}^M(\mathbf{k}_2))=0.$

in which the expansion Eq. (2.22) has been made, includes Eq. (2.23) as the first contribution. The second one arises from the matrix element of the electric charge density. The substitution

$$[m^2+\tfrac{1}{4}k^2(1-v^2)]^{-1}=i\int_0^\infty ds\exp[-is(m^2+\tfrac{1}{4}k^2(1-v^2))] \quad (6.3)$$

brings Eq. (6.2) into a form corresponding precisely to Eq. (4.21). The space, momentum, and proper time integrations are carried out in exectly the same way as there. The final integral involves only the parameter v:

$$M^P=\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2(Z\alpha^2/2\pi)\times\int_0^1 dv\,v^2(1-\tfrac{1}{3}v^2)[-4(1-v^2)^{-\frac{1}{2}}-4(1-v^2)^{-\frac{3}{2}}]$$
$$=-\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2\{\tfrac{3}{4}Z\alpha^2\}. \quad (6.4)$$

VII. SUMMARY

The two contributions to the Fermi formula are, Eqs. (5.12) and (6.4),

$$\bar{M}=-\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2\{(\alpha/2\pi)+\tfrac{1}{4}Z\alpha^2(-13+4\ln 2)\}$$

and

$$M^P=-\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2\{\quad+\tfrac{3}{4}Z\alpha^2\}.$$

Hence the Fermi formula Eq. (2.16) becomes[18]

$$\Delta E=-\tfrac{2}{3}\mu_0\langle\boldsymbol{\sigma}\cdot\boldsymbol{\mu}\rangle|\varphi_0(0)|^2\{1+\alpha/2\pi-\tfrac{1}{2}Z\alpha^2(5-2\ln 2)-2.97\alpha^2/\pi^2\}(1+\tfrac{3}{2}(Z\alpha)^2)$$

when the fourth-order correction[2] to the magnetic moment is included.

With this change, the deduction of the value of the fine structure constant α from experimental quantities is modified. In terms of the notation of Dumond and Cohen,[5]

$$\Gamma=1.807\alpha^2+2\times10^{-6};$$
$$\delta(\alpha^{-1})=-0.880\alpha-1.5\times10^{-4}=-0.0065.$$

Hence

$$\alpha^{-1}=137.0364\pm0.0009.$$

Recent experimental and theoretical investigations of the deuteron[6,19] suggest that our treatment of the proton as a point magnetic dipole of infinite mass is inaccurate. An estimate of the necessary corrections is contained in reference 19.

It is a pleasure to acknowledge numerous stimulating and enlightening discussions with Julian Schwinger and with Norman M. Kroll.

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[18] This result has been already reported by Karplus, Klein, and Schwinger, Phys. Rev. 84, 597 (1951). The same result has been obtained by a different method by N. M. Kroll and F. Pollock, Phys. Rev. 84, 594 (1951).

[19] F. E. Low and E. E. Salpteter, Phys. Rev. 83, 478 (1951).