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Invariance Conditions on the Scattering Amplitudes for Spin $\frac{1}{2}$ Particles

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The most general form of the scattering matrix in spin space for two spin $\frac{1}{2}$ particles is derived subject to invariance under rotation, reflection, and time-reversal. The result may be used to prove a relationship important for polarization experiments and previously stated without proof.

A COMPLETE description of the scattering of two particles of spin $\frac{1}{2}$ is given by a matrix specifying the amplitude of any outgoing spin and momentum state as a function of the incident spin and momentum. This matrix will be written

$$M(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{k}_i, \mathbf{k}_f)$$

and is to be considered as an operator in the four-dimensional spin space operating on the initial spin state. Here $\mathbf{k}_i$ and $\mathbf{k}_f$ are the initial and final momenta of one of the particles (in the c.m. system) and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the Pauli spin operators for the two particles. The treatment is nonrelativistic.

I. RELATION OF M TO EXPERIMENT

A polarization state for a system of two spin $\frac{1}{2}$ particles is, in general, described as a mixture of many separate pure spin states in which the system may be found at any particular time. If the polarization state of the particles before the collision is specified, the most general scattering problem is that of finding the outgoing polarization state and intensity as a function of the angle of scattering.

A polarization state of the system can be completely specified by the average values of a suitable array of 16 observables[1] pertaining to the spins of the two particles. This is most easily shown with the help of the von Neumann density matrix ρ for a mixture of states:

$$\rho = \sum_n P_n \chi_n \chi_n^\dagger,$$

where χ_n is a column vector with four components representing one of the spin states present in the mixture, $\chi_n^\dagger$ is the adjoint row vector, and P_n is the relative probability of finding the system in state χ_n. Knowledge of the matrix ρ suffices to obtain the average value $\langle S\rangle$ of any spin operator S, for the mixture, through the relation

$$\langle S\rangle\, \mathrm{Tr}(\rho) = \mathrm{Tr}(\rho S). \tag{1}$$

Since the matrix ρ is Hermitian and 4×4, it is fixed by giving 16 real numbers which may be chosen as the average values of a complete set of 16 Hermitian operators S^μ in the spin space. A set of operators will be called complete if it obeys the orthogonality relations

$$\mathrm{Tr} S^\mu S^\nu = 4\delta_{\mu\nu}. \tag{2}$$

For example, the operators $1, \sigma_{1x}, \sigma_{1y}, \sigma_{1z}, \sigma_{2x}, \sigma_{2y}, \sigma_{2z}$, and the nine products of one of the Pauli spin operators for particle 1 with one of those for particle 2 form such a set of operators. Any matrix must be expressible linearly in the S^μ. Thus

$$\rho = \tfrac{1}{4} \sum_\mu S^\mu\, \mathrm{Tr}(\rho S^\mu) = \tfrac{1}{4}\, \mathrm{Tr}(\rho) \sum_\mu \langle S^\mu\rangle S^\mu \tag{3}$$

showing how ρ is characterized by the $\langle S^\mu\rangle$.

The desired relation between ingoing and outgoing polarization states is obtained by giving averages $\langle S^\mu\rangle_f$ in the final state in terms of the $\langle S^\mu\rangle_i$ for the initial state. By the definition of the matrix M,[2] the density matrix for the final polarization state at any angle is

$$\rho_f = \sum_n P_{ni} M \chi_n \chi_n^\dagger M^\dagger = M \rho_i M^\dagger. \tag{4}$$

Using Eqs. (1) through (4) we get, therefore,

$$\langle S^\mu\rangle_f I = \tfrac{1}{4} \sum_\nu \langle S^\nu\rangle_i\, \mathrm{Tr}(M S^\nu M^\dagger S^\mu), \tag{5}$$

[1] It is convenient throughout to include the unity operator as one of the operators whose average values specify the polarization state; that is, the normalization of the state will be considered as part of the specification of the polarization state.

[2] Matrix notation is used only for the spin dependence of M and not for the momentum dependence. Thus $\mathbf{k}_i$ and $\mathbf{k}_f$ are fixed parameters in Eq. (4) and those that follow.

where

$$I = \mathrm{Tr}(\rho_f)/\mathrm{Tr}(\rho_i) \tag{5a}$$

is the differential scattering cross section at the angle in question.

Two special cases of interest are (a) the polarization of particles 1 produced by the scattering of an unpolarized beam on an unpolarized target, and (b) the differential scattering cross section of initially polarized particles 1 against initially unpolarized particles 2. In the first case all $\langle S^\nu\rangle_i$ equal zero in Eq. (5) except for the unity operator, yielding

$$\langle\boldsymbol{\sigma}_1\rangle_f I_0 = \tfrac{1}{4}\,\mathrm{Tr}M^\dagger\boldsymbol{\sigma}_1 M. \tag{6}$$

In the second case one obtains

$$I = \tfrac{1}{4}\,\mathrm{Tr}M^\dagger M + \tfrac{1}{4}\langle\boldsymbol{\sigma}_1\rangle_i\cdot\mathrm{Tr}M^\dagger M\boldsymbol{\sigma}_1, \tag{7}$$

or, for the case of a beam completely polarized in the direction $\mathbf{N}(\langle\boldsymbol{\sigma}_1\rangle_i=\mathbf{N})$,

$$I = I_0 + I_p = \tfrac{1}{4}\,\mathrm{Tr}M^\dagger M + \tfrac{1}{4}\,\mathrm{Tr}M^\dagger M\boldsymbol{\sigma}_1\cdot\mathbf{N}, \tag{7a}$$

where I_0 is the scattering cross section for an unpolarized beam and I_p is the contribution to the cross section due to initial polarization. In previous papers[3] it is stated as obvious that the quantities in Eqs. (6) and (7a) are related by

$$I_p/I_0 = \mathbf{N}\cdot\langle\boldsymbol{\sigma}_1\rangle_f \tag{8}$$

implying that

$$\mathrm{Tr}M^\dagger\boldsymbol{\sigma}_1 M = \mathrm{Tr}M^\dagger M\boldsymbol{\sigma}_1. \tag{8a}$$

Since the $\boldsymbol{\sigma}$ matrices do not commute with M, this is not a mathematical identity; it will be shown herein, however, that this relation follows from the condition on M of invariance under time reversal.

II. MOST GENERAL FORM OF M

The most general form of the matrix M may be found following a procedure similar to that used by Eisenbud and Wigner[4] to find the most general form of the interaction Hamiltonian for two particles of spin $\frac{1}{2}$. Conditions placed on the matrix M are invariance under space rotations and reflections and time reversal. The matrix M must be a scalar obtained by combining the sixteen linearly independent matrices in spin space:

$$\begin{array}{ll} 1 & \text{(scalar)} \\ (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-1) & \text{(scalar)} \\ (\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) & \text{(axial vector)} \\ (\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2) & \text{(axial vector)} \\ (\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2) & \text{(axial vector)} \\ t_{\alpha\beta}=(\sigma_{1\alpha}\sigma_{2\beta}+\sigma_{1\beta}\sigma_{2\alpha}) & \text{(symmetric tensor)} \end{array}$$

with functions of the momenta:

$$\begin{array}{ll} 1 & \text{(scalar)} \\ \mathbf{k}_f-\mathbf{k}_i=\mathbf{K} & \text{(polar vector)} \\ \mathbf{k}_f\times\mathbf{k}_i=\mathbf{n} & \text{(axial vector)} \\ \mathbf{n}\times\mathbf{K}=\mathbf{P} & \text{(polar vector)} \\ \left.\begin{array}{c} K_\alpha K_\beta,\ n_\alpha n_\beta, \\ P_\alpha P_\beta,\ K_\alpha P_\beta+K_\beta P_\alpha \end{array}\right\} & \text{(symmetric tensors).} \end{array}$$

Any of these functions of momenta may be multiplied by an arbitrary function of the scalars $\mathbf{k}_i\cdot\mathbf{k}_f$ and $k_i^2(=k_f^2)$. Each of the symmetric tensors above have only five independent components, since the sum of the diagonal elements of the tensor is one of the scalars. Among the functions of momenta, only those symmetric tensors are listed which do not change sign under space inversion.

The resulting forms invariant under space rotation and reflection are:

$$1,\ (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-1),\ (\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{n}, \tag{I}$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{n}, \tag{II}$$

$$(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\mathbf{n}, \tag{III}$$

$$\left.\begin{array}{c}\sum_{\alpha\beta} t_{\alpha\beta}K_\alpha K_\beta,\ \sum_{\alpha\beta} t_{\alpha\beta}n_\alpha n_\beta, \\ \sum_{\alpha\beta} t_{\alpha\beta}P_\alpha P_\beta, \\ \sum_{\alpha\beta} t_{\alpha\beta}(K_\alpha P_\beta+K_\beta P_\alpha). \end{array}\right\} \tag{IV}$$

The set (IV) is equivalent to

$$\boldsymbol{\sigma}_1\cdot\mathbf{K}\boldsymbol{\sigma}_2\cdot\mathbf{K},\ \boldsymbol{\sigma}_1\cdot\mathbf{n}\boldsymbol{\sigma}_2\cdot\mathbf{n},\ \boldsymbol{\sigma}_1\cdot\mathbf{P}\boldsymbol{\sigma}_2\cdot\mathbf{P}, \tag{IVa}$$

$$(\boldsymbol{\sigma}_1\cdot\mathbf{K}\boldsymbol{\sigma}_2\cdot\mathbf{P}+\boldsymbol{\sigma}_1\cdot\mathbf{P}\boldsymbol{\sigma}_2\cdot\mathbf{K}). \tag{IVb}$$

Only two of the forms (IVa) are really independent, since the three may be combined to give $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$.

If time-reversed quantities are indicated by a prime superscript, the effects of time reversal[5] may be summarized by

$$\boldsymbol{\sigma}'=-\boldsymbol{\sigma},\quad \mathbf{k}_i'=-\mathbf{k}_f,\quad \mathbf{k}_f'=-\mathbf{k}_i,$$

and therefore,

$$\mathbf{K}'=\mathbf{K},\quad \mathbf{n}'=-\mathbf{n},\quad \mathbf{P}'=-\mathbf{P}.$$

It is seen that (III) and (IVb) change sign under time-reversal and these are therefore ruled out. Thus the most general form of M is

$$M = A + B(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-1) + C(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{n} + D(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{n} + E(\boldsymbol{\sigma}_1\cdot\mathbf{K})(\boldsymbol{\sigma}_2\cdot\mathbf{K}) + F(\boldsymbol{\sigma}_1\cdot\mathbf{P})(\boldsymbol{\sigma}_2\cdot\mathbf{P}), \tag{9}$$

where the coefficients are functions of the scalars k_i^2 and $\mathbf{k}_i\cdot\mathbf{k}_f$, that is, of the energy and $\cos\theta$, where θ is the c.m. scattering angle.

Substituting from Eq. (9) into Eqs. (6) and (7a) one finds

$$\mathbf{N}\cdot\langle\boldsymbol{\sigma}_1\rangle_f I_0 = I_p = \mathbf{N}\cdot\mathbf{n}\ 2\ \mathrm{Re}\{CA^*+D(A^*-2B^*)\}, \tag{10}$$

thus proving Eq. (8). If the forms (III) or (IVb) were

[3] L. Wolfenstein, Phys. Rev. **75**, 1664 (1949), Phys. Rev. **76**, 541 (1949). One of us (L.W.) takes this opportunity to apologize for having stated as obvious a relation which is far from being so.

[4] L. Eisenbud and E. Wigner, Proc. Nat. Acad. Sci. **27**, 281 (1941).

[5] E. P. Wigner, Göttinger Nachr. p. 546 (1932).

included in Eq. (9), this would no longer be true; thus the condition of invariance under time-reversal is necessary in order to prove Eq. (8). The important consequences of invariance under space reflections (and rotations) have been previously noted;[3] these include: (a) the polarization $\langle\boldsymbol{\sigma}_1\rangle_f$ resulting from an initially unpolarized beam is normal to the plane containing the initial and final momenta; and (b) the contribution I_p to the scattering due to the initial polarization of one particle is proportional to the component of $\langle\boldsymbol{\sigma}_1\rangle_i$ perpendicular to the initial direction of motion and contains as a factor $\sin\theta\cos\varphi$, where φ is the azimuthal angle measured from the normal to the plane defined by the initial momentum and polarization.

Of special interest is the 3×3 submatrix of M which involves scattering in triplet states only. The usual representation[6] of this matrix is $\mathcal{S}_{m_s'm_s}$, where m_s' and m_s are the final and initial z-components of the total spin, respectively. Each of the conditions imposed here on M may be translated into this representation if it concerns matrix elements which are nonzero only between triplet states. For example, since the matrix (IVb) may be regarded as one of a complete set of matrices in terms of which M may be expanded, the condition that (IVb) cannot enter into the expansion is expressible by the vanishing of the trace of the product of M and (IVb). Computation of this trace yields

$$\mathcal{S}_{11}-\mathcal{S}_{00}-e^{2i\varphi}\mathcal{S}_{1-1}=\sqrt{2}\cot\theta(e^{-i\varphi}\mathcal{S}_{01}+e^{i\varphi}\mathcal{S}_{10}). \quad (11)$$

This equation, which is seen to follow from the time-reversal condition, is not apparent in explicit formulations and thus may be used as a check on the accuracy of numerical calculations. Similarly the condition of invariance of M under space reflections may be translated into this representation:

$$\begin{aligned}\mathcal{S}_{11}=\mathcal{S}_{-1-1},\quad e^{-i\varphi}\mathcal{S}_{-10}=-e^{i\varphi}\mathcal{S}_{10},\\ e^{-i\varphi}\mathcal{S}_{01}=-e^{i\varphi}\mathcal{S}_{0-1},\quad e^{2i\varphi}\mathcal{S}_{1-1}=e^{-2i\varphi}\mathcal{S}_{-11}.\end{aligned} \quad (12)$$

III. GENERAL PROOF OF EQUATION (8)

Equation (8) may be shown to be valid in the more general case of the scattering of a spin $\frac{1}{2}$ particle from a scatterer of spin I. As a complete set of operators in the combined spin spaces of the two particles one may use all the products of the form $(\sigma^\alpha\times s^\beta)$, where σ^α are the four spin operators of the spin $\frac{1}{2}$ particle and s^β are a complete set of $(2I+1)^2$ operators in the spin space of spin I particle. It is important to note that all the operators $(\sigma^\alpha\times s^\beta)$ have zero trace except the unity operator (1×1). The most general form of M may be written

$$M=a+b\boldsymbol{\sigma}_1\cdot\mathbf{n}+c\boldsymbol{\sigma}_1\cdot\mathbf{K}+d\boldsymbol{\sigma}_1\cdot\mathbf{P},$$

where a, b, c, and d are linear combinations of operators of the form s^β. To prove Eq. (8), one evaluates

$$\begin{aligned}4I_0(I_p/I_0-\mathbf{N}\cdot\langle\boldsymbol{\sigma}_1\rangle_f)&=\mathrm{Tr}(M^\dagger M-MM^\dagger)\boldsymbol{\sigma}_1\cdot\mathbf{N}\\ &=2i\,\mathrm{Tr}\{\boldsymbol{\sigma}_1\cdot(\mathbf{n}\times\mathbf{P})(b^\dagger d-bd^\dagger)+\boldsymbol{\sigma}_1\cdot(\mathbf{n}\times\mathbf{K})(b^\dagger c-c^\dagger b)\\ &\qquad+\boldsymbol{\sigma}_1\cdot(\mathbf{P}\times\mathbf{K})(d^\dagger c-c^\dagger d)\}\boldsymbol{\sigma}_1\cdot\mathbf{N}.\end{aligned} \quad (13)$$

Here contributions to the commutator of $M^\dagger$ and M from the noncommutation of a, b, c, and d have been omitted, since these terms must contain an s^β factor other than unity and so will have zero trace. Furthermore, for our considerations the factors $b^\dagger d-bd^\dagger$, $b^\dagger c-c^\dagger b$, and $d^\dagger c-c^\dagger d$ are not operators in the spin space of the spin I particle, but depend only on the vectors $\mathbf{k}_i$ and $\mathbf{k}_f$, because any term containing an operator s^β other than unity will yield zero trace. Following the arguments used previously one finds the transformation properties indicated in Table I. Since it is impossible to construct from $\mathbf{k}_i$ and $\mathbf{k}_f$ quantities having the transformation properties required for $bd^\dagger$, $bc^\dagger$, $dc^\dagger$, etc., it follows that each of the terms in Eq. (13) is zero.

TABLE I. Transformation properties of factors in Eq. (13). *S*—scalar. *PS*—pseudoscalar.

	Reflection-rotation	Time reflection
b	*S*	+
c	*PS*	−
d	*PS*	+
$b^\dagger d$, $bd^\dagger$	*PS*	+
$c^\dagger b$, $bc^\dagger$	*PS*	−
$d^\dagger c$, $dc^\dagger$	*S*	−

[6] J. Ashkin and T. Y. Wu, Phys. Rev. 73, 973 (1948).