Neutron Scattering Lengths*

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The schematic treatment of nuclear reactions is applied to the case $E_n \to 0$ as a means of interpreting measurements of the scattering length a. It is found that the quantity $\Delta = (a-R)$ rather than a is significant for determining the resonance properties of the compound nucleus, where R is the phenomenological nuclear radius. In some cases this treatment can resolve ambiguity in the evaluation of a without the necessity of polarization measurements and assign the J value of a low-lying resonance or two, but it can only be used successfully for separated isotopes. For practically pure isotopes it can be used to determine rough values of level spacing D from the measured position of the levels nearest $E_n=0$, or conversely. Values of D obtained in this way are plotted against A to yield an estimate of a parameter in the statistical formula for level density; this estimate suggests a somewhat more rapid increase in level spacing with decreasing excitation than given by the simplest statistical formula.

INTRODUCTION AND SUMMARY

A COMPLETE interpretation of the scattering length[1] for slow neutrons has been hampered by difficulty in separating the effects of near-lying resonance levels from the "potential scattering." The present note indicates how a schematic treatment of nuclear reactions[2] resolves this difficulty and allows the extraction of semi-quantitative information about the compound nuclear levels nearest $E_n=0$.

A simple formula is presented that reduces in most cases of interest to a relation among the scattering length, nearest resonance energy, and average level spacing D in the compound nucleus (or alternatively the reduced neutron width γ^2), so that from any two quantities the third can be estimated. This relation is applied to susceptible data, and further measurements of possible interest are indicated. A very tentative order-of-magnitude curve for D (or γ^2) *vs* mass number A is extracted from presently analyzable data; this curve is then used in turn to suggest order-of-magnitude positions for unobserved levels.

I. FORMULATION

The scattering length a can be defined from the wave function of the neutron beam as $k\to 0$. Since only s-neutrons are involved, the wave function exterior to the nucleus is $\varphi=(kr)\psi=\sin k(r-a)$ and is determined by fitting its logarithmic derivative[2] at a suitable radius R:

$$R\varphi'/\varphi = kR\cot k(R-a) = f(E_n), \tag{1}$$

where the function f is determined by the nuclear forces interior to R. As $k\to 0$, this becomes

$$a-R=\Delta=-R/f(0). \tag{2}$$

All the resonance properties of the nucleus are expressed by f, so that according to (2) the significant quantity for interpretation is Δ rather than the measured scattering length. The radius R represents the point at which the incident neutrons are subject to a large, abrupt change in refractive index of the medium, which produces potential scattering. Of course this change is not perfectly sharp, and there is some uncertainty δR in the definition of R, although generally $\delta R \ll R$. As a standard value $\delta R = 1\times 10^{-13}$ cm is assumed throughout, which implies that (2) is significant for analysis only if

$$|\Delta| > 10^{-13}\ \text{cm}. \tag{3}$$

Failure to satisfy condition (3) means that there are no resonances available to s-neutrons that lie much closer than D to the position $E_n=0$.

The function $f(E)$ has simple zeros at the resonance energies E_r, so that

$$1/f(E)=\Sigma_r[1/f'(E_r)][E-E_r], \tag{4}$$

where $f'=[\partial f/\partial E]_{E_r}=-\pi KR/D_r$ in the notation of reference 2. Here D_r is the effective level spacing at the rth resonance and may decrease rapidly with increasing E_r; K is a wave number appropriate to the motion of the neutron inside the nucleus. Thus

$$\Delta=-(1/\pi K)\Sigma_r D_r/E_r = -\Sigma_r \gamma_r^2/E_r, \tag{5}$$

where $\gamma_r^2=\Gamma_n/2k$ is the "reduced width"[3] of the rth resonance level. The value of K is not precise and may vary considerably among the light nuclei; however, $K=1.0\times 10^{13}$ cm^{-1} should be in error by less than 50 percent for any particular case and will be assumed throughout.

If the level nearest $E_n=0$ has $|D_0/E_0|\gg 1$, it is sufficient to approximate (5) by the corresponding single term, since for all other r, $|D_r/E_r|\ll 1$ unless there is an exceptional fluctuation in D_r. Thus most of the cases below are analyzed with the approximation

$$\Delta\approx -(1/\pi K)(D_0/E_0), \tag{5a}$$

although it may occasionally be necessary to include more terms of (5).

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Actually, E_r has an imaginary component $-\frac{1}{2}i\Gamma_a$, which has been neglected above as very small in comparison with E_r. Since $D_r/\Gamma_a \sim 10^3$ to 10^6, this is *a priori* reasonable; any exceptions would distinguish themselves as experimentally anomalous, and none is known.

For a bound resonance level the binding energy $-E_b$ does not equal E_r, for E_b is defined so that $f(-E_b) = -(2M'E_b/\hbar^2)^{\frac{1}{2}}R$, where M' is the reduced neutron mass. Assuming that a single term of (4) is sufficient in the neighborhood of a resonance, one has

$$-E_r = E_b + (M'E_b/\hbar^2)^{\frac{1}{2}}\gamma_r^2. \qquad (6)$$

II. APPLICATION TO LIGHT NUCLEI

The simplest case is that of n-p scattering, for which the triplet and singlet scattering lengths are[4] $a_t = 5.3 \times 10^{-13}$ cm, $a_s = -23.8 \times 10^{-13}$ cm. The radius R should be chosen as the intrinsic range of the force, which is[5] $R_t = 1.8 \times 10^{-13}$ cm for the triplet square well. In this case $f = -KR\tan(KR - \pi/2)$, so that D_r can be computed exactly; it turns out that $D_r \approx 125$ Mev, $E_r \approx -11$ Mev before correction by (6). For the singlet system D_r and R are presumably about the same, although Δ_s is relatively insensitive to the choice of R_s. This insensitivity has been remarked in the interpretation of low-energy scattering measurements;[5] from the present point of view its cause is the great excess of resonant over "potential" scattering in the neighborhood of the low-lying resonance, which extends over a wide region $\frac{1}{2}\Gamma_n \approx 10$ Mev. Using the same R and D_r as for the triplet system, the singlet resonance is at $E_r \approx 1.9$ Mev. In comparison with the usual formulation, it must be remembered that this formulation[6] deals not with E_r but with E_v, which is defined in exact analogy to E_b, so that $f(E_v) = +(2M'E_v/\hbar^2)^{\frac{1}{2}}R$. Correction by a formula equivalent to (6) but with a reversal of sign for E_r yields $E_v \approx 0.06$ Mev, the usually quoted value. The difference between the two values of E_r represents the difference in depth of the corresponding square wells, or about 13 Mev.

The next heavier target is the deuteron, which has been studied by a number of investigators;[7–9] the most precise values[9] are

$$\left.\begin{matrix} a_{\frac{1}{2}} \\ a_{\frac{3}{2}} \end{matrix}\right\} = \left\{\begin{matrix} 0.7 \pm 0.3 \\ 6.38 \pm 0.06 \end{matrix}\right. \times 10^{-13} \text{ cm}, \qquad (7a)$$

or

$$\left\{\begin{matrix} 8.26 \pm 0.12 \\ 2.6 \pm 0.2 \end{matrix}\right. \times 10^{-13} \text{ cm}, \qquad (7b)$$

where polarization measurements are the only direct means to decide this characteristic ambiguity that occurs whenever the target spin is $I \neq 0$. An immediate but not surprising conclusion from these data is that an effective two-body square well interaction between neutron and deuteron[10] does not give a reasonable description of the triton. If (7b) is chosen, the two-body square well can give agreement only if there is a second bound doublet state in addition to the ground state; for (7a) this second level becomes unbound by about 1.3 Mev, which still implies an incredible doublet S level spacing of $\lesssim 10$ Mev. In both cases the well parameters are not very reasonable physically, with $R \approx 8.0 \times 10^{-13}$ cm, $V \approx 10$ Mev. One must therefore revert to somewhat less specific arguments: for the triton ground state with $E_b = -6.2$ Mev, E_r is probably on the order of -15 Mev; since the doublet S level spacing is presumably $\gg 15$ Mev, $a_{\frac{1}{2}}$ should reflect mainly the properties of the ground state. Thus $\Delta_{\frac{1}{2}}$ is positive, and the set of values (7b) is indicated. From this set, the lowest quartet S level is virtual with very uncertain position probably on the order of 10 Mev. Somewhat less uncertain is the feature that its half-width will be on the same order of magnitude as its position.

It would also be of interest to determine the singlet and triplet scattering lengths on H^3; if any indication of a resonance level in H^4 is found, the corresponding state would exist in He^4, differing only by Coulomb energy. For scattering on He^4 it is known that there are no bound S states, so that[11] $R \geq a = +(\sigma/4\pi)^{\frac{1}{2}} = 3.5 \times 10^{-13}$ cm.

The scattering length of lithium has been determined[12] for the separated isotope Li^7; because the initial spin $I \neq 0$, there is a twofold ambiguity in the assignment of $a_{I\pm\frac{1}{2}}$. If $R = 4.1 \times 10^{-13}$ cm is assumed, the two possible sets of Δ_J are $\Delta_{1,2} = (-10.8, -4.1)$ or $(-2.4, -9.1)$ in 10^{-13} cm units. Two resonances are reported[13] in neutron scattering on lithium, at 270 kev and ~ 1.15 Mev, with respective half-widths of 45 kev and ~ 2.4 Mev. If both are attributed to s-neutrons, the corresponding $\gamma_r^2/E_r = 0.7$, $\sim 4.5 \times 10^{-13}$ cm. The first of these values does not approach any of the observed Δ, so that the corresponding level is probably due[13] to neutrons with $l > 0$. The larger value of γ_r^2/E_r indicates a level spacing $D_r \approx 17$ Mev, but cannot be unambiguously associated with any particular Δ. In any case, the measured Δ indicate the presence of two such levels with dissimilar characteristics.

The respective scattering lengths on C^{12} and O^{16} targets are 6.4 and 5.8×10^{-13} cm. Comparison with the scattering of protons by these targets indicates[14] that

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the bound S levels in C^{13} have $E_b=-1.85$ Mev, $\gamma_r^2\approx 7.6\times 10^{-13}$ Mev cm, and in O^{17} $E_b=-1.09$, -3.24 Mev, $\gamma_r^2\approx 0.05\times 10^{-13}$, 2×10^{-13} Mev cm, respectively. Inserting these values in (6) and (5), one finds the phenomenological radius $R=4.4$, 5.2×10^{-13} cm for the two cases. If these are compared with the formula $R=r_0(A^{\frac{1}{3}}+1)$, where A is the mass number of the target nucleus, the value of the parameter is $r_0=1.41\pm 0.07\times 10^{-13}$ cm. This expression is used to compute R for the other light targets considered in this section, Li^7 and N^{14}.

For N^{14}, $\Delta_{\frac{3}{2},\frac{1}{2}}=(5.7, -0.3)$ or $(1.7, 7.7)$ in 10^{-13} cm, indicating at least one bound level. If the probable value of γ_r^2 suggested by neighboring light nuclei is taken as $\gamma_r^2\sim 3\times 10^{-13}$ Mev cm, the bound level has $E_b\sim 2$ Mev.

III. APPLICATION TO HEAVIER NUCLEI

Here "heavier" nuclei are arbitrarily defined as those with $A>18$; there is less detailed knowledge of their level positions and parameters than for light nuclei. The interpretation of scattering lengths is frequently obscured by the presence of many isotopes and spins, when

$$a=\Sigma_k\rho_k a_k,\quad \sigma_{sc}/4\pi=\Sigma_k\rho_k a_k^2, \tag{8}$$

where the statistical factors are $\rho_k=\rho_I(J+\frac{1}{2})/(2I+1)$, ρ_I is the abundance of a given isotope of spin I, and $J=I\pm\frac{1}{2}$. It is clear that analysis is impossible for mixtures with two or more isotopes of comparable abundance; and even for practically single isotopes a twofold ambiguity exists in the assignment of $a_{I\pm\frac{1}{2}}$ unless $I=0$. Frequently, however, some information can be obtained in spite of the ambiguity if an independent measure of γ_r^2 or J is available for the nearest virtual states.

The cases in which known neutron resonances can be correlated with a are discussed below, referred to the compound nucleus. The one-level approximation (5a) is used throughout, and the effective R is taken from analysis of high energy neutron scattering[15] with some smoothing of fluctuations. All lengths are in units of 10^{-13} cm and γ_r^2 in units of Mev$\times 10^{-13}$ cm. The effective level spacings are estimated from $D_r^J=-\pi K E_r\Delta_J$, where Δ_J is obtained from the scattering length measurements.

Na^{24}: $R=4.6$, $\Delta_{1,2}=(-6.3, +2.0)$ or $(+4.1, -4.2)$. A scattering resonance is observed[16] at $E_r=0.06$ Mev, apparently with $\Gamma_n\lesssim 20$ kev; hence $\gamma_r^2/E_r\lesssim 3$. This agrees best with the second set of Δ from the scattering length measurements, indicating $J=2$, $D_r^2=0.8$ Mev. A lower resonance has been found[17] at $E_r\approx 3$ kev; but it does not show the characteristic dip below resonance and is therefore presumably because of neutrons with $l\neq 0$.

Al^{28}: $R=4.6$, $\Delta_{2,3}=(-1.1, -1.1)$. The two lowest scattering resonances[18] are at $E_r\approx 0.04$, 0.095 Mev, $\Gamma_n\lesssim 6$, 15 kev, whence $\gamma_r^2/E_r\lesssim 1.6$, 1.2. This is satisfactory agreement with the observed Δ, so that the two levels may be assigned $J=2$, 3 in indeterminate order; $D_r=0.1$, 0.3 Mev.

S^{33}: $R=4.5$, $\Delta_{\frac{1}{2}}=-1.3$. The lowest resonance[19] has $E_r=0.11$ Mev, $\Gamma_n\approx\Gamma\approx 18$ kev. Thus $\gamma_r^2/E_r=1.1$, in excellent agreement with the observed Δ; $D_r^{\frac{1}{2}}\approx 0.4$ Mev.

V^{52}: $R=5.4$, $\Delta_{3,4}(-12.6, +0.2)$ or $(1.8, -11.0)$. The lowest reported resonance[20] is at $E_r=2.7$ kev with a measured value of

$$(J+\tfrac{1}{2})/(2I+1)\gamma_r^2/E_r=5.6;$$

from this measurement $\Delta_{3,4}=(-12.7, -9.9)$. Best agreement with the observed Δ is obtained for $J=3$; $D_r^3=0.1$ Mev.

Mn^{56}: $R=5.5$, $\Delta_{2,3}=(-6.4, -10.5)$ or $(-11.2, -7.1)$, with[21] $\sigma_{sc}=1.8$ barn. Observed[19] resonances are at $E_r=0.35$, 2.4 kev; the measured values of $(J+\frac{1}{2})/(2I+1)\gamma_r^2/E_r$ lead to $\Delta_{2,3}=(-10.8, -4.2)$ or $(-5.9, -7.7)$ according as the lower level has $J=2$ or $J=3$. The agreement or disagreement with measured Δ is about the same for either choice of J for the lower level, so no assignment can be made on this basis. Independent considerations[22] indicate $J=3$ for the lower level; accordingly, $D_r^{2,3}=0.05$, 0.01 Mev.

Co^{60}: $R=5.7$, $\Delta_{3,4}=(-9.3, +2.1)$ or $(3.5, -7.9)$. The prominent resonance at $E_r=115$ ev has a peak value compatible only with[23] $J=4$. The value of $\Gamma_n\approx\Gamma$ is given[20] as 3.5 to 4.5 ev, whence $\gamma_r^2/E_r=6$ to 9, this range straddles the observed Δ only if $J=4$; $D_r^4=3$ kev.

As^{76}: $R=6.3$, $\Delta_{1,2}=(-5.3, +3.1)$ or $(5.2, -3.1)$. A resonance is reported[24] only as 115 ev$<E_r<$300 ev; combined with the ambiguity in Δ, this yields $D_r=1$ to 5 kev.

Ag^{110}: $R=6.7$, $\Delta_{0,1}=(6.9, -5.5)$ or $(-11.1, +0.7)$. The prominent resonance is reported[25] to have $E_r=5.2$ ev, $\Gamma_n\approx 0.01$ ev, whence $\gamma_r^2/E_r\approx 2$. This agrees with the measured Δ if $J=1$; $D_r^1=0.1$ kev. Although this assignment seems the most plausible, it should be remembered that Ag has at least two higher resonances[26] at 16 and 45 ev about which little is known.

I^{128}: $R=7.3$, $\Delta_{2,3}=(-4.3, -0.6)$ or $(+0.1, -3.7)$. In either case only one nearby virtual level is indicated, with $\Delta\approx -4$ and indeterminate spin. If Δ is associated with one of the strong resonances[27] at $E_r\approx 35$ ev, $D_r=0.4$ kev.

Ta^{182}: $R=8.0$, $\Delta_{3,4}=(2.1, -3.5)$ or $(-4.3, +1.5)$. If the negative scattering length is associated with the first resonance[28] at 4.1 ev, the level spacing (indeterminate J) is $D_r\approx 0.05$ kev.

Au^{198}: $R=7.8$, $\Delta_{1,2}=(-4.3, +2.5)$ or $(+4.2, -2.6)$. The scattering resonance is reported[29] to have $E_r=4.9$ ev, $\Gamma_n=0.021$ ev ($J=1$) or 0.016 ev ($J=2$). The corresponding values of γ_r^2/E_r are 8.6 and 6.6; best agreement with the measured Δ is for $J=1$, $D_r^1=0.07$ kev.

In those cases discussed above where some estimate of Γ_n is available, it is frequently possible to assign J values to low-lying resonances. These assignments generally agree with those made independently, although the present method of analysis appears to be the simplest when applicable.

The values of D_r are plotted against A of the com-

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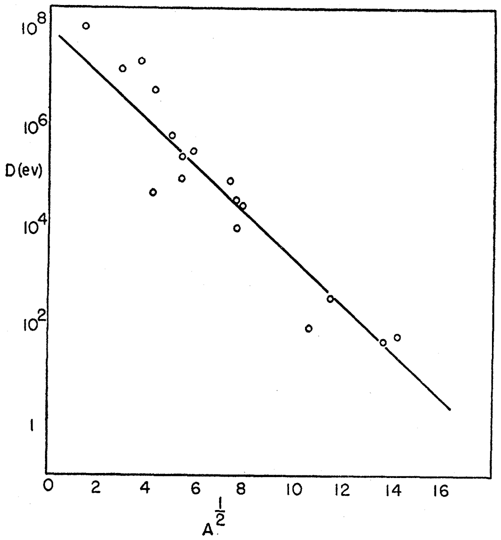

FIG. 1. Effective level spacing near $E_n=0$, regardless of J value.

pound nucleus in Fig. 1, where the straight line is drawn to fit the points without regard for the light nuclei. It must be emphasized that the curve drawn has at most order of magnitude validity; it is uncorrected for variation in J, in K, in even-odd character of A, in binding energy. Moreover, D_r is defined from the derivative $[\partial f/\partial E]_{E_r}$ and hence resembles only in order of magnitude the actual spacing between levels. The curve also yields an order of magnitude estimate for γ_r^2 and hence Γ_n for low energy neutron reactions. It provides an additional motive for associating s-neutrons with the 60-kev level in Na rather than the 3-kev level and suggests that one of the higher levels in Ag may make a major contribution to the scattering length, since this point is exceptionally far from the curve.

The form of the curve is suggested by the statistical expression[30] for the level density

$$\omega=1/D=C\exp\{(bAE)^{\frac{1}{2}}\} \tag{9}$$

where E is the excitation energy of the nucleus. Assuming that here the average $E\approx 7$ Mev, the curve of Fig. 1 corresponds to $C=0.01$ Mev^{-1}, $b=0.1$ Mev^{-1}. The inclusion of levels with all possible J values would increase C by an order of magnitude but would not be expected to alter b, which indicates a somewhat smaller level density in statistical nuclear reactions than the usual value[30] of $b'=0.2$ Mev^{-1}. Despite order of magnitude uncertainties in Fig. 1, it is unquestionable that $b<b'$. Since the excitation energy at which b' is determined generally exceeds that for b by some 5–10 Mev, this difference apparently represents a slight deviation from Eq. (8) in the sense of a relative increase of level spacing for lower excitation energies.

IV. UNOBSERVED LEVELS

Figure 1 permits an estimate of the positions of unobserved levels whose presence is indicated by the scattering length. These order of magnitude values for E_r (not E_b) are presented in Table I, opposite the compound nucleus involved.

Recent velocity spectrometer studies[31] of Fe and Ni show resonances at 0.95 and 5 kev, which can be associated with Fe^{58} and Ni^{61} according to Table I. The resonance resulting from Ni^{63} will be much weaker than that resulting from Ni^{61} on account of abundance and may not be apparent in the presence of Co and Mn impurities with strong resonances in this region. This example suggests that measurement of scattering length

TABLE I. Order of magnitude E_r for levels near $E_n=0$.

	Virtual			Bound	
K^{40}	14 kev		Na^{24}	-60 kev	$(J=1)$
A^{41}	20	$(J=\frac{1}{2})$	Fe^{57}	-3	$(J=\frac{1}{2})$
Ca^{45}	8	$(J=\frac{1}{2})$	Ni^{59}	-0.6	$(J=\frac{1}{2})$
Fe^{55}	7	$(J=\frac{1}{2})$	Co^{60}	-3	$(J=3)$
Fe^{58}	1		As^{76}	-0.3	
Ni^{61}	2	$(J=\frac{1}{2})$	Ag^{110}	-20 ev	
Ni^{63}	0.3	$(J=\frac{1}{2})$	Cs^{134}	-12	
As^{76}	0.3		Ta^{182}	-9	
Cs^{134}	5 ev		Au^{198}	-8	$(J=2)$

for separated isotopes may be useful in combination with velocity spectrometer studies, where measurement of separated isotopes is precluded by the large samples required.

For certain isotopes the scattering length does not indicate the presence of any nearby levels for s-neutrons, namely for target nuclei He^4, Be^9, F^{19}, Ca^{40}, Cb^{93}, Bi^{209}. Isotopic mixtures of elements that are certain to include at least one nearby level are those for which $\sigma_{sc}\gg 4\pi a^2$ or a is much different from R. Such elements include Cl, Ti, Cr, Se, Rb, Sr, Zr, Mo, W. A few elements that are practically pure isotopes have unknown scattering lengths: Ne, Si, P, Sc, Y, Rh, In, La.

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