The Angular Correlation of Two Successive Nuclear Radiations*

STUART P. LLOYD
Institute for Advanced Study, Princeton, New Jersey
(Received Novermber 12, 1951)

A general formula is derived, using group-theoretic methods, which gives the correlation as series in Jacobi polynomials whose argument depends on the angle between the radiations. The coefficients factor into quantities depending on the separate nuclear transitions, and the factors are worked out explicitly for the directional correlation cases where one of the interfering radiation angular momenta is the lowest or next-lowest allowed by conservation of angular momentum. Correction factors are given for converting lowest multipole $\gamma-\gamma$ correlations into an extensive class of directional correlations.

1. INTRODUCTION

THE theoretical aspects of the problem of the angular correlation of successive nuclear radiations have been treated by many authors, several of whom have presented results in numerically tractable form.[1-4] The problem is now well known, and we refer the reader to (FU) for a detailed general examination of the theory, as well as references to earlier work. Briefly, the problem is the following. A nucleus in an excited level E_1 decays to excited level E by emitting radiation R_1. The intermediate nucleus in E then decays to level E_2 by emitting radiation R_2, and one wants to find the relative angular distribution of R_2 with respect to R_1 in terms of the total angular momenta J_1, J, and J_2 (and perhaps other nuclear properties) of the successive levels E_1, E, and E_2, respectively. This relative angular distribution is called the angular correlation of R_1 and R_2.[5]

In the initial form for the angular correlation function, the form given by quantum-mechanical perturbation theory, the nuclear matrix elements appearing are those for emission of the radiations into plane wave states, but for computational purposes these matrix elements must be expanded in terms of those for emission into angular momentum eigenstates. The resulting expression for the angular correlation function contains summations involving many magnetic quantum numbers of various nuclear and radiation states; several of these summations are not trivial. In this paper we show that most of these summations can be eliminated by use of the formulas of Racah[6] for certain sums of products of vector-addition coefficients, so that the angular correlation function can be simplified to the point where the main considerations in its evaluation in a given case are physical in character, e.g., the dependence of a correlation on β-ray energy or on the relative amplitudes of simultaneously emitted γ-ray multipoles, etc. Put another way, the complicated sums over magnetic quantum numbers are geometrical entities, having to do with the properties of the three-dimensional rotation group,[7] and the geometrical problem of obtaining a closed form for these sums has already been solved by Racah in connection with the theory of complex spectra. Numerically tractable results obtained previously without the use of Racah's formulas have been in general limited to those cases where the angular momenta of the radiations do not exceed two units or where the initial or final nuclear spin is zero.

A general correlation formula is derived in Sec. 3, together with the directional correlation formula resulting from it. The directional correlation is set up to make use of existing $\gamma-\gamma$ numerical tables[8] as far as possible; formulas for coefficients not to be found in reference 8 are given in Secs. 4 and 5. The explicit formulas are limited to those cases where, in each transition, one of the interfering particle multipole orders is the lowest or next lowest allowed by conservation of angular momentum. The final directional correlation formula is Eq. (31), Sec. 6, where a summary of results is given for the benefit of those readers who are mainly interested in the application of the formula to practical problems.

2. PRELIMINARIES

We give here a general proof for the multipole expansion of nuclear matrix elements assumed in (FU).

* This paper is an exposition of results described in: S. P. Lloyd, Phys. Rev. **80**, 118 (1950) and Phys. Rev. **81**, 307 (1951). The preliminary stages of the work were performed while the author held an AEC Predoctoral Fellowship. Much of the present material is part of: S. P. Lloyd, thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics in the Graduate College of the University of Illinois, 1951.

[1] D. R. Hamilton, Phys. Rev. **58**, 122 (1940).

[2] D. S. Ling, Jr., and D. L. Falkoff, Phys. Rev. **76**, 1639 (1949), to be referred to as (LF).

[3] D. L. Falkoff and G. E. Uhlenbeck, Phys. Rev. **79**, 323 (1950), referred to hereafter as (FU).

[4] J. W. Gardner, Proc. Roy. Soc. (London) **A62**, 763 (1949). Our list is not complete.

[5] It is assumed that: (a) the initial nucleus is randomly oriented, (b) the intermediate nucleus is not disturbed by extranuclear fields before it emits R_2, and (c) effects caused by nuclear recoil are neglected.

[6] G. Racah, Phys. Rev. **62**, 439 (1942), referred to hereafter as (R).

[7] Or more precisely, with the properties of the group {P} of unitary unimodular two-dimensional matrices, the group of the Pauli electron. The group theory used in our derivations can be found in: E. Wigner, *Gruppentheorie usw* (Vieweg, Braunschweig, 1931), referred to hereafter as (W), or in: C. Eckart, Revs. Modern Phys. **2**, 305 (1930).

[8] S. P. Lloyd, Phys. Rev. **83**, 716 (1951).

The matrix element for the emission of radiation R into a state specified by a set of quantum numbers A in the nuclear transition $J_im_i \to J_fm_f$ will be denoted by $(J_fm_f|A|J_im_i)$. A set A includes the direction $\mathbf{\Omega}$ of the detected particle of R, and in the conversion electron case, for example, it includes the quantum numbers of the hole left by the ejected electron. We omit all nuclear quantum numbers except the total angular momentum quantum number J and its magnetic quantum number (z-projection) m.

Under the operations of $\{P\}$ these matrix elements transform according to

$$(J_fm_f|A|J_im_i)=\Sigma_{m_i'm_f'}(m_f|P^{-1}|m_f')^{J_f} \times(J_fm_f'|PA|J_im_i')(m_i'|P|m_i)^{J_i}, \quad (1)$$

where the $(m|P|m')^J$ are the usual $(2J+1)$-dimensional unitary irreducible representations of $\{P\}$.[9] Equation (1) is obtained simply by evaluating the matrix element in a rotated coordinate system and using the scalar property of the perturbation Hamiltonian describing the emission of R. The product of matrix elements $(\ |\ |\ )^J$ of $\{P\}$ in Eq. (1) can be reduced (Clebsch-Gordan series[10]):

$$\begin{aligned}(m_f|P^{-1}|m_f')^{J_f}(m_i'|P|m_i)^{J_i} &=(-1)^{m_f-m_f'}(-m_f'|P|-m_f)^{J_f}(m_i'|P|m_i)^{J_i}\\ &=(-1)^{m_f-m_f'}\Sigma_{LMM'}(J_fJ_i-m_f'm_i'|J_fJ_iL-M')\\ &\qquad\times(J_fJ_i-m_fm_i|J_fJ_iL-M)(-M'|P|-M)^L\\ &=\Sigma_{LMM'}[(2L+1)/(2J_f+1)](J_iLm_i'M'|J_iLJ_fm_f')\\ &\qquad\times(J_iLm_iM|J_iLJ_fm_f)(M'|P|M)^{L*} \quad (2)\end{aligned}$$

to give

$$(J_fm_f|A|J_im_i) =\Sigma_{LM}(2J_f+1)^{-\frac{1}{2}}(J_iLm_iM|J_iLJ_fm_f)G_L^M(A)^*, \quad (3)$$

where

$$\begin{aligned}G_L^M(A)^*&\equiv\Sigma_{m_i'm_f'M'}(2J_f+1)^{-\frac{1}{2}}(2L+1)\\ &\qquad\times(J_iLm_i'M'|J_iLJ_fm_f')(M'|P|M)^{L*}\\ &\qquad\times(J_fm_f'|PA|J_im_i')\\ &=\Sigma_{m_im_f}(2J_f+1)^{-\frac{1}{2}}(2L+1)\\ &\qquad\times(J_iLm_iM|J_iLJ_fm_f)(J_fm_f|A|J_im_i). \quad (4)\end{aligned}$$

Various symmetry properties given in (W) for $(\ |\ |\ )^J$ and in (R) for vector-addition coefficients have been used in Eqs. (2)–(4). It is easy to verify from their definition, Eq. (4), that the $G_L^M(A)$ belong to $(\ |\ |\ )^L$ as indicated by the indices, i.e.:

$$G_L^M(A)=\Sigma_{M'}G_L^{M'}(PA)(M'|P|M)^L. \quad (5)$$

Hence, if $\mathfrak{S}$ denotes the ($\{P\}$-invariant) summation over the angular variables in set A, the $G_L^M(A)$ are unitary-orthogonal under $\mathfrak{S}$, and one can write

$$\mathfrak{S}G_L^M(A)^*G_{L'}^{M'}(A)=\delta_{LL'}\delta_{MM'}|(J_f\|L\|J_i)|^2, \quad (6)$$

where

$$\begin{aligned}|(J_f\|L\|J_i)|^2&\equiv(2L+1)^{-1}\Sigma_M\mathfrak{S}|G_L^M(A)|^2\\ &=\mathfrak{S}|G_L^M(A)|^2, \quad \text{(all } M). \quad (7)\end{aligned}$$

Numbers $(J_f\|L\|J_i)$, independent of angles, are determined by Eq. (7) to within a phase factor, so that if one puts

$$G_L^M(A)=(J_f\|L\|J_i)^*\mathcal{Y}_L^M(A), \quad (8)$$

the $\mathcal{Y}_L^M(A)$ belong to $(\ |\ |\ )^L$ and are unitary orthonormal under $\mathfrak{S}$. Usually the $\mathcal{Y}_L^M(A)$ will be those whose phases are fixed by convention or convenience, e.g., the (CS) spherical harmonics, so that Eq. (8) is to be thought of as determining the phases of the $(J_f\|L\|J_i)$.

The rate of emission of R in the transition $J_i \to J_f$ will be proportional to

$$\begin{aligned}I&=(2J_i+1)^{-1}\Sigma_{m_im_f}\mathfrak{S}|(J_fm_f|A|J_im_i)|^2\\ &=(2J_i+1)^{-1}\Sigma_{LM}(2L+1)^{-1}\mathfrak{S}|G_L^M(A)|^2\\ &=(2J_i+1)^{-1}\Sigma_L|(J_f\|L\|J_i)|^2, \quad (9)\end{aligned}$$

so that $|(J_f\|L\|J_i)|^2$ is simply the relative probability that R is emitted with angular momentum L. It will be convenient in the following to use a different normalization for these scalar amplitudes. Put

$$(J_f{:}L{:}J_i)=(J_f\|L\|J_i)/(\Sigma_L|(J_f\|L\|J_i)|^2)^{\frac{1}{2}}.$$

Then $\Sigma_L|(J_f{:}L{:}J_i)|^2=1$, and the $|(J_f{:}L{:}J_i)|^2$ are actual probabilities.

3. A GENERAL TWO-STEP CORRELATION FORMULA

We start with the usual expression

$$\begin{aligned}W(A_1;A_2)&=a\Sigma_{m_1m_2}|\Sigma_m(J_2m_2|A_2|Jm)(Jm|A_1|J_1m_1)|^2\\ &=a\Sigma_{m_1m_2mm'}(J_2m_2|A_2|Jm)(Jm|A_1|J_1m_1)\\ &\qquad\times(J_2m_2|A_2|Jm')^*(Jm'|A_1|J_1m_1)^* \quad (10)\end{aligned}$$

for the angular correlation function $W(A_1;A_2)$.[11] The normalization constant a will be chosen later. Multipole expansion of the nuclear matrix elements after Eq. (3) gives:

$$W(A_1;A_2)=a\Sigma_{(LM)}\mathcal{G}\Sigma_{m_1m_2mm'}\mathcal{V}, \quad (11a)$$

where we abbreviate

$$\begin{aligned}(LM)&=L_1, M_1, L_1', M_1', L_2, M_2, L_2', M_2'\\ \mathcal{G}&=G_{L_1}^{M_1}(A_1)^*G_{L_1'}^{M_1'}(A_1)\\ &\qquad\times G_{L_2}^{M_2}(A_2)^*G_{L_1'}^{M_2'}(A_2),\\ \mathcal{V}&=[(2J_2+1)(2J+1)]^{-1}(J_1L_1m_1M_1|J_1L_1Jm)\\ &\qquad\times(J_1L_1'm_1M_1'|J_1L_1'Jm')\\ &\qquad\times(JL_2mM_2|JL_2J_2m_2)\\ &\qquad\times(JL_2'm'M_2'|JL_2'J_2m_2).\end{aligned} \quad (11b)$$

[9] The $(m|P|m')^J$ are the $D^{(J)}(P)_{mm'}$ of (W).

[10] We use the vector-addition coefficients of (W), but in the Condon and Shortley notation. See E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935), Chapter III. This work will be referred to as (CS).

[11] G. Goertzel, Phys. Rev. **70**, 897 (1946), treats the $\gamma-\gamma$ case and his results are easily generalized.

We have used the fact that the vector-addition coefficients are real. It is known that Eq. (11) can be considerably simplified by assuming that one of the particles is emitted along the z axis-of-quantization;[1,3] a further reduction is possible when the correlation is expressed in terms of the matrix elements for emission of both particles along the z axis, as follows.

The angular correlation is a scalar function of the whole set A_1 plus A_2, so that without loss of generality one can assume that the $(+z)$ axis lies along the direction $\mathbf{\Omega}_1$ of the detected particle in the first transition and that the z-$(+x)$ half-plane contains $\mathbf{\Omega}_2$ of the second-transition particle. Let P_{21} be the rotation by angle θ, $0\leq\theta\leq\pi$, around the $(-y)$ axis which rotates $\mathbf{\Omega}_2$ (in these axes) into the z axis, so that θ is the angle between $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$. Denote by $A_1{}^z$ the values of the A_1 in these axes and by $A_2{}^z$ the values of the A_2 after P_{21} has been performed on them: $A_2{}^z=P_{21}A_2$. The $G_{L_2}{}^{M_2}(A_2)$ belong to $(\ \ |\ \ |\ \ )^{L_2}$, so that[12]

$$\begin{aligned}\mathcal{G}&=G_{L_1}{}^{M_1}(A_1{}^z)^*G_{L_1'}{}^{M_1'}(A_1{}^z)G_{L_2}{}^{M_2}(A_2)^*G_{L_2'}{}^{M_2'}(A_2)\\&=\Sigma_{N_2N_2'}G_{L_1}{}^{M_1}(A_1{}^z)^*G_{L_1'}{}^{M_1'}(A_1{}^z)G_{L_2}{}^{N_2}(A_2{}^z)^*\\&\quad\times G_{L_2'}{}^{N_2'}(A_2{}^z)(N_2|P_{21}|M_2)^{L_2*}(N_2'|P_{21}|M_2')^{L_2'}\\&=\Sigma_{N_2N_2'}(-1)^{N_2-M_2}G_{L_1}{}^{M_1}(A_1{}^z)^*G_{L_1'}{}^{M_1'}(A_1{}^z)\\&\quad\times G_{L_2}{}^{N_2}(A_2{}^z)^*G_{L'}{}^{N_2'}(A_2{}^z)\textstyle\sum_{\lambda\mu_1\mu_2}(L_2L_2'\\&\quad-N_2N_2'|L_2L_2'\lambda\mu_2)(L_2L_2'-M_2M_2'|L_2L_2'\lambda-\mu_1)\\&\qquad\qquad\times(\mu_2|P_{21}|-\mu_1)^{\lambda}.\end{aligned}\quad(12)$$

Two more vector-addition coefficients have been introduced into the formula for $W(A_1;A_2)$, but they are the ones that make Racah's formulas applicable. The sums over (M_2, M_2', m_2) and (m_1, m, m') can be eliminated, in that order, using Eq. (R41) and others, to give:

$$\begin{aligned}W(A_1;A_2)&=a\Sigma_{(L)}(-1)^{L_2'-L_2}(-1)^{J_1-L_1-J_2+L_2}\\&\times\textstyle\sum_{\lambda\mu_1\mu_2}(-1)^{\lambda+\mu_1}d^{(\lambda)}(\theta)_{\mu_2-\mu_1}\cdot\{W(L_1JL_1'J;J_1\lambda)\\&\times\textstyle\sum_{M_1}(-1)^{L_1+M_1}(L_1L_1'-M_1M_1'|L_1L_1'\lambda\mu_1)\\&\times G_{L_1}{}^{M_1}(A_1{}^z)^*G_{L_1'}{}^{M_1'}(A_1{}^z)\}\cdot\{W(L_2JL_2'J;J_2\lambda)\\&\times\textstyle\sum_{M_2}(-1)^{L_2+M_2}(L_2L_2'-M_2M_2'|L_2L_2'\lambda\mu_2)\\&\qquad\times G_{L_2}{}^{M_2}(A_2{}^z)^*G_{L_2'}{}^{M_2'}(A_2{}^z)\},\end{aligned}\quad(13)$$

where $(L)=(L_1, L_1', L_2, L_2')$, with the sum unrestricted, and $W(abcd;ef)$ is defined in Eqs. (R36).[13]

The $d^{(\lambda)}(\theta)_{\mu_2,-\mu_1}$ are the Jacobi polynomials given in (W):

$$\begin{aligned}d^{(\lambda)}(\theta)_{\mu_2,-\mu_1}&\\&=\sum_\kappa\frac{(-1)^\kappa[(\lambda+\mu_1)!(\lambda-\mu_1)!(\lambda+\mu_2)!(\lambda-\mu_2)!]^{\frac12}}{(\lambda-\mu_2-\kappa)!(\lambda-\mu_1-\kappa)!\kappa!(\kappa+\mu_1+\mu_2)!}\\&\qquad\times(\cos\tfrac12\theta)^{2\lambda-\mu_1-\mu_2-2\kappa}(\sin\tfrac12\theta)^{2\kappa+\mu_1+\mu_2};\end{aligned}$$

the sum on κ is over all integral values for which the arguments of the factorials are not negative.

Let us consider the $(2\lambda+1)$quantities

$$\begin{aligned}U_\lambda{}^\mu(A)=\Sigma_M(-1)^{L+M}&(LL'-MM'|LL'\lambda\mu)\\&\times G_L{}^M(A)^*G_{L'}{}^{M'}(A),\end{aligned}\quad(14)$$

$\mu=-\lambda, -\lambda+1, \cdots, \lambda$. From their definition and the transformation properties of the $G_L{}^M(A)$, it can be shown easily that the $U_\lambda{}^\mu(A)$ belong to $(\ \ |\ \ |\ \ )^\lambda$, as indicated by the notation. Let $\mathcal{S}$ denote the ($\{P\}$-invariant) summation over all angular variables in set A except for the direction $\mathbf{\Omega}$ of the detected particle. Since $\mathcal{S}$ is invariant, the quantities $\mathcal{S}U_\lambda{}^\mu(A)$ belong to $(\ \ |\ \ |\ \ )^\lambda$ or vanish identically. Their angular dependence is on $\mathbf{\Omega}$ only, and from parity considerations they are even functions of $\mathbf{\Omega}$. Also, since $L+L'$ will always be an integer, λ is an integer. These conditions fix the $\mathcal{S}U_\lambda{}^\mu(A)$ rather thoroughly, and one must have[14]

$$\begin{aligned}\mathcal{S}U_\lambda{}^\mu(A)&=u_\lambda Y_\lambda{}^\mu(\mathbf{\Omega}),\quad \lambda\text{ even},\\&=0,\qquad\qquad\ \ \lambda\text{ odd};\\\mathcal{S}U_0{}^0(A)&=(2L+1)^{\frac12}|(J_f\|L\|J_i)|^2\delta_{LL'}/(4\pi),\end{aligned}$$

where $Y_\lambda{}^\mu(\mathbf{\Omega})$ are the (CS) spherical harmonics and u_λ is a constant independent of μ. When $\mathbf{\Omega}$ is along the z axis the spherical harmonics $Y_\lambda{}^\mu(\mathbf{\Omega})$ vanish unless $\mu=0$. The terms $\mu_1\neq0$ or $\mu_2\neq0$ in Eq. (13) thus describe only the dependence of the correlation on the polarization of the first or second radiation, respectively, since these terms drop out when the correlation is summed over the corresponding polarization.

For directional correlations [the function $W(\theta)=\mathcal{S}_1\mathcal{S}_2W(A_1;A_2)$] with which the following will be mainly concerned, one needs only then that $d^{(\lambda)}(\theta)_{0,0}=P_\lambda(\cos\theta)$, so that the directional correlation has been obtained as a series of even Legendre polynomials. One sees from Eq. (13) that the coefficients of $P_\lambda(\cos\theta)$ factor into quantities depending on each transition separately. It is convenient to put

$$\begin{aligned}a_\lambda(LL';JJ')_M&=a_\lambda(L'L;JJ')_M\\&=(-1)^{L'-L}a_\lambda(LL';JJ')_{-M}\\&=(2-\delta_{LL'})(-1)^{J'-J+M+\frac12\lambda}[(2L+1)(2L'+1)(2J+1)/\\&\quad(2\lambda+1)]^{\frac12}W(LJL'J;J'\lambda)(LL'-MM|LL'\lambda0)\end{aligned}\quad(15)$$

[12] J. A. Spiers, Phys. Rev. 80, 491 (1950), gives an equivalent form which differs considerably in appearance from Eqs. (11)–(12), but which can also be used to obtain Eq. (13), following. The essential point is the reduction of the outer product of $(\ \ |\ \ |\ \ )^J$ which appear. This is a generalization of the method used by Gardner to obtain nonrelativistic conversion electron (spinless particle) correlations, reference 4.

[13] The derivation to this point is similar to one shown to the author by Professor Racah (private communication). The summations were performed in a different order in Lloyd's thesis (see footnote to title). [*Note added in proof.*—See G. Racah, Phys. Rev. 84, 910 (1951), where a semiclassical interpretetion of the various quantities appearing in Eq. (13) (Eq. (8) of Racah's paper) is given.]

[14] Suppose $f_\lambda{}^\mu(\mathbf{\Omega})$ belong to $(\ \ |\ \ |\ \ )^\lambda$. By considering rotations around the z axis, $f_\lambda{}^\mu(\mathbf{\Omega}^z)=\delta_{\mu0}f_\lambda{}^0(\mathbf{\Omega}^z)$. Also, $f_\lambda{}^\mu(\Omega)=\Sigma_{\mu'}f_\lambda{}^{\mu'}(P\mathbf{\Omega})(\mu'|P|\mu)^\lambda=f_\lambda{}^0(\mathbf{\Omega}^z)(0|P|\mu)^\lambda$, where P is any rotation of $\mathbf{\Omega}$ which brings it to the z axis. From the known form of the $(\ \ |\ \ |\ \ )^\lambda$, $f_\lambda{}^\mu(\mathbf{\Omega})=[4\pi/(2\lambda+1)]^{\frac12}f_\lambda{}^0(\mathbf{\Omega}^z)Y_\lambda{}^\mu(\mathbf{\Omega})$.

and also, after (LF),

$$\mathcal{S}\,\mathcal{Y}_L^M(\mathrm{A})^*\,\mathcal{Y}_{L'}^M(A) = [(2L+1)(2L'+1)]^{\frac{1}{2}}(4\pi)^{-1}F_{LL'}{}^M(\beta), \quad (16)$$
$$F_{LL'}{}^M = F_{LL'}{}^M(\beta=0) = F_{L'L}{}^{M*},$$

where β is the angle the detected particle makes with the z axis. This notation is intended to be that of (FU), except that the specific normalization

$$\Sigma_M F_{LL'}{}^M(\beta) = \delta_{LL'},$$
$$\int F_{LL'}{}^M(\beta)d(\cos\beta) = (L+\tfrac{1}{2})^{-1}\delta_{LL'}, \quad (\text{all } M) \quad (17)$$

is required.[15] In terms of these the directional correlation takes the form[16]

$$W(\theta) = 1 + \sum_{L_1\leq L_1'}\sum_{L_2\leq L_2'}(-1)^{L_2'-L_2}\sum_\lambda(2\lambda+1)$$
$$\times P_\lambda(\cos\theta)\cdot\Sigma_{M_1}a_\lambda(L_1L_1';JJ_1)_{M_1}$$
$$\times \mathrm{R.P.}[(J{:}L_1{:}J_1)(J{:}L_1'{:}J_1)^*F_{L_1L_1'}{}^{M_1}]$$
$$\cdot\Sigma_{M_2}a_\lambda(L_2L_2';JJ_2)_{M_2}$$
$$\times \mathrm{R.P.}[(J_2{:}L_2{:}J)(J_2{:}L_2'{:}J)^*F_{L_2L_2'}{}^{M_2}]. \quad (18)$$

The normalization constant has been given the value $a=(4\pi)^2((2J_1+1)I_1I_2)^{-1}$ to normalize $W(\theta)$ to unit average. The angular correlation selection rules appear as properties of Racah's coefficient $W(abcd;ef)$; the sum on λ in Eq. (18) is over the even integers satisfying simultaneously: $L_1'-L_1\leq\lambda\leq L_1+L_1'$; $L_2'-L_2\leq\lambda\leq L_2+L_2'$; $2\leq\lambda\leq 2J$. There is no correlation if $J=0$ or $J=\frac{1}{2}$. (These are the multipole mixture modifications of Yang's rules.[17])

For spinless particles the $\mathcal{Y}_L^M(A)$ can be only the spherical harmonics $Y_L^M(\mathbf{\Omega})$, so that $F_{LL}{}^M=\delta_{M0}$. For γ-rays, from (LF) $F_{LL}{}^M=\frac{1}{2}\delta_{|M|1}$; also, $(LL-11|LL\lambda0) = (LL1-1|LL\lambda0)$, λ even. In most cases of physical interest the L's will be integers, i.e., spinless particle, β-ray, γ-ray, conversion electron (exception: single nucleons), so that in Eq. (18), considering only the pure multipole terms, the quantities in square brackets are the fractions of the radiations that behave as 2^L-pole: spinless particles ($M=0$), γ-rays ($|M|=1$), massless quanta of spin-two ($|M|=2$), etc.—at any rate for directional correlation purposes. The quantities in square brackets are the physics of the problem and involve through the $F_{LL'}{}^M$, the type of radiation, and through the $(J_f{:}L{:}J_i)$, the particular nucleus emitting it. The quantities $a_\lambda(LL';JJ')_M$ are simply numerical coefficients and can be obtained from formulas in (R). The calculations for $a_\lambda(LL';JJ')_M$ are rather simple if one of the radiation angular momenta L, L' is the lowest ($L=|J-J'|$) or next lowest ($L=|J-J'|+1$) allowed by conservation of angular momentum, provided also that $|M|$ is not too large. The coefficients for these cases are given explicitly in the next section. The quantities in square brackets in Eq. (18) are discussed in Sec. 5.

4. THE GEOMETRICAL COEFFICIENTS

(a) Dependence on M

We observe first that the $a_\lambda(LL';JJ')_M$ as defined in Eq. (15) depend on M through the vector-addition coefficient only. Since the $\gamma-\gamma$ correlation coefficients ($|M_1|=|M_2|=1$) are already tabulated as functions of nuclear angular momentum,[8] it is convenient to introduce, for integral L and L', the ratios

$$\begin{aligned}\xi_\lambda(LL')_M &= (-1)^{L'-L}\xi_\lambda(LL')_{-M}\\ &= a_\lambda(LL';JJ')_M/a_\lambda(LL';JJ')_1\\ &= (-1)^{M-1}(LL'-MM|LL'\lambda0)/(LL'-11|LL'\lambda0). \quad (19)\end{aligned}$$

From Eqs. (R3) and (R5),

$$\xi_\lambda(LL')_0 = \frac{2[L(L+1)L'(L'+1)]^{\frac{1}{2}}}{L(L+1)+L'(L'+1)-\lambda(\lambda+1)}, \quad L+L'=\text{even}$$

$$\xi_\lambda(LL')_0 = 0, \quad L+L'=\text{odd}$$

$$\xi_\lambda(LL')_2 = \frac{(L-1)L(L+1)(L+2)+(L'-1)L'(L'+1)(L'+2) - \lambda(\lambda+1)[2(L-1)(L+2)+2(L'-1)(L'+2)-(\lambda-2)(\lambda+3)]}{((L-1)(L+2)(L'-1)(L'+2))^{\frac{1}{2}}[L(L+1)+L'(L'+1)-\lambda(\lambda+1)]}, \quad L+L'=\text{even}, \quad (20)$$

$$\xi_\lambda(LL')_2 = \frac{(\lambda-1)(\lambda+2)-(L-1)(L+2)-(L'-1)(L'+2)}{((L-1)(L+2)(L'-1)(L'+2))^{\frac{1}{2}}}, \quad L+L'=\text{odd},$$

[15] The summation in Eq. (17) is $-\min(L,L')\leq M\leq\min(L,L')$.
[16] The pure multipole terms in Eq. (18) have been given by K. Alder, Phys. Rev. **83**, 1266 (1951), and, for a special case, by Biedenharn, Arfken, and Rose, Phys. Rev. **83**, 586 (1951), who also consider triple cascade correlations.
[17] C. N. Yang, Phys. Rev. **74**, 764 (1948).

and in particular, for the pure multipole terms $L'=L$:

$$\xi_\lambda(LL)_0=\left[1-\frac{\lambda(\lambda+1)}{2L(L+1)}\right]^{-1},$$

$$\xi_\lambda(LL)_2=\left[1-\frac{2\lambda(\lambda+1)}{L(L+1)}+\frac{(\lambda-2)\lambda(\lambda+1)(\lambda+3)}{2(L-1)L(L+1)(L+2)}\right]\xi_\lambda(LL)_0, \tag{21}$$

$$\xi_\lambda(LL)_3=\left[1-\frac{9\lambda(\lambda+1)}{2L(L+1)}+\frac{3(\lambda-2)\lambda(\lambda+1)(\lambda+3)}{(L-1)L(L+1)(L+2)}-\frac{(\lambda-4)(\lambda-2)\lambda(\lambda+1)(\lambda+3)(\lambda+5)}{2(L-2)(L-1)L(L+1)(L+2)(L+3)}\right]\cdot\xi_\lambda(LL)_0.$$

The $\xi_\lambda(LL')_0$, $\xi_\lambda(LL')_2$, $\cdots$ are the factors which convert the L,L' terms in a γ-ray correlation into the corresponding terms in spinless particle, spin-two quantum, . . . correlations, and, as it turns out, the ones given in Eqs. (20)–(21) cover the spinless particle, β-ray to second-forbidden, and K and L conversion electron cases.

(b) The γ-Ray Coefficients

The lowest and next lowest γ-ray coefficients are, from Eq. (R36) and others, with $L\leq L'$ and $L=\pm(J-J')$ or $L=\pm(J-J')+1$:

$L+L'=$even:

$$a_\lambda(LL';JJ')_1=(2-\delta_{LL'})(-1)^{\frac{1}{2}(L'-L)}a_\lambda(LL')u(LL';JJ')v_\lambda(LL';JJ')[w_\lambda(J)]^{\pm\frac{1}{2}} \tag{22}$$

with

$$a_\lambda(LL')=\frac{\left(\frac{\lambda+L+L'}{2}\right)!(\lambda+L'-L)!\left(\frac{(2L'+1)(L+L')(L'-L+1)(L+L')!(2L+1)!}{(2L)(L'-L)!}\right)^{\frac{1}{2}}}{\left(\frac{\lambda+L+L'}{2}\right)!\left(\frac{\lambda+L'-L}{2}\right)!\left(\frac{L+L'-\lambda}{2}\right)!(\lambda+L+L'+1)!\xi_\lambda(LL')_0},$$

$$u(LL';JJ')=\left[\frac{(J+J'-L)!(J+J'+L+1)!}{(J+J'-L')!(J+J'+L'+1)!}\right]^{\pm\frac{1}{2}}, \tag{23}$$

$$w_\lambda(J)=\frac{(2J-\lambda)!(2J+\lambda+1)!}{(2J)!(2J+1)!},$$

$$v_\lambda(LL';JJ')=1 \text{ for } L=\pm(J-J'),$$

$$v_\lambda(LL';JJ')=1-\frac{(J'+\frac{1}{2}\mp\frac{1}{2})\lambda(\lambda+1)}{(J+\frac{1}{2}\pm\frac{1}{2})(L+L')(L'-L+1)} \text{ for } L=\pm(J-J')+1;$$

in particular, the pure multipole coefficients are

$$a_\lambda(LL;JJ')_1=a_\lambda(LL)v_\lambda(LL;JJ')[w_\lambda(J)]^{\pm\frac{1}{2}},$$

$$a_\lambda(LL)=\left[1-\frac{\lambda(\lambda+1)}{2L(L+1)}\right]\frac{(2L+1)!\lambda![L+(\lambda/2)]!}{(2L+\lambda+1)![(\lambda/2)!]^2[L-(\lambda/2)]!},$$

$$v_\lambda(LL;JJ')=1 \text{ for } L=\pm(J-J'), \tag{24}$$

$$v_\lambda(LL;JJ')=1-\frac{(J'+\frac{1}{2}\mp\frac{1}{2})\lambda(\lambda+1)}{2L(J+\frac{1}{2}\pm\frac{1}{2})} \text{ for } L=\pm(J-J')+1.$$

$L+L'=$odd:

$$a_\lambda(LL';JJ')_1=\zeta_\lambda(LL';JJ')a_\lambda(LL'';JJ')_1$$

where $L''=L'-1$ (and $L\leq L''$), and

$$\zeta_\lambda(LL';JJ')=\frac{1}{(2-\delta_{LL''})}x(LL';JJ')y_\lambda(LL')z_\lambda(LL';JJ'), \tag{25}$$

with

$$x(LL';JJ')=\pm\left[\frac{J+J'-L'+1}{J+J'+L'+1}\right]^{\pm\frac{1}{2}} \text{ for } L=\pm(J-J') \text{ or } =\pm(J-J')+1,$$

$$y_\lambda(LL')=(\lambda+L-L'')(\lambda+1+L''-L)\left[\frac{(L+L''+1)(2L''+3)}{L(L+1)(L''+1)(L''+2)(2L''+1)(L''+1-L)}\right]^{\frac{1}{2}}\xi_\lambda(LL'')_0,$$

$$z_\lambda(LL';JJ')=1 \text{ for } L=\pm(J-J'),$$

$$z_\lambda(LL';JJ')=\left[\frac{(L+L')(L'-L+1)}{(L+L'')(L''-L+1)}\right]^{\frac{1}{2}}\frac{v_\lambda(LL';JJ')}{v_\lambda(LL'';JJ')}, \text{ for } L=\pm(J-J')+1.$$

In particular, if $L'=L+1$

$$y_\lambda(LL+1)=\frac{\lambda(\lambda+1)}{(L+1)}\left[\frac{(2L+3)}{L(L+2)}\right]^{\frac{1}{2}}\left[1-\frac{\lambda(\lambda+1)}{2L(L+1)}\right]^{-1},$$

$$x(LL+1;JJ')=\mp\left[\frac{J'+\frac{1}{2}\mp\frac{1}{2}}{J+\frac{1}{2}\pm\frac{1}{2}}\right], \qquad (26)$$

$$x(12;JJ)z_2(12;JJ)=[(2J-1)(2J+3)/3]^{-\frac{1}{2}},$$

this last giving the dipole-quadrupole interference in $J'=J\rightarrow J$. Thus the $L+L'=$odd type interference terms are given in terms of the lower $L+L'-1=L+L''=$even type terms.

Note that the factor $v_\lambda(LL;JJ')$ is all that is needed to obtain pure next lowest multipole terms from a complete tabulation of pure lowest multipole terms. For example, if $L_1=\pm(J-J_1)+1$, the $J_1(L_1)J(X)J_2$ correlation, with $X=$any radiation, is

$$W(\theta)=1+\Sigma_\lambda v_\lambda(L_1L_1;JJ_1)C_\lambda P_\lambda(\cos\theta)$$

if the coefficients of the $J_1\mp1(L_1)J(X)J_2$ correlation are C_λ. Similarly, the coefficients C_λ' of the X-(2^{L_2}-pole) correlation for $J_1\rightarrow J\rightarrow J_2\mp1$ yield the $J_1(X)J(L_2)J_2$ correlation

$$W(\theta)=1+\Sigma_\lambda v_\lambda(L_2L_2;JJ_2)C_\lambda'P_\lambda(\cos\theta)$$

when $L_2=\pm(J-J_2)+1$.

Equations (19)–(26) will give most of the correlations of physical interest in the γ-ray, β-ray, and conversion electron cases. In the α-particle case the $(J_f\vdots L\vdots J_i)$ are not always sharply decreasing with increasing L;[18] and comparison with experiment might require the use of several of the higher terms in: $|J_i-J_f|\leq L, L'\leq J_i+J_f$.

5. THE PHYSICAL PARAMETERS

The analysis in Sec. 3 shows that the natural parameters for $F_{LL'}{}^M(\beta)$ are simply $F_{LL'}{}^M=F_{LL'}{}^M(\beta=0)$. Different parameterizations have been used elsewhere, however, so we collect here various properties of the $F_{LL'}{}^M(\beta)$ which will prove useful in obtaining, for example, β-ray correlations from the work of Fuchs[19] or Falkoff and Uhlenbeck.[20]

In the first place, the $F_{LL'}{}^M$ always occur in the combination $\Sigma_M(-1)^M(LL'-MM|LL'\lambda0)F_{LL'}{}^M$, and it is convenient to introduce a special notation for it. Put, for integral L and L',

$$B_\lambda(LL')=\sum_{-\nu}^{\nu}\xi_\lambda(LL')_M F_{LL'}{}^M,\ \nu=\min(L,L'). \qquad (27)$$

Particular cases are

$$B_2(11)=(4\lambda)/(3+\lambda)=(-2B)/(3A+B) \qquad (28a)$$

if $F_{11}{}^0(\beta)\sim A+B\cos^2\beta$ and λ is the $L=1$ parameter of (FU).

$$B_2(22)=(14+12\mu_2)/\nu_1=(2-42k_2)/\nu_2=(14B+12C)/\nu_3,$$
$$B_4(22)=(-2\mu_2)/\nu_1=2/\nu_2=(-2C)/\nu_3, \qquad (28b)$$
$$\nu_1=5+15\mu_1+3\mu_2,\quad \nu_2=2-15k_1-30k_2,\quad \nu_3=15A+5B+3C,$$

if $F_{22}{}^0(\beta)\sim A+B\cos^2\beta+C\cos^4\beta$ and (μ_1,μ_2) or (k_1,k_2) are the $L=2$ parameters of (FU). [Equation (FU20) should read

$$F_2{}^{\pm2}(\vartheta)=\mu_1+\tfrac{1}{2}\mu_2+\tfrac{2}{3}-(\mu_2+1)\cos^2\vartheta+\tfrac{1}{6}\mu_2\cos^4\vartheta.]^{21}$$

For $L=L'=3$, if

$$F_{33}{}^0(\beta)=A+B\cos^2\beta+C\cos^4\beta+D\cos^6\beta,$$
$$B_2(33)=10(7B+6C+5D)/\nu,$$
$$B_4(33)=8(11C+15D)/\nu, \qquad (28c)$$
$$B_6(33)=-16D/(5\nu),$$
$$\nu=105A+35B+21C+15D.$$

For γ-rays, the polarized spherical harmonics of Eq. (8) are, with the direction $\mathbf{\Omega}^z$ of the quantum along

[18] See H. A. Bethe, Revs. Modern Phys. 9, 69 (1937), Secs. 66–73; in particular, Table XXXIII, p. 172 and TABLE XXXIV, p. 179.

[19] M. Fuchs, thesis, University of Michigan, 1951.

[20] D. L. Falkoff and G. E. Uhlenbeck, Phys. Rev. 79, 334 (1950).

[21] As a result of this misprint the results involving the (μ_1,μ_2) parameters given in Lloyd's thesis (see footnote to title) are incorrect.

the z axis,

$$\mathcal{Y}_L^{\pm 1}(\mathbf{\Omega}^z, \boldsymbol{\varepsilon}) = \mp\tfrac{1}{2}[(2L+1)/(4\pi)]^{\frac{1}{2}} i^L e^{\pm i\varphi}, \text{ (el)}$$
$$\mathcal{Y}_L^{\pm 1}(\mathbf{\Omega}^z, \boldsymbol{\varepsilon}) = \tfrac{1}{2}[(2L+1)/(4\pi)]^{\frac{1}{2}} i^{L-1} e^{\pm i\varphi}, \text{ (mag)} \quad (29)$$
$$\mathcal{Y}_L^{\pm 1}(\mathbf{\Omega}^z, \boldsymbol{\varepsilon}) = 0, \quad |M| \neq 1, \text{ (el or mag)},$$

where φ is the angle the linear polarization vector $\boldsymbol{\varepsilon}$ of the quantum makes with the x axis; in Eq. (13), φ is the angle that $\boldsymbol{\varepsilon}$ (in either transition) makes with the plane containing the directions of both particles. The $\mathcal{S}$ process is simply:

$$\mathcal{S}f(\mathbf{\Omega}^z, \varphi) = f(\mathbf{\Omega}^z, \varphi) + f(\mathbf{\Omega}^z, \varphi + \tfrac{1}{2}\pi).$$

One finds that

$$F_{LL'}{}^{\pm 1} = \tfrac{1}{2}(-1)^{\frac{1}{2}(L'-L)}, \quad L+L' = \text{even } (e-e \text{ or } m-m)$$
$$F_{LL'}{}^{\pm 1} = \pm\tfrac{1}{2}(-1)^{\frac{1}{2}(L_e-L_m-1)},$$
$$L+L' = \text{odd } (e-m \text{ or } m-e),$$

so that, for γ-rays,

$$B_\lambda(LL') = (-1)^{\frac{1}{2}(L'-L)}, \qquad L+L' = \text{even},$$
$$B_\lambda(LL') = (-1)^{\frac{1}{2}(L_e-L_m-1)}, \quad L+L' = \text{odd}. \quad (30)$$

The $(J_f{:}L{:}J_i)$ in the γ-ray case are real,[22] and theoretical and experimental evidence indicates that all except the one for smallest L are small or negligible.[23]

6. SUMMARY

It has been shown that the angular correlation function can be expanded naturally as a series in the matrix elements $(\ \ |\ \ |\ \ )^\lambda$ of the Pauli spin group (Jacobi polynomials of $\sin^2\frac{1}{2}\theta$) naturally in the sense that the coefficients are factored into quantities which depend on the first and second transitions in $J_1 \to J \to J_2$ separately. This feature makes it easy to modify, e.g., $\gamma-\gamma$ directional correlations to get other correlations of physical interest.[24] The general formula is Eq. (13), Sec. 3.

The directional correlation is obtained as a series of even Legendre polynomials, the formula intended for numerical use being:

$$W(\theta) = 1 + \sum_{L_1 \leq L_1'} \sum_{L_2 \leq L_2'} \sum_{\lambda} \{(-1)^{L_2'-L_2}(2\lambda+1)$$
$$\times a_\lambda(L_1L_1'; JJ_1)_1 a_\lambda(L_2L_2'; JJ_2)_1\} P_\lambda(\cos\theta)$$
$$\cdot \text{R.P.}[(J{:}L_1{:}J_1)(J{:}L_1'{:}J_1)^* B_\lambda(L_1L_1')]$$
$$\cdot \text{R.P.}[(J_2{:}L_2{:}J)(J_2{:}L_2'{:}J)^* B_\lambda(L_2L_2')]. \quad (31)$$

The coefficients in curly brackets are the coefficients for $\gamma-\gamma$ correlations, given in Sec. 4, and tabulated for pure lowest multipoles in Lloyd's thesis (see footnote to title). The quantities $(J_f{:}L{:}J_i)$, introduced in Sec. 3, are relative scalar amplitudes for emission of the radiation in $J_i \to J_f$ with angular momentum L, normalized to $\Sigma_L |(J_f{:}L{:}J_i)|^2 = 1$. They appear as unknown nuclear parameters in the correlation unless the corresponding radiation is emitted as a pure multipole. The coefficients $B_\lambda(LL')$ are the physical parameters of the emitted radiation. In the γ-ray case they are (± 1), after Eq. (30). In the scalar particle case they have the value $\xi_\lambda(LL')_0$, Eqs. (27) and (20), and again are simply numerical coefficients. The $B_\lambda(LL')$ for conversion electrons are functions of nuclear charge and electron energy, different for each atomic shell; the pure multipole $B_\lambda(LL)$ up to $L=5$ obtained by using Dirac wave functions for the atomic electrons have been given by Rose, Biedenharn, and Arfken for the K shell for certain values of the energy and z.[25] The angular distributions $F_{LL}{}^0(\beta)$ of (FU) have been given by Fuchs[19] for β-decay up to second forbidden; the $B_\lambda(LL)$ are then to be had from Eqs. (28). They are, again, functions of particle energy and nuclear charge.

7. ACKNOWLEDGMENTS

I wish to express my reverence for deeply cherished memories of association with the late Sidney M.

---

[22] The phases and normalization used in S. P. Lloyd, Phys. Rev. **81**, 161 (1951) have been changed to conform more closely with those of (FU), but the proof and results remain essentially unchanged. The $(J_f\|L\|J_i)$ for γ-rays in the present article are $(-1)^{2J_f-L}(2L+1)^{-\frac{1}{2}}$ times those of Phys. Rev. **81**, 161 (1951). This leads to a change in sign of the interference terms, Eqs. (25)–(26) above, compared to those of reference 7, so that the phases are now the same as those of (LF), i.e., $(J_f\|L-1\|J_i)=\alpha$, $(J_f\|L\|J_i)=\beta$. A point overlooked by Ling and Falkoff is that the matrix $\alpha\beta^*$ is antiHermitian, so that the interference terms for the first and second transitions must have opposite signs, whatever the choice of phase. The signs in the (LF) tables are those which hold when the first γ-ray is mixed, the second pure, although they claim the converse.

[23] M. Goldhaber and A. W. Sunyar, Phys. Rev. **83**, 906 (1951).

The dipole-quadrupole mixture case may be an exception, and we conclude by remarking that (X)-(mixed dipole-quadrupole γ-ray) correlations can be obtained from the (LF) quadrupole-(dipole-quadrupole) tables as follows. (1) For $J-\Delta j \to J \to J+\Delta J$, express $W(\theta) \doteq Q + R\cos^2\theta + S\cos^4\theta$ as a Legendre series: $W(\theta) = 1 + A_2P_2(\cos\theta) + A_4P_4(\cos\theta)$, where

$$A_2 = [\tfrac{2}{3}R + (4S/7)][Q + \tfrac{1}{3}R + (S/5)]^{-1},$$
$$A_4 = (8S/35)(Q + \tfrac{1}{3}R + \tfrac{1}{5}S)^{-1}.$$

(2) Multiply these A_λ by

$$B_\lambda(L_1L_1')\, a_\lambda(L_1L_1'; JJ_1)_1 / a_\lambda(22; JJ-\Delta j)_1.$$

One then has the $(X-2^{L_1}-2^{L_1'}$-pole)-(dipole-quadrupole γ-ray) coefficient of $P_\lambda(\cos\theta)$ for $J_1 \to J \to J+\Delta J$. This is independent of the value of Δj chosen, of course, and the calculations will be easiest for $\Delta j = \pm 2$. (3) Multiply by the appropriate scalar matrix elements (taking the real part of the product if necessary), and sum over L_1 and L_1'. (4) Change the sign of the $\alpha\beta^*$ term for the second γ-ray. (See reference 22.) If the (dipole-quadrupole)-γ-ray is first, the procedure is essentially the same, except that the signs in the (LF) tables are applicable as they stand; one obtains $J+\Delta J \to J \to J-\Delta j$, with ΔJ mixed, as $J-\Delta j \to J \to J+\Delta J$ from (LF), keeping the sign, and then performs steps (2) and (3), preceding, but not (4).

[24] Polarization-polarization $\gamma-\gamma$ correlations, including interference terms, can be obtained from the preceding as soon as one knows $(LL'11|LL'\lambda 2)/(LL'-11|LL'\lambda 0)$.

[25] Rose, Biedenharn, and Arfken, to be published. [*Added in proof.*—See Phys. Rev. **85**, 5 (1952). The $b_\nu(Z, k, \pi, L)$ of (RBA) are the $B_\nu(LL)$ of the present article.]

Dancoff, Professor of Physics at the University of Illinois, who stood as constant and friendly guide throughout my graduate studies, as well as in the preparation of my thesis. He was, in every sense, a good and inspiring man.

The many discussions with and the advice of Professor J. M. Blatt of the University of Illinois have been of great benefit. Thanks are also due A. Pais and R. Jost of the Institute for Advanced Study for criticism of the manuscript.

PHYSICAL REVIEW VOLUME 85, NUMBER 5 MARCH 1, 1952

On the Quantum Theory for a Finite-Sized Electron. I

GEORGE J. YEVICK
Stevens Institute of Technology, Hoboken, New Jersey
(Received June 7, 1951)

Section I discusses briefly different types of models for a finite-sized electron from the classical point of view with each model being characterized by a particular type of charge-current distribution. Each of these models can be quantized by the Feynman Lagrangian procedure and the new problems encountered are analyzed in Sec. II. These problems are illustrated in Sec. III which goes into the detailed calculations for the first-order perturbation calculation of a finite-sized electron interacting with an external field.

INTRODUCTION

OUR point of view has been to place emphasis not so much on the elegance of the physical model but rather on the exposing and the understanding of some of the problems of the new quantum-mechanical framework developed by Feynman. We should like to stress the fact that this framework is the most ideally suited at the present time to handle, in a relativistically covariant manner, the Lorentz type of finite-sized electron.

We can consider a finite-sized electron as a worthy subject for investigation for several physical reasons. In reality the electron may not be a point singularity but actually extends over a region in space-time. Or, it may be that the notions which we have concerning space and time, in particular their continuity properties, actually are incorrect in the very small but because there is no suitable theory that one can use, the finite-sized electron may, for the time being, introduce the necessary fuzziness. Finally there is the physical possibility that the electron is a complex structure in the extremely small regions of space surrounding its center. This structure may be connected with meson particles concerning whose nature we know nothing at present. The finite-sized electron can serve us then as a useful mechanism for calculating physical processes by lumping our ignorance into a suitable structure function with the hope that an explicit evaluation of the structure function by experiment will give rise to new clues for a deeper insight into the electron.

The present theory of quantum electrodynamics still is in a basically unsatisfactory state because of the existence of well-known legitimate problems which the theory is inherently incapable of solving. These problems perhaps can be separated into two categories which may have little to do with each other. The first category contains the infinities of mass and charge and it is the thesis of this work that a finite-sized electron will solve these problems. The second category pertains to the deeper problems of the fine structure constant, the relationship of the electron to the mesons, and the reason for the great stability of the electron. There has been no indication so far as we know from experiment or theory up to the present time as to how to proceed with the second category of problems and we have not attempted to consider these problems in our analysis.

With regard to the first kind of problems, i.e., the mass and charge infinities, the following point ought not to be ignored. The existence of these infinities is in itself unsatisfactory. But techniques for calculating a physical process involving electrons and photons by subtracting out the infinities have now been developed. Nevertheless, the crucial question still remains open whether this calculated value will agree with experiment at extremely high energies. For example, in the scattering of a moving electron against an electron at rest, we have in mind energies of several billion electron volts, for it is only at such high energies that the classical radius of the electron comes into play.

Now undoubtedly the reason for the infinities in the present theory is that the electron is treated as a point singularity. Our assumption is that if one does not have a point but an extended electron we would not have such infinities. By an extended or finite-sized electron we mean that the electron charge-current density is no longer a point singularity but has some sort of smeared-out distribution. This smeared-out distribution is not rigidly fixed with respect to the center of the electron as the electron moves along its world line, but will change its shape as the motion of the electron changes because of the fundamental requirement of conservation of charge-current density.

The difficulty which one immediately encounters is how to treat quantum mechanically such an extended