Meson Tail Effect and Treatment of Proton-Proton Scattering Data*†

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The fit of experimental material on proton-proton scattering by phenomenological potentials is reviewed with the inclusion of observations by Heydenburg-Little in the 300-kev region and of Cork at 30 Mev. Weighting of data by a criterion based on internal consistency is employed. Data obtained by means of Van de Graaff machines are used to determine the best fits for different shapes of potential energy curves. Other data are compared with the resultant (f, E) plots graphically. A trend towards agreement with the long-tailed Yukawa and exponential well potentials is noticeable among the supposedly more accurate measurements of Rouvina and of Cork. The meson potential $-Ce^{-r/a}/(r/a)$ with $C=(93.4\pm1.0)mc^2$, $a=(0.412\pm0.002)e^2/mc^2$ fits experiment and corresponds to a meson mass of $333\pm2m$, the limits of error being stated above somewhat arbitrarily.

The effect of the tail of the Yukawa potential on scattering is studied with particular attention to its influence on the nearly linear functions of energy used in the analysis of experiment, and on the coefficients of powers of the energy in the expansion of this function (f).

The convergence of the power series is better if the tail of the potential is chopped off. Calculation of coefficients in this series is also easier for the potential without tail. The more practical arrangements of calculations are: (a) direct calculation of f for a number of energies, the tail effect being treated by a first-order approximation; and (b) calculation of coefficients of powers of E in the series for f for potential without tail followed by first-order calculation of effect of tail on f. Results obtained in these ways are compared.

I. INTRODUCTION

SINCE the appearance of the comprehensive review of experimental and theoretical aspects of proton-proton scattering by Jackson and Blatt[1] there appeared some additional measurements by Rouvina,[2] Bruce Cork,[3] Panofsky and Fillmore,[4] Faris and Wright,[5] Zimmerman and Kreuger,[6] and by Mather.[7] Some preliminary measurements of Heydenburg and Little[8] also have become available. It appeared desirable, therefore, to compare the enlarged experimental material with expectation for different phenomenological potentials. It will be seen that there is now a slight trend towards agreement with the Yukawa or exponential type of potential. This circumstance may perhaps be significant since it has been found possible to reconcile the simple hypothesis of the symmetric Hamiltonian with neutron scattering experiments through the employment of long-tailed potentials[1] and since the exactness of the equality of n-p and p-p forces is even improved by making relativistic and magnetic effects corrections as has been found by Schwinger.[9]

In view of the usual difficulty of assigning relative weights to different sets of experimental data on a rational basis the experiment has been tried of employing general agreement with the majority of the better measurements as a criterion for establishing the relative weights. This is made possible by the employment of the f function of Breit, Condon, and Present,[10] which varies approximately linearly with energy, a circumstance successfully used by Schwinger,[11] Blatt,[12] Bethe,[13] Chew and Goldberger,[14] and others in analysis of scattering experiments. In order not to prejudice the determination of the range parameter, the criterion used disregards the manner in which the centroid of the f values falls on the f, E curve but pays attention to the way in which the slope of f, E curve is reproduced by the data of one set of observers. This criterion is manifestly unjust to data which contain an error varying systematically with energy and is present only at some energies. Such an error need not affect the position of the centroid very strongly. This is the case for the observations of Heydenburg, Hafstad, and Tuve,[15] which show other evidence of being quite accurate at their highest energy. Since a great many points are now available, such an occasional unfairness has probably an insignificant effect on the final result. The values of Heydenburg, Hafstad, and Tuve[16] obtained in the vicinity of the scattering minimum and sometimes[17] shown in surveys are not included in the analysis since the object of this work was to ascertain the existence of the minimum rather than to provide accurate phase shift values. Other questions

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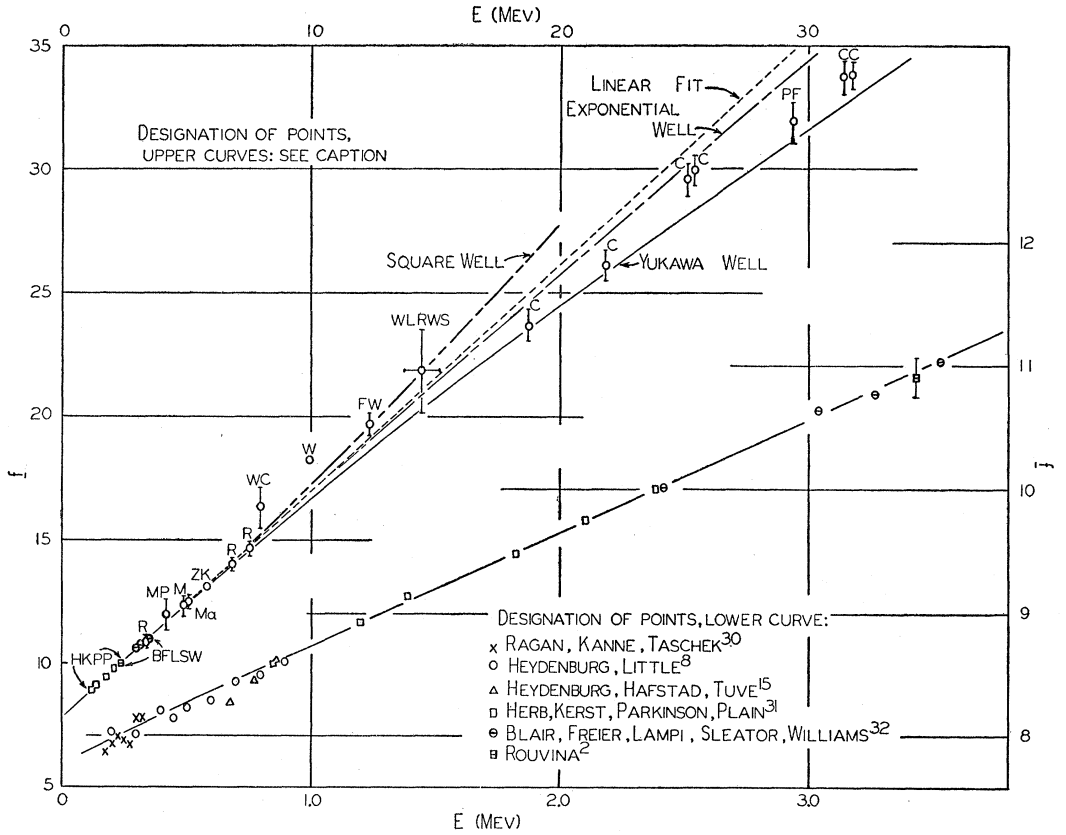

FIG. 1. *Lower curve:* Plot of f *vs* E for best fits with experimental points as designated. Use scales on right and lower margins. *Upper curves:* Plots of f *vs* E for best fits as labeled with experimental points having the following designations: HKPP—Herb, Kerst, Parkinson, and Plain (reference 31); BFLSW—Blair, Freier, Lampi, Sleator, and Williams (reference 32); R—Rouvina (reference 2); MP—May and Powell (reference 29); M—Meagher (reference 36); Ma—Mather (reference 7); ZK—Zimmerman and Kreuger (reference 6); WC—Wilson and Creutz (reference 37); W—Wilson (reference 38); FW—Faris and Wright (reference 5); WLRWS—Wilson, Lofgren, Richardson, Wright, and Shankland (reference 39); C—Cork (reference 3); PF—Panofsky and Fillmore (reference 4). Use scales on left and upper margins.

concerned with the use of data are discussed in a more detailed account contained in the body of the paper.

The differences caused by fitting data in different ways are examined as follows with special reference to the distinction between linear fits to the f, E curve and fits employing the deviations from linearity which are expected for an assumed shape of the potential energy well. These have been previously mentioned in a more limited way by Breit and Hatcher[18] in relation to the quadratic terms but are more fully covered here with a consideration of the effect of cubic and quartic terms. In addition to comparison of power series and directly computed f values mentioned below, an analytic discussion of the effects of linear fits shows that if the curvature of the (f, E) curve is produced by a quadratic term, a linear visual fit will give the same initial slope as a linear least squares fit, but will fail to reproduce the initial value $(f, E=0)$ by an amount proportional to the square of the maximum energy used in the fit. If a quadratic term is accounted for properly, but a cubic addition is fitted linearly, the addition to the slope is nearly determined, but an error in the initial value proportional to the cube of the maximum energy is left out of account.

In order not to be completely dependent on least squares analyses of the data and on semi-empirical formulas relating the potential parameters to the results of such an analysis, a further study was undertaken wherein theoretical f, E plots for various sets of parameters were compared directly with the experimental data and with an f, E plot resulting from a least squares analysis of the data. In such plots a change in the range parameter appears approximately as a rotation of the f, E plot, while a change in the depth parameter appears as change in the value of f by nearly the same amount at each energy. The comparison, therefore, allows a determination of the changes in the parameters needed to rotate and translate one of the theoretical curves into the one derived from experiment. The parameters obtained from both methods of analysis differ by much less than the estimated errors.

[18] G. Breit and R. D. Hatcher, Phys. Rev. **78**, 110 (1950), referred to in the text as BH.

TABLE I. Values of the $f^{(i)}$ in their different interpretations for the Yukawa potential with $a'=0.40$, $C'=99$.

Line	$f^{(0)}$	$f^{(1)}$	$f^{(2)}$	$f^{(3)}$	$f^{(4)}$	Tail presence	How obtained
A	7.9917	0.905	−0.005	0	0	No	HAB representations.[a]
B	7.9900	0.910	−0.006	0.00005	0	No	Least squares fit to f in Table I of HAB.[b]
C	...	0.917	−0.0063	0.000053	0	No	Least squares fit to $\partial f/\partial E$.[c]
D	7.9684	0.9295	−0.0090	0.00026	-5.3×10^{-6}	Yes	Graphical tail corrections applied to preceding entry.[d]
E	7.9900	0.9153	−0.00659	0.000051	5.4×10^{-7}	No	Integrals involving u_i.[e]
F	7.9684	0.9269	−0.00890	0.000270	-5.3×10^{-6}	Yes	Integrals involving u_i.[f]

[a] Direct substitution of $a'=0.40$, $C'=99$ into Eqs. (3) to (3.8) of HAB.
[b] Least squares fit to values of f for different values of maximum energy from 9 to 40 Mev. Value of $f^{(3)}$ proved necessary to insure energy independence of other $f^{(i)}$. Value of $f^{(0)}$ obtained agrees with that from Eq. (8.6) of BB.
[c] Calculation of $\partial f/\partial E$ by the BH extension of the Schwinger formula to arbitrary E; least squares fits at $E=0$, 1, 2.6, 6, 8, 10, 12, 14, and 16 Mev.
[d] Graphical determination of $\delta(\partial f/\partial E)$ from Fig. 2; power series fits to these values.
[e] Jackson-Blatt equations involving energy derivatives of radial functions applied to potential without tail; the value of $f^{(4)}$ was required to represent directly computed f's of HAB up to 32 Mev. It is probably not accurate since it may be including effects of $f^{(5)}$.
[f] Obtained from values given in immediately preceding entry with inclusion of tail effect corrections calculated by means of Eq. (2) employing energy differentiation of asymptotic Bessel function expansions (see reference 26) at $E=0$. A least squares fit of f up to $E=22$ Mev determined $f^{(4)}$ which is probably not accurate and may include effects of $f^{(5)}$.

The section on the treatment of data is preceded by a consideration of the effect of including the tail of the Yukawa potential at distances greater than $3e^2/mc^2$. This tail effect is found to be represented poorly by a power series in the energy E, and it has a marked influence on the coefficients $f^{(i)}$ in the power series in E which represents f. The difference between representing f by a polynomial with a small number of terms and by a power series is studied and is found to be appreciable. The power series representation of f at 32 Mev is found to require terms in E^n with $n=4$ as is seen by inspection of Table I and Fig. 1. The power series representation of the values of f without tail converges more rapidly, however, and could be used in conjunction with a directly computed tail effect obtained as in Sec. II.

The notation is the same as in Hatcher, Arfken, and Breit[19] with the following additions or changes:

r_e = interparticle separation in units e^2/mc^2.
$\mathbf{K}=\frac{1}{2}f-0.15443$ is the function introduced by Jackson and Blatt[1] in their analysis of experimental data.
$u^s=C_0\mathfrak{F}/\sin K_0$, where $\mathfrak{F}$ is the Coulomb function $F\cos K_0+G\sin K_0$.
$y=1/(C'a'^2)^{\frac{1}{3}}$.
E = energy in Mev unless otherwise stated.
δK_0 = change in K_0 caused by an addition to the potential. In the present case, the addition is taken to be the tail of the Yukawa potential from $r=3e^2/mc^2$ to $6e^2/mc^2$.

II. EFFECT OF THE TAIL OF THE YUKAWA POTENTIAL ON PHASE SHIFT

In the work reported on which follows, use is made of the fact that the effect of the tail extended from $r_e=3$ to infinity is given with an accuracy of at least 0.0003 degrees by the first-order perturbation formula

$$\delta K_0=\int_{r_e=3}^{6}(\delta V/E')\mathfrak{F}^2 d\rho, \qquad (1)$$

where δK_0 = change in phase shift as a result of the extension of the Yukawa potential beyond $3e^2/mc^2$ (δK_0 is to be added to K_0 of Table II of HAB to give the effect of the Yukawa potential extended to infinity), $\delta V=Ce^{-r/a}/(r/a)$ = Yukawa potential, E' = energy of relative motion, and $\mathfrak{F}$ = Coulomb function with phase shift K_0. The error is usually much less than 0.0003°. Since the tail effect is calculated here for use with the phase shift table of HAB[19] which has an accuracy of 0.01°, this error is negligible. Inaccuracies in $\mathfrak{F}$, the nature of which is explained in detail later, may give at most an additional error of 0.0002°.

The smallness of the effects of these errors in applications may be judged by observing that a one percent error in scattering at an observation angle of 45° usually introduces errors in the phase shift having absolute values between 0.01° and 0.4°. The dependence of the errors in K_0 and f on energy caused by such an error in scattering is seen in Table II. This table will be found useful for comparison with values of the tail effect arrived at later in the article.

The inherent errors introduced by the use of Eq. (1) are to be assigned to two main causes. First there is the neglect of the tail beyond $6e^2/mc^2$. The maximum effect of cutting off the Yukawa potential at $r_e=6$ was estimated by evaluating the integral of Eq. (1) from $r_e=6$ to infinity with $\mathfrak{F}=1$. Since $|\mathfrak{F}|\leqq 1$, this provides an upper limit on the error which is found to be less than $8.5\times10^{-5}/\sqrt{E}$ degrees with E in Mev. For all energies used this is less than 0.0001 degree. The inaccuracy introduced by using the square of the Coulomb function instead of the product of the Coulomb and Yukawa functions was investigated. For the higher energies, an analytical argument shows this to be less than $0.22/\sqrt{E}$ percent of δK_0, that is less than about 2×10^{-5} degree. At the low energies, numerical estimates indicate an upper limit to this type of error of about 0.0002 degree.

In order to calculate δK_0 from Eq. (1) it is necessary to obtain the wave function $\mathfrak{F}=F\cos K_0+G\sin K_0$. The phase shifts used were taken directly from Table II of HAB. Calculation of the regular and irregular

[19] Hatcher, Arfken, and Breit, Phys. Rev. **75**, 1389 (1949), referred to in the text as HAB.

Coulomb functions, F and G, proceeded in several different ways.

For energies below 2 Mev, Eqs. (13) through (21) of Yost, Wheeler, and Breit[20] were used. In this region inaccuracies in F and G present in the calculations introduced errors of less than 3×10^{-5} degree in δK_0.

For energies above and including 9 Mev, $\mathfrak{F}$ was approximated by a sine function, $A\sin[Q(b)r_e+\delta]$ where Q is defined by the Coulomb equation, $d^2\mathfrak{F}/dr_e^2+Q^2(r_e)\mathfrak{F}=0$, and b is some suitably chosen value of r_e slightly greater than 3. A and δ were found by fitting the sine function in slope and value at $r_e=3$ to $\tilde{a}\sin(\varphi+K_0)$; $\tilde{a}$ and φ at $r_e=3$ were obtained from previous unpublished work at this laboratory by Hatcher who used Eqs. (4(b)), (9), and (10) of Wheeler.[21] Estimates of the error introduced by this approximation for $E>9$ Mev were made by integrating

$$\int_{r_e=3}^{6}[e^{-r/a}/(r/a)][\tilde{a}^2\sin^2(\varphi+K_0)-A^2\sin^2(Q(b)r_e+\delta)]d\rho,$$

taking into account first-order effects in K_0. The resultant upper limit for the error is $\sim3\times10^{-3}/E_{\text{Mev}}^{\frac{3}{2}}$ which is <0.0002 degree. Numerical checks on the sine approximation were made in a number of cases by comparison with accurate values of $\mathfrak{F}$. The agreement was always better than 0.0001 degree in δK_0.

For 2 Mev$<E<$9 Mev a sine function of the form $A\sin[Qr_e+\delta]$ was joined to $\mathfrak{F}$ at $r_e=3$. F and G and their derivatives at $r_e=3$ were obtained from Table I in Thaxton and Hoisington.[22] At larger values of r_e, the difference between the sine function and the desired Coulomb function was calculated by means of a Taylor's series in r_e-3, the successive derivatives of which were obtained from the Coulomb equation. The series converges rapidly requiring only two or three terms. The error in $\mathfrak{F}$ where it mattered was not more than 0.001 and the error in δK_0 was less than 0.0002 degree.

TABLE II. Errors in K_0 in degrees and f produced by a one percent error in scattering at 45° for $C=84.422mc^2$ and $a=0.4323e^2/mc^2$.

E	ΔK^0	Δf
0.25	−0.05	0.03
0.50	0.03	−0.01
0.75	0.08	−0.02
1	0.13	−0.03
2	0.23	−0.05
4	0.33	−0.09
10	0.37	−0.16
20	0.35	−0.24
30	0.33	−0.30
40	0.32	−0.35

[20] Yost, Wheeler, and Breit, Phys. Rev. **49**, 174 (1936).
[21] J. A. Wheeler, Phys. Rev. **52**, 1123 (1937).
[22] H. M. Thaxton and L. E. Hoisington, Phys. Rev. **56**, 1194 (1939), referred to in the text as TH.

TABLE III. Values of tail correction δK_0 in hundredths of degrees for meson potential. These should be added to the K_0 in Table I of HAB to allow for the effect of the potential beyond $r=3e^2/mc^2$.

E (Mev) \ a' / C'	0.40 / 102	0.40 / 99	0.40 / 96	0.42 / 89.648	0.45 / 84	0.45 / 81	0.45 / 78
0.8	5.8	5.3	4.8	7.5	14.9	13.6	12.2
1.4	4.3	4.3	4.2	6.1	8.7	9.4	9.6
3	0.8	1.0	1.2	1.5	2.0	2.8	3.7
5	0.1	0.2	0.1	0.2	0.5	0.3	0.3
6	0.3	0.2	0.2	0.3	0.9	0.7	0.5
7	0.5	0.4	0.3	0.6	1.5	1.2	0.9
9	0.9	0.8	0.7	1.2	2.4	2.1	1.9
16	1.0	0.9	1.0	1.4	2.3	2.3	2.2
22	0.4	0.4	0.5	···	0.9	1.0	1.0
28	0.2	0.2	0.2	0.3	0.4	0.4	0.4
32	0.2	0.2	0.2	0.3	0.5	0.5	0.4
40	0.4	0.4	0.4	0.6	1.0	1.0	0.9

Values of δK_0 for Yukawa parameters close to the best fit to experiment are listed in Table III.

Values of δf were obtained as first-order effects of δK_0 from the definition of f and are shown in Table IV.

At zero energy, it was convenient to use

$$\delta f=-2\int_3^6(\delta V/mc^2)(u^s)^2dr_e, \tag{2}$$

which follows from Eq. (1) and the relation between δf and δK_0, where at any energy

$$u^s=C_0\mathfrak{F}/\sin K_0, \tag{2.1}$$

an expression, the limit of which for $E=0$ is according to Breit and Hatcher,[18]

$$u^s=-x[K_1(x)+(\gamma-\tfrac{1}{2})I_1(x)]+(x/4)I_1(x)f, \tag{2.2}$$

where

$$x^2=8r/a_B. \tag{2.21}$$

By differentiation of Eq. (2) the energy derivatives of δf at $E=0$ can be obtained in terms of energy derivatives of u^s at zero energy which can be expressed in terms of Bessel functions. The values of δf and $\partial(\delta f)/\partial E$ at $E=0$ for various meson parameters are included in Tables IV and V. The values of δf given in Table IV are to be added to the values of f for the meson potential extended to $r_e=3$ in order to give the effect for a meson potential extending to infinity.[23]

The uncertainty in δf is roughly constant over the complete energy range and is probably not greater than about 0.0003 in most cases. At zero energy estimates indicate that the uncertainty in δf is probably not greater than 0.0005.

Tail corrections for the Yukawa parameters C'

[23] A knowledge of the phase shifts enables one to calculate f from Eq. (7.6) of BCP for various energies. For different Yukawa parameters one can calculate f up to 10 Mev to within about 0.01 from the representations given in Eqs. (3) through (3.8) of HAB. An uncertainty of 0.01 in f, for almost all energies, corresponds to an uncertainty in scattering at 45° of less than 1 percent as indicated in Table II.

TABLE IV. Values of δf caused by tail of meson potential. These should be added to f to allow for the effect of the potential beyond $r=3e^2/mc^2$.

a'	0.40	0.40	0.40	0.42	0.45	0.45	0.45
E (Mev) \ C'	102	99	96	89.648	84	81	78
0	−0.0179	−0.0216	−0.0257	−0.0299	−0.0287	−0.0366	−0.0455
0.8	−0.0110	−0.0134	−0.0166	−0.0188	−0.0169	−0.0223	−0.0286
1.4	−0.0073	−0.0093	−0.0115	−0.0128	−0.0107	−0.0147	−0.0194
3	−0.0016	−0.0024	−0.0032	−0.0035	−0.0034	−0.0057	−0.0086
5	−0.0004	−0.0006	−0.0005	−0.0005	−0.0011	−0.0009	−0.0009
6	−0.0008	−0.0006	−0.0005	−0.0009	−0.0025	−0.0021	−0.0016
7	−0.0016	−0.0014	−0.0012	−0.0020	−0.0045	−0.0039	−0.0033
9	−0.0034	−0.0033	−0.0031	−0.0047	−0.0085	−0.0080	−0.0076
16	−0.0055	−0.0058	−0.0061	−0.0083	−0.0119	−0.0126	−0.0134
22	−0.0029	−0.0032	−0.0035	⋯	−0.0062	−0.0069	−0.0077
28	−0.0014	−0.0014	−0.0015	−0.0022	−0.0034	−0.0035	−0.0038
32	−0.0018	−0.0017	−0.0017	−0.0026	−0.0046	−0.0045	−0.0044
40	−0.0045	−0.0044	−0.0044	−0.0064	−0.0107	−0.0106	−0.0105

$=89.648$, $a'=0.42$, can be used to determine the tail corrections for other sets of slightly different parameters within ± 0.0007 in f. The results can be summarized as follows. For $0<E\leqq 9$ Mev,

$$(\delta f-\delta f^{(0)})/(\delta f_0-\delta f_0{}^{(0)}) = 1+15(a'-0.42)[1+3.7(a'-0.42)] +0.248[1+12(a'-0.42)] \times[15.81-C'a'^2][1-0.0855E], \quad (3)$$

$$\delta f^{(0)}/\delta f_0{}^{(0)}=1+14.6(a'-0.42)[1+4.6(a'-0.42)] +0.344[1+10.5(a'-0.42)][15.81-C'a'^2]. \quad (3.1)$$

For $E\geqslant 9$ Mev,

$$\delta f/\delta f_0=1+18.3(a'-0.42)[1+5.7(a'-0.42)] +0.150[1+16.7(a'-0.42)][15.81-C'a'^2] \times\sin[0.0331E^{\frac{1}{2}}(E-12)]. \quad (3.2)$$

Here E means the energy in the laboratory system in Mev, $\delta f^{(0)}$ is the first term in a power series in energy for δf, and the subscript 0 means that δf and $\delta f^{(0)}$ for $a'=0.42$, $C'=89.648$ have been taken. Values of δf_0 and $\delta f_0{}^{(0)}$ are obtainable from Table IV, fourth column, with $\delta f_0{}^{(0)}=\delta f_0$ at $E=0$.

It may be pointed out that Eq. (3) can be used to obtain $\delta f^{(1)}$, since $[\partial(\delta f)/\partial E]_{E=0}=\delta f^{(1)}$, and $[\partial(\delta f_0)/\partial E]_{E=0}=\delta f_0{}^{(1)}$ is 0.0161 from Table V.

III. COEFFICIENTS OF E IN THE POWER SERIES OF f

The quantities $f^{(0)}, f^{(1)}, \cdots$ are defined as coefficients of powers of E in the power series representing f. In the work of HAB the primary object was to represent f as a function of the energy with an accuracy sufficient for preliminary comparison with experiment. Simplicity of representation was therefore put ahead of accuracy in the values of the $f^{(i)}$. The procedure followed by HAB was that of fitting directly computed values of f by a quadratic

$$f^{(0)}+f^{(1)}E+f^{(2)}E^2,$$

for values of E from 0.2 to 10 Mev, and while the fact that the values of $f^{(0)}$, $f^{(1)}$, and $f^{(2)}$ will be affected by the presence of $f^{(i)}$ with $i>2$ has been realized and even brought out in HAB's paper, no explicit account of them has been taken. It is clear that the effect of the omitted $f^{(i)}$ is a function of the energy range within which the fits of f are made. In the work of Jackson and Blatt,[1] on the other hand, the function f is represented by a power series in E, and the coefficients $f^{(i)}$ are used in their strict mathematical sense. The agreement between the two sets of the $f^{(i)}$ is, therefore, not as good as that between the values of f. The situation is somewhat similar to that encountered in the analysis of experimental material by means of the f function. The best that can be done without assuming a shape of the potential well is presumably to fit the experimental values of f by a polynomial using least squares with appropriate weights. The values of the $f^{(i)}$ arrived at are then functions of the assumed number of significant $f^{(i)}$. The values could be appreciably in error if the f curve contains some large enough $f^{(i)}$ with large enough i. The procedure of HAB is sufficiently close to what has to be done in treating data with respect to effects of neglected $f^{(i)}$ to make it at least comparable in usefulness with that of a mathematically rigorous representation. It sacrificed some numerical accuracy in order to cover a large range of parameters, and a direct comparison with Jackson and Blatt's work would be difficult for any one who has not carried out similar calculations. It appeared desirable, therefore, to take into account the effect of the tail of the potential energy curve on the $f^{(i)}$ both in order to make the comparison more immediate and in order to ascertain the effect of the inclusion of the tail on the $f^{(i)}$. It will be seen that the effect of the tail is not negligible and that the

TABLE V. Values of $\partial(\delta f)/\partial E$ at $E=0$. E is in Mev.

$a'=$	0.40	0.40	0.40	0.42	0.45	0.45	0.45
$C'=$	102	99	96	89.648	84	81	78
$\partial(\delta f)/\partial E=$	0.0102	0.0116	0.0133	0.0161	0.0177	0.0211	0.0248

inclusion of the higher i is especially important in representing its effect. This is especially clear by inspection of Fig. 2 which shows the tail effect δf plotted against E. An expansion of this quantity in powers of E is obviously not the easiest way of representing it. The values of the $f^{(i)}$ corresponding to different conventions and purposes of employment are summarized in Table I. The fundamental constants enter the calculations for Table I in the combinations $(M/m)(e^2/\hbar c)^2 = 0.097714$ and $(mc^2/\text{Mev}) = 0.5114$. The numerical values used here are consistent with those in Breit, Thaxton, and Eisenbud,[24] Breit and Bouricius,[25] HAB, BH. The last line of Table I is in good agreement with the values of Jackson and Blatt.[1] The first four coefficients obtained from their work and adjusted with their formulas to correspond to a Yukawa well with $a' = 0.40$, $C' = 99$ and $f^{(0)} = 7.9666$, $f^{(1)} = 0.9268$, $f^{(2)} = -0.0089$, and $f^{(3)} = 0.00026$.

Table I shows that the same set of values of f can be represented in a number of ways by polynomials and that the rather tempting interpretation of data by means of the almost rectilinear plots of f against energy has its limitations. This fact has been brought out by BH in connection with the influence of quadratic terms on straight line fits to data. It is seen in Table I that cubic and quartic terms also have an appreciable influence. While the least squares fitting was made to values of f computed for a definite model rather than to values obtained from experiment, the indefiniteness in conclusions could not have been made worse by doing so. One may conclude that the mere fact that a set of experimental values of f shows no recognizable curvature in the (f, E) plot does not justify one in fitting it by a straight line if it is desired to obtain the potential well parameters.

The evaluation of f can be either made directly for a number of energies as in the work of HAB or else by the calculation of the $f^{(i)}$ as in the work of Jackson and Blatt. For a small range of energies in the immediate vicinity of $E=0$, the latter method is more efficient if one wishes to avoid the use of Coulomb functions and prefers to employ tabulations of Bessel functions (I, K) instead. For wider ranges of E the direct calculation of f or K_0 is definitely the easier approach because of the severe cancellation in the integral expressions for $f^{(i)}$ with higher i. When the meson wave function, expanded as a power series in the energy to yield coefficients which are functions of r analogous to the Bessel function coefficients of the Coulomb function,[26] is joined at $r_e = 3e^2/mc^2$ to the Coulomb function, the limits of integration for the computation of the $f^{(i)}$ are 0 and 3 on r_e, and the two terms in the integrand which are subtracted are closely equal in value over the latter part of the range of integration. The cancellation is

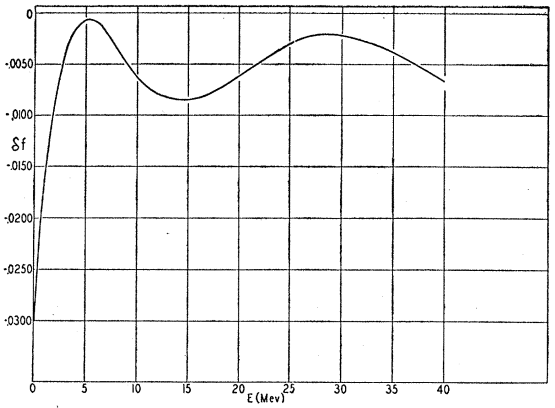

FIG. 2. A typical plot of δf vs E for the parameters $C' = 89.648mc^2$, $a' = 0.42e^2/mc^2$. Similar curves result for the other sets of parameters appearing in Table IV.

minimized, therefore, by reducing the range of integration. It appears that integration over 60 percent of the range reproduces 99 percent of $f^{(1)}$ while cancellation leaves 20 percent of $\int (u_0^s)^2 dr_e$. Integration over 85 percent of the range is necessary to give 98 percent of $f^{(2)}$ and only 2 percent of $\int u_0^s u_1^s dr_e$ is left, while in order to obtain 96 percent of $f^{(3)}$ the integration must be carried over 94 percent of the range and 0.06 percent of $\int (u_1^s)^2 dr_e$ is left. Here $u_i^s = (\partial^i u^s/\partial E^i)_{E=0}$, where u^s is the Coulomb function. This may be viewed differently by saying that if one wishes only 1 percent error in $f^{(1)}$, the integrals must be known to 0.2 percent, a 2 percent error in $f^{(2)}$ requires knowledge of the integrals to 0.04 percent, and a 4 percent error in $f^{(3)}$ is obtained if one knows the integrals to 0.0025 percent. Such numerical accuracy requirements would be too severe and could make the calculation of the $f^{(i)}$ very impractical if it were not for the fact that when the two integrals representing each $f^{(i)}$ are combined into one, the resultant integrand is a relatively smooth function of r. Numerical quadrature performed on this integrand, which changes sign within the range of integration, has not shown any special difficulties regarding the necessity of using high order quadrature formulas. Nevertheless one may have a feeling of caution regarding numerical work with this integrand, since it arises as a difference of two much larger quantities which are sufficiently complicated mathematically to make conclusions regarding the smoothness of the difference rather difficult. It may be of interest to note that computation of f by the power series method throws all of the weight on the accurate calculation of the $f^{(i)}$ while direct calculation of f distributes the accuracy requirements. In employing the power series greater care has to be taken concerning accuracy because for the larger E the errors become magnified. Direct computation has the advantage of providing a number of checks such as regularity of differences against energy and integral relations enabling one to expand logarithmic derivatives and related quantities in powers of energy differences or other

[24] Breit, Thaxton, and Eisenbud, Phys. Rev. **55**, 1018 (1939), referred to in the text as BTE.
[25] G. Breit and W. G. Bouricius, Phys. Rev. **75**, 1029 (1949), referred to in the text as BB.
[26] G. Breit and M. H. Hull, Jr., Phys. Rev. **80**, 392 (1950).

parameter value differences. Such checks have been described by BTE and related ones by HAB. The question of the number of coefficients needed for a given energy range does not enter in the method of direct calculation while it cannot be avoided if one uses a power series in E.

The direct method of calculating f is not entirely free of stringent accuracy requirements, however. In this case Eqs. (7.1) and (7.6) of BCP are used together with quantities tabulated in the appendix of BTE, in TH, or obtained some other way for the Coulomb functions, and logarithmic derivatives of the meson functions obtained by a numerical integration of the suitable Schroedinger equation or an equivalent method. The tabulated Coulomb functions are given to four decimals in the places mentioned, and an assumed uncertainty 5×10^{-5} in each quantity at, say, $E=2.6$ Mev, introduces an uncertainty in f of 0.003. If there is an error in one of the quantities, for example $\Phi_0\Theta_0$, of 1.5×10^{-4}, there is an error in f of roughly the same amount. While these uncertainties are small, they entered within the accuracy of the graphs discussed later in this paper. There is also a further mitigating circumstance in the use of the integral formulas in the fact that the tail effect contributes the major part of the coefficients $f^{(i)}$ for $i>2$, and this part may be computed directly with the aid of the first-order formula, Eq. (2), where the wave function is suitably differentiated with respect to E under the integral. Thus for the entry in Table I, line E, the error in $f^{(3)}$ without tail is estimated at ±10 percent, while in line F, the error in $f^{(3)}$ with tail is only about 3 percent, since the tail effect is known to about 0.5 percent and is four-fifths of the value. The values of $f^{(1)}$ and $f^{(2)}$ with or without tail are known to ±0.03 percent and ±0.5 percent, respectively.

It has, therefore, been found helpful in the present work to use both direct calculations of f and energy expansions with coefficients obtained by integral formulas in order that independent checking could be effected.

In deriving the values of meson well parameters from experiment, it is convenient to have available the approximate equations

$$\begin{aligned}\Delta a' &= -0.0138\Delta f^{(0)}+0.411\Delta f^{(1)}\\ \Delta[(C')^{-\frac{1}{2}}/a'] &= 0.00309\Delta f^{(0)}+0.0166\Delta f^{(1)},\end{aligned} \tag{4}$$

which follow from the representations of the $f^{(i)}$ by HAB. These formulas relate small changes in $f^{(0)}$ and $f^{(1)}$ to corresponding changes in C' and a' on the supposition that $f^{(2)}$ is varied with $f^{(0)}$ and $f^{(1)}$ in a manner required by the meson well. The effect of $f^{(i)}$ beyond $f^{(2)}$ is neglected and the variation of $f^{(2)}$ is obtainable from

$$\Delta f^{(2)}=0.000170\Delta f^{(0)}-0.0108\Delta f^{(1)}. \tag{5}$$

The tail of the meson potential beyond $r=3e^2/mc^2$ has the effect of increasing the range of nuclear force and of producing an increase in $f^{(1)}$ of approximately one percent. The change in phase shift, of course, is always positive. Since δf is proportional to $-\delta K_0$, δf is always negative, and in particular, $\delta f^{(0)}$ is negative. The curvature of the f-curve is increased in absolute value because of the tail by about 35 percent in a typical case.

The absolute value of the tail effect is greatest for $f^{(0)}$ and decreases with the order of the f-coefficients. However, in percentage, the tail has the greatest effect on the higher order coefficients. For example, the tail increases $f^{(3)}$ by about 500 percent and $f^{(4)}$ by about -1000 percent.

As would be expected, the tail has the greatest effect on the phase shift for low energies because of the Coulomb barrier. The oscillatory nature of δf arises because of the way in which the "wavelength" of the wave function decreases with energy, in combination with the exponentially decreasing factor in δV.

In the course of this work it has been found to be simpler to determine f curves by direct calculation of phase shifts rather than by determining the $f^{(i)}$ by means of integral formulas.[1,11,18,27] It is not meant, however, to deny the advantages of this latter method in showing qualitative features of the behavior of the $f^{(i)}$ with changes in potential well parameters especially for $f^{(1)}$. It is somewhat cumbersome, however, for the $f^{(i)}$ with higher i which are necessary to represent f at higher energies.

The large number of coefficients needed is illustrated by comparing results of using the values in Table I, line F at 20 Mev with a value of f computed from the HAB phase shift and corrected for tail effect. When $f^{(1)}$ was included, the series yielded a value too large by 2.11 or 9.5 percent, inclusion of $f^{(2)}$ led to an f which was too small by 5 percent, and the series with four coefficients was too large by 3.8 percent. The convergence is definitely faster, however, when the coefficients without tail of line E of Table I are used to compute values of f not corrected for the tail effect, as may be seen from the fact that at 20 Mev the series f is high by 9 percent for two terms, low by 2 percent for three terms, and low by 0.5 percent for four terms. With the same coefficients, the series disagrees with the tail-corrected f at 40 Mev by 20 percent and with the tail-uncorrected f by only 2 percent.

This comparison coupled with earlier discussions of cancellations encountered in integral formulas for the $f^{(i)}$ for $i\geq3$ leads one to the conclusion that the expansions in powers of E can most profitably be used for calculations not including the tail effect, followed by a direct calculation of the tail effect by means of Eq. (2) or interpolations in Table IV. Since the tail contributes most of the value of the coefficient from $f^{(3)}$ on, such a plan reduces the need for extreme accuracy in computing the $f^{(i)}$ without tail, and in

[27] G. Breit, Revs. Modern Phys. **23**, 238 (1951).

addition, the more rapid convergence of the series representing the tail-uncorrected f can be fully utilized.

The tail effect illustrates an extreme condition of variability with bombarding energy of sensitivity of scattering to changes in potential energy. This variability is clearly seen in Fig. 2. At energies of 30 Mev the tail effect is small, and it is small again at 5 Mev while in the region 0–1 Mev it has its largest values. The calculations by means of which the tail has been computed involve the same approximation as has been used by Hoisington, Share, and Breit[28] and BBH for the discussion of sensitivity of scattering to changes in potential in general. Figure 2 shows in conjunction with Table II that the tail effect is equivalent to a change in 45° scattering by roughly 0.7 percent at 1 Mev bombarding energy and is over 1 percent at 0.5 Mev. While small, the tail effect is seen to be not negligible from an experimental viewpoint.

It appears appropriate at this place to clear up a misunderstanding concerning statements in HSB[28] as interpreted by Jackson and Blatt. The word "shape" was used by HSB in two senses: (a) the one according to which changes in range and depth parameters do not alter the shape of the potential well (this convention has become universal since); (b) the one according to which the addition of a potential anywhere changes the shape. The latter convention was a natural one to use in a part of the discussion since the paper of HSB was written as an illustration of the method of calculating effects of such changes. A different shape in sense (b) is also a different shape in sense (a) in most cases, but in special cases it is the same shape in sense (a). By "exponential well" the well with specific values of depth and range parameters of Rarita and Present was meant throughout as is clear from references to the work on binding energies with these parameters in HSB and from the discussion of their comparison of the meson and exponential wells, which applies to what HSB considered to be the best fit to scattering data by the Yukawa potential with what appeared to be at the time the best fit to binding energies of the Rarita-Present exponential potential. The distinction in the two uses of the word "shape" should have been more clearly brought out in the HSB paper. At the time at which it was written the interest in the Rarita-Present well and the absence of binding energy calculations for the meson potential made elaborate statements unnecessary. The statement by HSB concerning the fact that the data then under consideration were in better agreement with some shapes of potential energy wells (in the currently accepted sense) than with others was correct and can be seen in Fig. 15 in Jackson and Blatt's paper and in Fig. 1 of the present paper. The points of HHT which could not have been disregarded suggested strongly a preference for the meson potential and the trend toward convexity of the (f, E) plot for the HKPP points when viewed from below is also clear. It should be stated that the comparison of the best 1939 meson potential fit with the Rarita-Present exponential well was correct. The superiority of the (f, E) plot over the (K_0, E) plot claimed by Jackson and Blatt is more esthetic than practical as may be seen from the agreement of "best fits" obtained by the two methods. It is probable that the safest method is a direct plot of experimental data such as Fig. 11 of BTE. It is more laborious than either of the other two but frees one considerably from the confusion caused by the customary and frequently rather meaningless assignment of probable errors to their results by experimenters. The inclusion of data at various angles and diagrammatic presentation of experimental rather than theoretical quantities makes judgment concerning relative goodness of different fits more concrete.

[28] Hoisington, Share, and Breit, Phys. Rev. **56**, 884 (1939), referred to in the text as HSB.

IV. COMPARISON OF THEORETICAL f WITH EXPERIMENTAL DATA

Heydenburg and Little[8] have recently performed proton-proton scattering experiments at low energies using a Van de Graaff generator. These data were analyzed using the tables in Breit, Thaxton, and Eisenbud[24] and were found, at each energy, to be consistent among themselves to within 0.04° assuming only S wave scattering. This would indicate a minimum uncertainty in f of between 0.01 and 0.04.

In order to obtain the phase shift K_0 as a function of energy two types of least squares fits were made to these data. In the first, K_0 was determined by a fit at each energy to all the observation angles. In the second, it was supposed that at each energy there is present an undetected systematic error which affects scattering yields by the same factor at all angles. The least squares calculation included an adjustment of this factor at each energy in such a way as to enforce best agreement with theory on the assumption of a pure S scattering anomaly. The value of the phase shift is not assumed but is derived from the fit to adjusted experimental values. This factor in all cases turns out to differ from unity by less than 2 percent, and for all but two energies it leaves values unchanged to within less than 1 percent. The two types of fits give results that are almost identical, differing generally by only a few hundreths of a degree. Since all of the present experimental evidence indicates the absence of any deviations from S scattering at the energy of these experiments which could have been detected, and since absolute values of scattering yields could conceivably have errors depending on energy, the second type of fit appeared preferable and is used in the following. The results are given in Table VI along with the corresponding values of f.

The experimental results are analyzed by means of the f function of Breit, Condon, and Present.[10] In terms of this function the subsequently introduced f

TABLE VI. S wave phase shifts according to experiments of Heydenburg and Little.

E(Mev)	K_0(degrees)	f
0.2	6.66	8.053
0.3	11.16	8.031
0.4	15.02	8.216
0.45	17.16	8.163
0.5	18.82	8.261
0.6	22.32	8.304
0.7	25.13	8.450
0.8	27.97	8.504
0.9	30.35	8.607

of Bethe, f_{Be}, is

$$f_{Be}=f+2-2\gamma=f+0.84557,$$

and the function used by Jackson and Blatt is

$$\mathbf{K}=\tfrac{1}{2}f+1-2\gamma=\tfrac{1}{2}f-0.15443.$$

Here γ is the Euler-Mascheroni constant. The original notation is adhered to on account of the simplicity of the relation between the logarithmic derivative of the wave function and f which is apparent in Eq. (7.5) of BCP.

The determination of $f^{(0)}, f^{(1)}, f^{(2)}$ proceeded from the more accurately measured phase shifts. In addition to the work of Heydenburg and Little, the same values as used by BH, with the exception of the value at 4.2 Mev by May and Powell,[29] were considered. The energies and the experimental f's used are shown in Table VII. The interpretation of the data is as mentioned in BH.

The proper weighting to be assigned to the work of a set of experimenters is uncertain. The use of the nominal probable error as a criterion may be of significance, but it does not necessarily correspond to the relative value of different sets of observations, since the reluctance of an experimenter to claim too much accuracy enters the nominal error in a strong but unknown manner. Nor does this criterion consider the self-consistency of a given set of data. It was desired to obtain weightings for the different observations which were free in some measure from arbitrariness. This was partially accomplished by employing a criterion based on the internal consistency of the data of a given set of observers. It is clearly impossible to devise a criterion which is completely adequate and fair, since the comparison of observations at different energies presupposes the knowledge of variation of f with energy, and this variation is one of the things which must be found. On the other hand the data of one group of observers cover a reasonably limited range of energy values, and the collection of all data with any reasonable assignment of relative weights to the observations determines df/dE with fair definiteness. It is possible, therefore, to reduce the data of a single group of observers (SGO) to one energy by correcting for the effect of df/dE and to test the quality of the data by the consistency with which the reduced values check each other. Specifically this plan was carried out as follows. A preliminary fit to data similar to that obtained by BH was used to determine approximate values $f^{(0)}=7.788$, $f^{(1)}=0.940$, $f^{(2)}=-0.0053$. The value of $f^{(2)}$ used here corresponds to that expected for the meson potential according to the representations of $f^{(0)}$, $f^{(1)}$, $f^{(2)}$ in terms of the depth parameter C' and range parameter a' of the meson potential. The corresponding fit in the BH paper gave the very similar values $f^{(1)}=0.939$, $f^{(2)}=-0.0057$, even though the weights used in the least squares calculation were very different. The fit was next corrected for tail effect employing 0.41 as the value of the range parameter. The inclusion of the tail effect is primarily responsible for the markedly different value of $f^{(2)}=-0.00935$. The values $f^{(0)}$ and $f^{(1)}$ were then adjusted so as to correspond to the best linear fit for $f-f^{(2)}E^2$ resulting in $f^{(0)}=7.780$, $f^{(1)}=0.954$. According to Appendix A, the three $f^{(i)}$ thus obtained should correspond to the best meson fit rather closely except for changes arising from the assignments of different weights to observations. Since these values were obtained by fitting what are believed to be all of the more accurate measurements, the values of $f^{(1)}$ and $f^{(2)}$ were considered to be good enough for making corrections for energy of data within the material of any SGO. A constant $f^{(0)}{}_l$ was then determined by subtracting $f^{(1)}E+f^{(2)}E^2$ from the observed f of a SGO and averaging the results. This $f^{(0)}{}_l$ is a local $f^{(0)}$ for the SGO, and the subscript l is meant for "local" in local energy. The mean deviation of the observed f from $f^{(0)}+f^{(1)}E+f^{(2)}E^2$ was calculated for the SGO, and a relative weight for any observation of the SGO proportional to the reciprocal of the square of the mean deviation was then used in a least squares adjustment of $f^{(0)}$, $f^{(1)}$, $f^{(2)}$ to the observations of all of the SGO included in the analysis.

It appears pertinent to mention that the criterion is not capable of penalizing the data on account of the presence of a systematic error which would affect all the f of an SGO by the same amount. The desirable feature of this circumstance is that all observations are on the same footing regarding the determination of the f curve as long as the quality of observations is the same in the sense tested by the criterion of internal consistency. An undesirable feature is the possibility of leaving systematic errors undetected and of occasionally permitting statistical accidents in which data with poor statistics receive a high rating because the sample happens to show small dispersion. The criterion does not in any sense provide a figure of merit for the data of a given group of observers from such viewpoints as accuracy within a certain percentage of error in the cross section. In principle poor data at an energy especially favorable for the determination of f can receive a relatively high rating. Other possibilities of unfairness to data are mentioned in the introduction.

[29] A. N. May and C. F. Powell, Proc. Roy. Soc. (London) **A190**, 170 (1947).

Since several groups of observers are involved, a degree of statistical compensation of errors committed in weight assignment may be expected. The weights per observation w, determined by this criterion, are Ragan, Kanne, and Taschek[30]: 0.00353; HL: 0.0233; HHT: 0.00843; Herb, Kerst, Parkinson, and Plain[31]: 0.08338; and Blair, Freier, Lampi, Sleator, and Williams[32]: 0.06438. This method of obtaining weights will be referred to as criterion $\mathcal{I}$ to indicate that it is primarily based on internal consistency.

In order not to depend on this criterion too heavily, a second criterion was used in which the deviation of the f values of a SGO from the f curve as determined by criterion $\mathcal{I}$ was also used to determine the quality of data. The obvious danger of relying on such considerations is apparent in the example of sampling a Gauss error distribution. Were one to omit the observations outside the probable error, perfectly valid measurements would be left out of account. The difference between the Gauss error distribution and the statistical sample for the determination of the f curve is, however, that the statistical equivalence of all observations of the Gauss error distribution can hardly be expected to apply to data obtained with different equipment in different energy regions. Observations which are removed from the f curve by several times the nominal probable error would have to be supposed to contain a systematic error with a large probability, and their weight would have to be considered as small. A partial account of this circumstance was taken by calculating a second set of relative weights w', proportional to the reciprocal square deviation from the f curve determined by criterion $\mathcal{I}$. The weights obtained were RKT: 0.00315; HL: 0.01981; HHT: 0.00441; HKPP: 0.09242; and BFLSW: 0.06113. The least squares fit was then repeated with relative weights $(w+w')/2$. The second criterion is referred to as the $\mathcal{I}\mathcal{E}$ criterion. The values obtained are

$$\begin{aligned} &f^{(0)}=7.7846,\quad f^{(1)}=0.9564,\quad f^{(2)}=-0.01041 \quad (\text{criterion } \mathcal{I}),\\ &f^{(0)}=7.7864,\quad f^{(1)}=0.9558,\quad f^{(2)}=-0.01044 \quad (\text{criterion } \mathcal{I}\mathcal{E}). \end{aligned} \tag{6}$$

In addition, a linear least squares fit to the data results in

$$\begin{aligned} &f^{(0)}=7.8107,\quad f^{(1)}=0.9173 \quad (\text{criterion } \mathcal{I}),\\ &f^{(0)}=7.8132,\quad f^{(1)}=0.9165 \quad (\text{criterion } \mathcal{I}\mathcal{E}). \end{aligned} \tag{7}$$

This procedure is admittedly arbitrary and has been tried in a somewhat experimental spirit. It is apparent, however, that the assigned weights cannot make too much difference in the least squares fit. Using their weighting, BH found by a linear least squares fit that

TABLE VII. Values of f used in the least squares analysis. These are in addition to those for the Heydenburg-Little data.

E(Mev)	f	Source
0.25	7.984	RKT[a]
0.30	8.174	RKT
0.670	8.279	HHT[b]
0.776	8.465	HHT
0.867	8.616	HHT
0.860	8.601	HKPP[c]
1.200	8.944	HKPP
1.390	9.134	HKPP
1.830	9.478	HKPP
2.105	9.749	HKPP
2.392	10.004	HKPP
2.42	10.022	HKPP
3.04	10.637	BFLSW[d]
3.27	10.781	BFLSW
3.53	11.029	BFLSW

[a] See reference 30. [b] See reference 15. [c] See reference 31. [d] See reference 32.

$f^{(0)}=7.82$, $f^{(1)}=0.916$. These values are very close (with $\frac{1}{2}$ percent) to those obtained similarly by Jackson and Blatt. The present linear least squares fit gives values which differ from those of BH by about 0.1 percent for $f^{(1)}$ and less for $f^{(0)}$, as shown in Eq. (7).

The importance of considering the curvature in a fit of this type has already been discussed.[33] It is, however, not a large effect, and the fit to data offered by Eqs. (6) is perhaps just significantly better than the fit as offered by Eqs. (7). For the quadratic fit, the root mean square (rms) residual[34] is 0.0332 for the criterion $\mathcal{I}$ and 0.0309 for that using the weights of criterion $\mathcal{I}\mathcal{E}$. The rms residual for any quadratic fit to these data with $f^{(2)}$ fixed, of which the linear fit is a special case, is $0.0320(1+\Delta)$, where the average quadratic fit residual is used as the basic value. The quantity Δ will always be a positive number and is in fact $\cong 390(f^{(2)}+0.01)^2$. For a linear fit, $\Delta=0.04$. The improvement in the fit to the data which is produced by employing a quadratic rather than a linear fit is thus seen to be only 4 percent as measured by the rms residual. The rms residual apparently is sufficiently large to preclude the precise determination of three coefficients. Another condition to limit the variation of $f^{(2)}$ is needed.

Strictly speaking, one must consider the problem from the viewpoint of fitting the data by a polynomial in E such as $f^{(0)}+f^{(1)}E+f^{(2)}E^2$ with $f^{(0)}$, $f^{(1)}$, $f^{(2)}$ regarded as functions of the potential well parameters such as C' and a' for the meson well. Since $f^{(0)}$, $f^{(1)}$ can be used to determine C', a', this amounts to subjecting $f^{(2)}$ to the condition of being a definite function $f^{(2)}(f^{(0)}, f^{(1)})$ of $f^{(0)}$ and $f^{(1)}$ and regarding $f^{(0)}$, $f^{(1)}$ as independent variables. The weighted sum of squares of deviations must then be minimized with respect to $f^{(0)}$ and $f^{(1)}$. It is shown in Appendix A that in a good approximation

[30] Ragan, Kanne, and Taschek, Phys. Rev. **60**, 628 (1941), referred to in the text as RKT.

[31] Herb, Kerst, Parkinson, and Plain, Phys. Rev. **55**, 998 (1939), referred to in the text as HKPP.

[32] Blair, Freier, Lampi, Sleator, and Williams, Phys. Rev. **74**, 553 (1948), referred to in the text as BFLSW.

[33] See reference 18, Eq. (7) *et seq.*, and amplified discussions in Sec. I and Appendix A of the present paper.

[34] The residual is defined as the difference between the experimentally determined value and the value obtained from the fitted curve. The root mean square or rms residual represents an "average" deviation from the curve.

such a procedure gives nearly the same answers as are obtained by the following requirements: (A) the equations obtained by minimizing the weighted sum of squares with respect to $f^{(0)}$ and $f^{(1)}$, neglecting the entrance of $f^{(0)}$, $f^{(1)}$ into $f^{(2)}$; (B) restricting the value of $f^{(2)}$ so obtained by the requirement that $f^{(2)}=f^{(2)}(f^{(0)}, f^{(1)})$. The solution of the modified problem can be obtained in practice by an iteration procedure in which step (A) is taken with an approximate $f^{(2)}$, the resultant values of $f^{(0)}$, $f^{(1)}$ are used to compute an improved $f^{(2)}$, and the cycle repeated if necessary. The first step in this procedure is essentially what has been done in the work of Jackson and Blatt[1] and of Breit and Hatcher.[18] The question arises regarding the advisability of fitting data by least squares in preference to the simpler visual adjustment of a straight line or curve. Without trying to provide a complete answer, one may note nevertheless some pitfalls in the method of visual fits. These have been partly discussed by HAB in connection with the error involved in neglecting the curvature of the (f, E) plot. More specifically one can estimate the effect of an error of judgment in estimating the curvature of this plot. If the curvature is a result of a term in x^2 with $x=E/E_{\max}$, then the least squares representation of an error x^2 by a linear function has exactly the slope of the chord between $x=0$ and the maximum x used. A replacement of the least squares fit by a chord between end points would in this case give a reliable value of $f^{(1)}$. However, the value of $f^{(0)}$ would be seriously affected because the least squares straight line is lower than the chord by $x^2/6$, and for an error resulting in the representation of $-0.0040E^2$ by a linear fit there results a false contribution to $f^{(0)}$ of amount $0.00067E_{\max}{}^2$. For $E=3$ this contribution is 0.006 which is uncomfortably close to the desired accuracy.

Similarly, data analysis taking no account of the presence of $f^{(3)}E^3$ and assigning a correct $f^{(2)}$ will reproduce the value of the slope of the chord to within 9/10 of the true value of the omitted part and will make a false contribution to $f^{(0)}$ of $-f^{(3)}E^3/5$. For $E=10$ and $f^{(3)}=0.000270$ the error in slope is equivalent to 0.27 percent effect on range, while for $E=3$ this error is negligible. For $E=10$ the effect on $f^{(0)}$ is an objectionably large 0.05 while for $E=3.5$ it is a tolerable but nevertheless undesirable 0.002. All of the numbers quoted in connection with the cubic error are for uniform weighting of the observations. It is presumably difficult to make visual straight line fits with allowance for curvature to a degree much better than establishing the chord between end points followed by an approximate and roughly parallel displacement. The resultant errors may be appreciably greater than those arrived at in the estimates just quoted because of the confusion presented to the eye by points not following a regular line and because the cancellation of errors for the slope is accidentally small for uniform weighting.

Accordingly, therefore, the least squares equations were solved to give the best fit consistent with various types of potential wells. In order to make the solution consistent with the meson potential, for example, one can correct for the tail effect and then use values of $f^{(0)}$, $f^{(1)}$, $f^{(2)}$ consistent with the representations of these quantities in terms of depth and range parameters by formulas in HAB. For the square well, Eqs. (8.6) through (8.84) of BB were used, with Eq. (8.42) of that paper, to give an approximate value of $f^{(2)}=0.00475$ when a range was fixed. The value $f^{(2)}=-0.0015$ given by Jackson and Blatt[1] for the exponential well was used. The changes in $f^{(0)}$, $f^{(1)}$, and $f^{(2)}$ are so small that they are proportional to each other, and it is thus unnecessary to solve the least squares equations again once the solutions given by Eqs. (6) and (7) have been obtained. By the use of Eq. (5) along with Eqs. (6) and (7), one is able to find the best meson fit at once. Equations (8.6) through (8.42) of BB are used in place of Eq. (5) for the square well, while Eq. (11.3E) of Jackson and Blatt serve for the exponential well.

The results are:

(a) best meson fit,

$$\begin{aligned} &f^{(0)}=7.787\pm0.009,\quad f^{(1)}=0.953\pm0.005,\\ &\qquad\qquad f^{(2)}=-0.00949\quad(\text{criterion } \mathcal{g}),\\ &f^{(0)}=7.789\pm0.009,\quad f^{(1)}=0.952\pm0.005,\\ &\qquad\qquad f^{(2)}=-0.00947\quad(\text{criterion } \mathcal{g}\mathcal{E}). \end{aligned} \tag{8}$$

Although the values of the $f^{(i)}$ listed in Eq. (8) are referred to as "the best meson fit," the procedure followed in obtaining them contains an approximation which must now be explained. It was assumed that approximate proportionality of $f^{(2)}$ to the square of the range parameter holds for $f^{(2)}$, including tail effect, even though the proportionality of $\delta f^{(2)}$ to a a'^2 has not been investigated. Since the standard reference value of $f^{(2)}$ with tail which was used here (line F, Table I) was for $C'=99$, $a'=0.40$ a pair of parameters lying close to the values representing experiment, this approximation can be expected to be satisfactory. The values of C', a' obtained by means of it will be referred to below as "preliminary." They have been checked by two other methods which will be described presently.

(b) approximate best exponential fit,

$$f^{(0)}=7.810,\quad f^{(1)}=0.922,\quad f^{(2)}=-0.0015\quad(\text{criterion } \mathcal{g}\mathcal{E});\tag{8.1}$$

(c) approximate best square well fit,

$$f^{(0)}=7.825,\quad f^{(1)}=0.899,\quad f^{(2)}=0.0048\quad(\text{criterion } \mathcal{g}\mathcal{E}).\tag{8.2}$$

The values of Δ are, respectively, 0.0001, 0.028, and 0.087 for the meson, exponential, and square well fits. The smallness of these values indicates that the data are not sufficiently accurate to distinguish satisfactorily among the fits. The entries preceded by $\pm$ signs such as ±0.010 for $f^{(0)}$ indicate probable errors (not standard deviations) and have been calculated from rms residuals

in the least square work. It is of interest to compare them with the error in f to be expected for one percent error in scattering which is shown for comparison in Table II. With 15 observations of equal weight and a probable error of ± 0.01 in $f^{(0)}$, the probable error of one observation can be expected to be ± 0.04, and this number is reasonable if one supposes that the scattering measurements have an accuracy of one percent and employs Table II. The approximate magnitude of the uncertainty in $f^{(1)}$ can be understood as being the uncertainty in $f^{(0)}$ at an energy distant 2 Mev from a reference energy. The distance of 2 Mev is a reasonable distance since the observations are distributed through 3.3 Mev. The uncertainty of $f^{(1)}$ is thus essentially understandable also in terms of a one percent error in scattering.

The parameters C', a' of the Yukawa potential which correspond to the f-coefficients of Eq. (8) are obtained in first approximation from Eq. (4) by the use of the values of line F, Table I for $C'=99$, $a'=0.40$ as base coefficients. Equation (4) resulted from the HAB representations, and thus does not contain the effect of the tail of the potential. The preliminary values of C' and a' resulting from Eq. (4) were used, therefore, in Eq. (3) to compute $f^{(0)}$ and $f^{(1)}$. These tail corrections were applied with opposite sign to the "experimental fit" coefficients of Eq. (8); thus a virtual chopping off of the physical tail of the potential was effected. The coefficients so obtained were used in Eq. (4) to obtain the final values of the parameters. The results are

$$\begin{aligned} &C'=93.44mc^2, \quad a'=0.412e^2/mc^2, \quad \text{(criterion } \mathcal{J}) \\ &C'=93.58mc^2, \quad a'=0.412e^2/mc^2, \quad \text{(criterion } \mathcal{J}\mathcal{E}). \end{aligned} \tag{9}$$

Since these values differed by less than 0.5 percent from the preliminary results, the calculations of $\delta f^{(0)}$ and $\delta f^{(1)}$ were not repeated. The ranges agree within the precision with which they are written for the two weightings and correspond to a meson mass of 333 electron masses.

An independent method of arriving at the meson parameters was also used. For this purpose plots of f, E for $a'=0.410$ and 0.415 and each of three values of $1/(C'a'^2)^{\frac{1}{2}} \equiv y$ (namely, 0.2514372, 0.2511965, and 0.2509653), were compared with an f, E plot made using the coefficients of Eq. (6). The f values for the theoretical curves were computed directly from the material used to obtain the HAB phase shifts and interpolated (linearly in y, parabolically or bilinearly in a') to the desired values of a' and y. Tail corrections interpolated in Table IV were applied. In each case the quantity actually plotted was $f-(7.7941+0.9643E)$ in order that a small variation in values over the energy range 0–5 Mev would allow a very large scale to be used. Figure 3 shows the one experimental and six theoretical curves thus obtained, together with the experimental points used in the least squares analysis (Tables VI and VII). The theoretical curves show that a change in a' produces a rotation in the curves, while a change in y translates the curves up or down, and that the rotation and translation is very nearly linear in a' and y, respectively, for the small changes illustrated in the figure. It is of interest to compare Figs. 6, 7, and cogent material of BTE, where K_0, E plots with similar properties are used. The slope of chords drawn to the theoretical curves between 0 and 3.5 Mev was used as a measure of the rotation and with that of the experimental curve to determine the value of a'. With this value of a', linear interpolations at several energies for constant values of y were carried out on the graph to give several points on a curve with the experimental value of a' but with the values of y used in plotting the theoretical curves. The experimental value of y was then interpolated. The results of this analysis are:

$$C'=93.11mc^2, \quad a'=0.412e^2/mc^2. \tag{10}$$

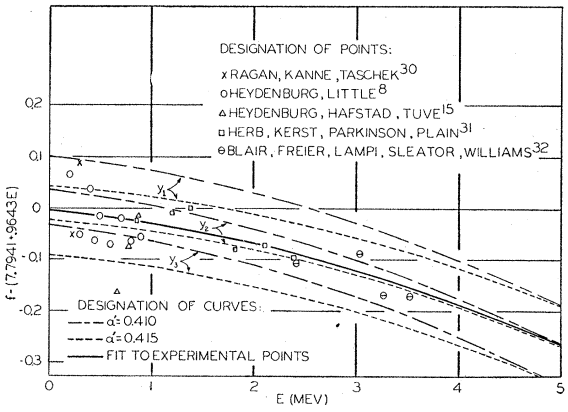

FIG. 3. Theoretical values of $f-(7.7941+0.9643E)$ for $a'=0.41$ and 0.415 with y_1, y_2, y_3 having the values 0.2514372, 0.2511965, and 0.2509653, respectively, *vs* E, together with the least squares fit to data of Eq. (6), criterion ($\mathcal{J}\mathcal{E}$), and experimental points as designated.

The close agreement between Eqs. (9) and (10) is satisfying, but is not to be taken as an indication of the precision of the present determination of the meson parameters. If one puts the probable errors of Eq. (8) into the first of Eqs. (4), an uncertainty in a' of 0.002 results. The scatter of the experimental points on Fig. 3 gives visual evidence of this uncertainty, as well as one in y. The probable errors give an uncertainty in y of 5×10^{-5}, which is actually larger than the spread in the three values of y obtained from Eq. (8) and in the graphical analysis, and is of the order of that which one might expect from the scatter (though somewhat smaller). As the most precise result of this analysis, therefore, the value of y may be stated as

$$y=0.25112\pm 0.00005.$$

Assuming this value of y and letting the uncertainty in C' be determined, therefore, by that in a', the parameters resulting from this analysis are

$$C'=93.4\pm 1mc^2, \quad a'=0.412\pm 0.002e^2/mc^2. \tag{11}$$

TABLE VIII. Values of f_0 for meson potential with parameters $C_0'=89.648$, $a_0'=0.42$, $y_0=0.2514667$, and coefficients for finding change in f with changes in a' and y.

E(Mev)	f_0	α_1	α_2	α_3
0.2	7.977	−11.350	275.5	−694
0.4	8.169	−10.889	277.4	−691
0.6	8.361	−10.436	279.2	−685
1.0	8.743	− 9.538	282.9	−673
1.4	9.122	− 8.679	286.7	−668
1.8	9.498	− 8.013	290.1	−662
2.2	9.872	− 7.033	293.7	−652
2.4	10.057	− 6.640	295.3	−647
2.6	10.242	− 6.212	297.2	−641
3.0	10.611	− 5.418	301.3	−640
4.0	11.526	− 3.467	310.3	−623
5.0	12.427	− 1.593	319.3	−610
6	13.315	0.158	327.5	−581
7	14.185	1.861	336.3	−578
8	15.056	3.523	344.6	−561
9	15.900	5.122	353.0	−547
10	16.684	6.613	359.8	−533
12	18.399	9.628	377.1	−512
14	20.012	12.457	392.6	−494
16	21.601	15.216	407.7	−470
18	23.142	17.757	422.4	−453
20	24.674	20.305	436.9	−433
22	26.181	22.778	451.0	−417
24	27.657	25.171	464.8	−399
26	29.120	27.536	478.1	−369
28	30.571	29.656	492.1	−358
30	32.000	31.909	513.0	−350
32	33.418	34.273	518.9	−312
36	36.203	38.342	544.1	−308
40	38.923	42.445	568.6	−269

From Eq. (8.1), the parameters of the best fit exponential well, if the potential of the interaction is written as $-Be^{-r/b}$, are, approximately,

$$B=214mc^2, \quad b=0.251e^2/mc^2. \tag{11.1}$$

The square well depth, D, and range, r, are determined from Eq. (8.2) to be about

$$D=12.60 \text{ Mev}, \; r=0.919e^2/mc^2, \tag{11.2}$$

$$D_c=13.50 \text{ Mev}. \tag{11.2'}$$

The square well depth, D_c, given in Eq. (11.2') includes the Coulomb potential inside the well. The correction from D of Eq. (11.2) has been made by Eq. (11) of BTE. A corresponding range correction, which is small, has not been made. These results agree well with those of BH and of Jackson and Blatt.

The series expansions of f with the coefficients of Eq. (6), (8), (8.1), and (8.2) are plotted in lower part of Fig. 1 up to 4 Mev with the experimental points included. The various sets of coefficients give essentially indistinguishable curves which fit the data equally well. It is of interest, therefore, to see how well a theoretical curve fits the higher energy data. In order to obtain the values of f needed for the meson well, an interpolation formula was obtained:

$$f=f_0+\alpha_1(a'-a_0')+\alpha_2(y-y_0)+\alpha_3(a'-a_0')(y-y_0), \tag{12}$$

where α_1, α_2, α_3 are functions of E and are given in Table VIII. Linear interpolation in E is quite accurate. The basic values of f, designated f_0 in Eq. (12), were computed for $C'=89.648$, $a'=0.420$, and corrected for tail effect from Table IV. The f_0 are also given in Table VIII. The values of a_0' and y_0 to be used are $a_0'=0.420$ and $y_0=0.2514667$. Accuracy of the order of 0.01 in f is expected if $(a'-a_0')$ and $(y-y_0)$ are $\sim$0.01 or less. Equation (12) with Table VIII may find more general application than in the present analysis. For use in converting experimental data to f values, there are also presented in Table IX values of C_0^2/η and $q_0/\eta-2ln\eta$. Equation (7.6) of BCP is then used with the experimental K_0, and this table to give f values.

For the exponential well, the value $f^{(3)}=1.2\times10^{-5}$ taken from the analysis of Jackson and Blatt was used together with the coefficients of Eq. (8.1) to plot the f curve at higher energies. The coefficients of Eq. (8.2) were used to plot the square well curve, and the plot was discontinued before the contribution from $f^{(3)}$ became significant, since $f^{(3)}>0$ for this well, and the curve had already begun diverging strongly from the experimental data at 20 Mev.

The f curves to 30 Mev corresponding to the parameters of Eqs. (11), (11.1), and (11.2) together with the linear fit of Eq. (7) and the available experimental data are shown in the upper part of Fig. 1. The data of Dearnley, Oxley, and Perry[35] have not been used because it is understood that the later work of Rouvina[2] at the same laboratory and using the same general method has a much higher accuracy. The correction to the phase shift of May and Powell,[29] the necessity of which has been noted by Jackson and Blatt,[1] has been used in preparing the graph. The other data plotted consist of work by Meagher,[36] Panofsky, and Fillmore,[4] Faris and Wright,[5] Zimmerman and Kreuger,[6] Mather,[7] Wilson and Creutz,[37] Wilson,[38] Wilson, Lofgren, Richardson, Wright, and Shankland,[39] and Cork.[3] For the value at 10 Mev by Wilson, the analysis of Peierls and Preston[40] was used. The values of f for the other two points of Wilson *et al.* were taken from Jackson and Blatt. The interpretation of Cork[41] was used for other values given by him.

Points at energies below $\sim$7 Mev do not favor any one of the potentials. Only the values of Rouvina

[35] Dearnley, Oxley, and Perry, Phys. Rev. **73**, 1290 (1949).
[36] R. E. Meagher, Phys. Rev. **78**, 667 (1950).
[37] R. R. Wilson and E. C. Creutz, Phys. Rev. **71**, 339 (1947).
[38] R. R. Wilson, Phys. Rev. **71**, 384 (1947).
[39] Wilson, Lofgren, Richardson, Wright, and Shankland, Phys. Rev. **71**, 560 (1947).
[40] R. E. Peierls and M. A. Preston, Phys. Rev. **72**, 250 (1947).
[41] The figure displaying agreement of data with theory in the paper just quoted refers to Jackson and Blatt's work for the theory used. Since the power series expansion given by Jackson and Blatt is not supposed to apply at such high energies, and since in the present work the employment of higher powers of E than the third was found necessary, there is some difficulty in understanding this statement. The curve drawn in Cork's paper differs only slightly from the f curve obtained by use of phase shifts tabulated by HAB for $a'=0.40$, $C'=99$ and could have been ascertained by substituting these phase shifts in the formula for f and applying a small correction for differences in values of potential well parameters.

indicate a slight preference for the Yukawa and exponential wells. The preference is not definite, but the value at 5.86 Mev from the measurements of Zimmerman and Kreuger[6] also appears to favor the longtailed potentials. It seems that some additional work at energies in this vicinity might be helpful in distinguishing between the phenomenologically postulated potentials. At the higher energies, the values of Cork[3] and that of Panofsky and Fillmore[4] definitely favor the longtailed potentials.

APPENDIX A

The justification for employing simple iteration in determining "best values" of potential well parameters will now be given. The complete problem with allowance for the entrance of $\partial f^{(2)}/\partial f^{(0)}$, $\partial f^{(2)}/\partial f^{(1)}$ in the answer will be considered, and it will then be shown that the effect of these quantities is insignificant. It is convenient to use the simplified notation

$$f^{(0)}=l,\quad E_{\max}f^{(1)}=m,\quad E_{\max}{}^2f^{(2)}=n,\quad E/E_{\max}=x$$
$$\partial n/\partial l=n_l,\quad \partial n/\partial m=n_m \tag{A-1}$$
$$E_{\max}=\text{maximum energy.}$$

The least squares equations are

$$\begin{aligned}\langle(1+n_lx^2)(l+mx+nx^2-f)\rangle&=0\\ \langle(x+n_mx^2)(l+mx+nx^2-f)\rangle&=0,\end{aligned} \tag{A-2}$$

where $\langle\ \rangle$ indicates a weighted mean. These equations are conveniently referred to the best quadratic fit (l_0, m_0, n_0) for the same relative weights. For this

$$\begin{aligned}\langle l_0+m_0x+n_0x^2-f\rangle&=0\\ \langle l_0x+m_0x^2+n_0x^3-fx\rangle&=0\\ \langle l_0x^2+m_0x^3+n_0x^4-fx^2\rangle&=0.\end{aligned} \tag{A-3}$$

Introducing $\lambda=l-l_0$, $\mu=m-m_0$, $\nu=n-n_0$ one obtains

$$\begin{aligned}n(l_0+\lambda, m_0+\mu)-n_0-\nu&=0\\ \langle(1+n_lx^2)(\lambda+\mu x+\nu x^2)\rangle&=0\\ \langle(x+n_mx^2)(\lambda+\mu x+\nu x^2)\rangle&=0.\end{aligned} \tag{A-4}$$

The λ, μ, ν are relatively small. The approximation corresponding to the iteration procedure described in the text is obtained by setting $n_l, n_m=0$ in Eq. (A-2). The solution of the modified equation will be called l', m', n'. To introduce $\lambda'=l'-l_0$, $\mu'=m-m_0$, $\nu'=n-n_0$, there results a way of obtaining (λ', μ', ν') as the solution of

$$\begin{aligned}n(l_0+\lambda', m_0+\mu', n_0+\nu')&=0\\ \langle\lambda'+\mu'x+\nu'x^2\rangle&=0\\ \langle\lambda'x+\mu'x^2+\nu'x^3\rangle&=0.\end{aligned} \tag{A-5}$$

On expanding $n(l_0+\lambda, m_0+\mu)$ in Eq. (A-4) in a Taylor series and neglecting quadratic and higher order terms one obtains

$$\begin{aligned}\lambda&=\{\langle x\rangle\langle x^3\rangle-\langle x^2\rangle^2+n_l[\langle x^3\rangle^2-\langle x^2\rangle\langle x^4\rangle]\\ &\quad+n_m[\langle x\rangle\langle x^4\rangle-\langle x^3\rangle\langle x^2\rangle]\}/D\\ \mu&=\{\langle x\rangle\langle x^2\rangle-\langle x^3\rangle+n_l\langle x\rangle\langle x^4\rangle-\langle x^2\rangle\langle x^3\rangle]\\ &\quad+n_m[\langle x^2\rangle^2-\langle x^4\rangle]\}/D\\ \nu&=\{\langle x^2\rangle-\langle x\rangle^2+n_l[\langle x^2\rangle^2-\langle x\rangle\langle x^3\rangle]\\ &\quad+n_m[\langle x^3\rangle-\langle x\rangle\langle x^2\rangle]\}/D,\end{aligned} \tag{A-6}$$

with

$$[n_0-n(l_0, m_0)]D=\langle x\rangle^2-\langle x^2\rangle+2n_l[\langle x\rangle\langle x^3\rangle-\langle x^2\rangle^2]+2n_m[\langle x\rangle\langle x^2\rangle-\langle x^3\rangle]+Q \tag{A-7}$$

$$Q=n_l{}^2[\langle x^3\rangle^2-\langle x^2\rangle\langle x^4\rangle]+2n_ln_m[\langle x\rangle\langle x^4\rangle-\langle x^2\rangle\langle x^3\rangle]+n_m{}^2[\langle x^2\rangle^2-\langle x^4\rangle]. \tag{A-8}$$

The λ', μ', ν' are obtained by setting $n_l=n_m=0$ in these relations. If the measurements have equal weights and are distributed uniformly in the range $0<x<1$, these formulas reduce to

$$\begin{aligned}\lambda&=(1/72-n_l/240+n_m/60)/D\\ \mu&=(-1/12+n_l/60-4n_m/45)/D\\ \nu&=(1/12-n_l/72+n_m/12)/D\end{aligned} \tag{A-9}$$

$$[n_0-n(l_0, m_0)]D=-1/12+n_l/36-n_m/6-n_l{}^2/240+n_ln_m/30-4n_m{}^2/45. \tag{A-10}$$

For the approximate best quadratic fit to data with one of the weightings that has been tried in a preliminary way $f^{(0)}=7.789$, $f^{(1)}=0.938$, $f^{(2)}=-0.0049$, and $E_{\max}=3.5$ Mev the values are $l_0=7.79$, $m_0=3.3$, $n_0=-0.060$.

TABLE IX. Quantities for computing f from phase shifts.

E(Mev)	$C_0{}^2/\eta$	$(q_0/\eta)-2ln(\eta)$
0.2	0.76448	1.50563
0.4	1.649211	2.06980
0.6	2.411586	2.42936
1.0	3.695285	2.90232
1.4	4.776889	3.22223
1.8	5.728617	3.46426
2.2	6.588139	3.65899
2.4	6.990691	3.74377
2.6	7.377889	3.82192
3.0	8.112484	3.96199
4	9.767629	4.24472
5	11.233633	4.46489
6	12.563461	4.64522
7	13.789206	4.79795
8	14.932039	4.93042
9	16.006798	5.04737
10	17.024373	5.15207
12	18.919171	5.33339
14	20.663649	5.48683
16	22.288750	5.61983
18	23.816065	5.73719
20	25.261372	5.84222
22	26.636624	5.93726
24	27.951095	6.02404
26	29.212215	6.10389
28	30.425987	6.17784
30	31.597390	6.24668
32	32.730549	6.31110
36	34.895729	6.42867
40	36.944136	6.53387

In the approximation $n=\text{const}m^2$ which takes into account the principal variations of n one has $n_l=0$, $n_m=-0.037$, and $n_m^2/45$ is quite negligible. All terms in n_m are seen to be sufficiently small to allow neglect of all but first-order effects in n_m. With this understanding and distinguishing quantities for the approximation l', m', n' by primes throughout, one obtains $D'/D=1-n_m=1.036$, $\lambda/\lambda'=0.9928$, $\mu/\mu'=0.9976$, $\nu/\nu'=1$. For the relative weights corresponding to the best quadratic fit under discussion the l', m', n' type of best meson fit to data gives $f^{(0)}=7.780$, $f^{(1)}=0.954$, $f^{(2)}=-0.00935$ which yields $\lambda'=-0.008$, $\mu'=0.056$ so that the correction factor to the l', m', n' type of solution arranged to obtain the l, m, n type is $1+\epsilon$ with $\epsilon=-0.0024(0.016/0.94)=-4\times10^{-5}$ for m', i.e., $f^{(1)}$ which amounts to a 0.004 percent correction in the range parameter. The effect on $f^{(0)}$ is a correction of the order -0.00006 in a total of 7.8 and is also of no interest. By employing the representations of the $f^{(i)}$ in HAB and taking into account n_l one estimates for the numbers under discussion that $n_m/n_l\cong-18$ so that n_m gives the main effects.

In the aforementioned estimates the effect of second- and higher order derivatives of n_0 has been neglected consistently for the complete problem corresponding to (l, m, n), and its approximation corresponding to (l', m', n'). Since these terms are small and since the difference in the two solutions, as considered, is negligible, the corrections for the second- and higher order derivatives may be considered as being practically identical and for the condition under discussion the corrections are of no practical interest.

In the more general case of arbitrary weights one has

$$\begin{aligned}\lambda/\lambda'&=1+n_l[(\langle x^3\rangle^2-\langle x^2\rangle\langle x^4\rangle)/\beta-\beta/\delta]\\&\quad+n_m(\alpha/\beta-\gamma/\delta),\\ \mu/\mu'&=1+n_l(\alpha/\gamma-\beta/\delta)\\&\quad+n_m[(\langle x^2\rangle^2-\langle x^4\rangle)/\gamma-\gamma/\delta],\end{aligned}\tag{A-11}$$

where

$$\begin{aligned}\alpha&=\langle x\rangle\langle x^4\rangle-\langle x^2\rangle\langle x^3\rangle, & \beta&=\langle x\rangle\langle x^3\rangle-\langle x^2\rangle^2\\ \gamma&=\langle x\rangle\langle x^2\rangle-\langle x^3\rangle, & \delta&=\langle x\rangle^2-\langle x^2\rangle.\end{aligned}\tag{A-12}$$

Substitution into these formulas for the fit discussed earlier in this Appendix employing values of the $\langle x^n\rangle$ which correspond to the relative weights employed in the least squares adjustment of the $f^{(i)}$, there result appreciably larger effects than in the estimates making use of uniform weighting. The values obtained by means of Eqs. (A-11), (A-12) are as follows

$$\lambda/\lambda'=0.9494,\quad \mu/\mu'=0.9802.$$

While larger than in the first estimates, the difference between the complete and approximate solutions is seen to be smaller than many other uncertainties in the determination of $f^{(0)}$, $f^{(1)}$, $f^{(2)}$.

The best linear fit corresponds to $l=l_0+\lambda''$, $m=m_0+\mu''$, $n=n_0+\nu''=0$ with

$$\lambda''+\mu''\langle x\rangle+\nu''\langle x^2\rangle=0,\quad \lambda''\langle x\rangle+\mu''\langle x^2\rangle+\nu''\langle x^3\rangle=0.$$

It follows from these relations and the result of setting n_l, n_m in Eqs. (A-6), (A-7) so as to obtain λ', μ', ν' that

$$\lambda''/\lambda'=\mu''/\mu'=-n_0/\nu'=\nu''/\nu',$$

which implies that difference between the values of l, m, n for the best quadratic and linear fits are proportional to corresponding differences between the best quadratic fit and any other fit with a preassigned $f^{(2)}$. According to the elementary relation for proportions this also implies that differences of l, m, n for two fits with arbitrarily assigned $f^{(2)}$ from the linear fit are proportional to each other. This fact follows directly from the least squares linear equations for the three fits. It gives a useful rule for avoiding repetition of similar calculations.